SOME NEW APPROXIMATIONS OF GLAISHER-KINKELIN CONSTANT

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Abstract. In this paper, we established some new approximation formulas for calculating Glaisher-Kinkelin constant.

1. Introduction

The Glaisher-Kinkelin Constant $A = 1.28242713\ldots$ is defined by

$$A = \lim_{n\to\infty} \frac{1^1 2^2 \cdots n^n}{n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}}}.$$

It is very important to construct new sequences which converge to these fundamental constants with increasingly higher speed. The Glaisher-Kinkelin Constant first appeared in Bares[1] and is also related to Riemann zeta function ζ, or the Euler-Mascheroni constant $\gamma = 0.5772$ such as

$$A = \exp\left\{\frac{1}{12} - \zeta'(-1)\right\} = \exp\left\{\frac{-\zeta'(2)}{2\pi^2} + \frac{\gamma + \ln(2\pi)}{12}\right\}.$$

Many useful formulas related to A exist, such as

$$\int_0^\infty \frac{x\ln x}{e^{2\pi x}-1}dx = \frac{1}{24} - \frac{1}{2}\ln A,$$

$$\int_0^1 x^2\psi(x)dx = \ln\left(\frac{A^2}{\sqrt{2\pi}}\right),$$

and

$$\int_0^1 \ln\Gamma(x+1)dx = -\frac{1}{2} - \frac{7}{24}\ln 2 + \frac{1}{4}\ln\pi + \frac{3}{2}\ln A.$$

in references[3, 4, 5].

To our knowledge, one of the useful sequences is

$$u_n = \sum_{k=1}^n k\ln k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\ln n + \frac{n^2}{4} \tag{1.1}$$

which converges constant $\ln A$. Up to now, many mathematicians made great efforts in the area of concerning the rate of convergence of these sequences and establishing faster sequences to converge to constant A. In [7], Mortici showed an inequality for constant A.

$$u_n - \frac{1}{720n^2} + \frac{1}{5040n^4} - \frac{1}{10080n^6} < \ln A < u_n - \frac{1}{720n^2} + \frac{1}{5040n^4}. \tag{1.2}$$

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Later, Lu and Mortici[6] also established a convergent sequence for the Glaisher-Kinkelin Constant as follows

$$v_n = \sum_{k=1}^{n} k\ln k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\ln\left(n + \frac{a_3}{n^3} + \frac{a_4}{n^4} + \frac{a_5}{n^5} + \cdots\right) + \frac{n^2}{4} \tag{1.3}$$

where $a_3 = \frac{1}{360}, a_4 = -\frac{1}{360}$ and $a_5 = \frac{29}{15120}$. Based on this sequence, they also gave a new inequality for constant A. Recently, You[10] established the following approximate sequence

$$w_n(i) = \sum_{k=1}^{n} k\ln k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\ln\left(n + \eta_0(n) + \eta_1(n) + \cdots \eta_i(n)\right) + \frac{n^2}{4} \tag{1.4}$$

where $\eta_0(n) = 0, \eta_1(n) = \frac{a_1}{n^3+b_2n^2+b_1n+b_0}, \cdots$ Hence, he proved the following inequality

$$\frac{1009}{25401600}\frac{1}{(n+1)^6} \le w_n(1) - \ln A < \frac{1009}{25401600}\frac{1}{(n-1)^6}. \tag{1.5}$$

In view of (1.1), we define the sequence $\{\alpha_n\}_{n\in N}$ and $\{\beta_n\}_{n\in N}$ by

$$\alpha_n = \sum_{k=1}^{n} k\ln k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\ln\left(n + \frac{q}{n+p}\right) + \frac{n^2}{4}, p \neq 0 \tag{1.6}$$

and

$$\beta_n = \sum_{k=1}^{n} k\ln k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\ln\left(n + \frac{r}{(n+1)} + \frac{s}{(n+1)^2} + \frac{t}{(n+1)^3}\right) + \frac{n^2}{4}. \tag{1.7}$$

We are devote to finding the values of the parameters p, q, r, s, t such that $\{\alpha_n\}_{n\in N}$ and $\{\beta_n\}_{n\in N}$ are the fastest sequences which would converge to zero. In fact, this provides the best approximations of the form (1.6) and (1.7).

2. Main results

The following Lemma is useful.

Lemma 2.1. *If the sequence $\{\lambda_n\}_{n\in N}$ converges to zero and if there exists the following limit*

$$\lim_{n\to\infty} n^k(\lambda_n - \lambda_{n+1}) = l, (k > 1).$$

Then

$$\lim_{n\to\infty} n^{k-1}\lambda_n = \frac{l}{k-1}.$$

Remark 2.1. Lemma 2.1 was firstly proved by Mortici in [8]. It is very effective for accelerating the speed of convergence of the sequence or in constructing some asymptotic expansions.

Theorem 2.1. *Let the sequence $\{\alpha_n\}_{n\in N}$ be defined by (1.6). Then for $p = 1, q = \frac{1}{6} \pm \frac{1}{15}\sqrt{5}$, we have $\lim_{n\to\infty} n^4(\alpha_n - \alpha_{n+1}) = \frac{1}{240}$ and $\lim_{n\to\infty} n^3\alpha_n = \frac{1}{720}$. That is the speed of convergence of the sequence $\{\alpha_n\}_{n\in N}$ is given by the order $O(n^{-3})$.*

Proof. We calculate the difference $\alpha_n - \alpha_{n+1}$ as the following power series in $\frac{1}{n}$:

$$\alpha_n - \alpha_{n+1} = \left(\tfrac{1}{2}qp - \tfrac{1}{2}q\right)\tfrac{1}{n^2} + \left(\tfrac{1}{360} + \tfrac{1}{3}q + \tfrac{1}{2}qp - qp^2 + \tfrac{1}{2}q^2\right)\tfrac{1}{n^3}$$
$$+ \left(-\tfrac{1}{240} - \tfrac{1}{4}q + \tfrac{3}{2}qp^3 - \tfrac{3}{2}q^2p - \tfrac{3}{4}qp\right)\tfrac{1}{n^4} + O\left(\tfrac{1}{n^5}\right).$$

Applying Lemma 2.1, the parameters p, q which produce the fastest convergence are given by

$$\begin{cases} \tfrac{1}{2}qp - \tfrac{1}{2}q = 0, \\ \tfrac{1}{360} + \tfrac{1}{3}q + \tfrac{1}{2}qp - qp^2 + \tfrac{1}{2}q^2 = 0. \end{cases}$$

Simple computation results in $p = 1, q = \frac{1}{6} \pm \frac{1}{15}\sqrt{5}$. Furthermore, we get

$$\alpha_n - \alpha_{n+1} = \frac{1}{240}\frac{1}{n^4} + O\left(\frac{1}{n^5}\right).$$

Using Lemma 2.1 again, we complete the proof. □

Theorem 2.2. *Let the sequence $\{\beta_n\}_{n\in N}$ be defined by (1.7). Then for $s = t = 0, r = \frac{1}{6} \pm \frac{1}{15}\sqrt{5}$, we have $\lim_{n\to\infty} n^4(\beta_n - \beta_{n+1}) = \frac{1}{80}$ and $\lim_{n\to\infty} n^3\beta_n = \frac{1}{240}$. That is the speed of convergence of the sequence $\{\beta_n\}_{n\in N}$ is given by the order $O(n^{-3})$.*

Proof. From (1.7), we can easily obtain $\beta_n - \beta_{n+1}$ and write the difference on power of $\frac{1}{n}$ as

$$\beta_n - \beta_{n+1} = \tfrac{s}{2}\tfrac{1}{n} - \left(\tfrac{3}{2}s + \tfrac{1}{2}t\right)\tfrac{1}{n^2} + \left(\tfrac{1}{2}r^2 - \tfrac{1}{6}r + \tfrac{1}{360}\right)\tfrac{1}{n^3}$$
$$+ \left(\tfrac{1}{2}r - \tfrac{49}{6}s - \tfrac{19}{12}t - \tfrac{3}{2}r^2 + \tfrac{1}{240} + 4rs + \tfrac{1}{2}rt - \tfrac{3}{4}s^2\right)\tfrac{1}{n^4} + O\left(\tfrac{1}{n^5}\right).$$

Following similar method used in the proof of Theorem 2.1, the parameters r, s, t satisfy the following equation:

$$\begin{cases} \tfrac{s}{2} = 0, \\ \tfrac{3}{2}s + \tfrac{1}{2}t = 0, \\ \tfrac{1}{2}r^2 - \tfrac{1}{6}r + \tfrac{1}{360} = 0. \end{cases}$$

So we have $s = t = 0$ and $r = \frac{1}{6} \pm \frac{1}{15}\sqrt{5}$. Applying Lemma 2.1, the proof is complete. □

Remark 2.2. The numerical computation were performed by using the Maple software.

Theorem 2.2 prompts us to pose the following open problem:

Open Problem 2.1. *Find the best constants $r_j, (j \in N)$ such that*

$$\ln A \sim \sum_{k=1}^{n} k\ln k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\ln\left(n + \sum_{j=1}^{\infty}\frac{r_j}{(n+1)^j}\right) + \frac{n^2}{4}.$$

Remark 2.3. It is worth noting that Chen[2] gave the asymptotic representation of the Glaisher-Kinkelin constant

$$1^1 2^2 \cdots n^n \sim A \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}} \exp\left\{\sum_{k=1}^{\infty}\frac{-B_{k+2}}{k(k+1)(k+2)}\frac{1}{n^k}\right\}$$

by using Euler-Maclaurin formula where B_k is Bernoulli number. Later, Wang and Liu[9] showed

$$1^1 2^2 \cdots n^n \sim A \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}} \left\{\sum_{k=1}^{\infty}\frac{g_k}{(n+h)^k}\right\}^{1/r}.$$

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