OSTROWSKI TYPE INTEGRAL INEQUALITIES FOR MULTIPLE INTEGRALS OF FUNCTIONS WITH LIPSCHITZIAN PARTIAL DERIVATIVES ON CONVEX BODIES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we establish some Ostrowski type inequalities for functions of n-variables defined on closed and bounded convex bodies of the Euclidean space $\mathbb{R}^n$ whose partial derivatives are Lipschitzian. Some examples for *n-hyper boxes* $R_n := [a_1, b_1] \times ... \times [a_n, b_n] \subset \mathbb{R}^n$, $n \geq 2$ and 3-dimensional balls are also provided.

1. Introduction

In paper [1], the authors obtained among others the following results concerning the difference between the double integral on the disk and the values in the center or the path integral on the circle:

Theorem 1. *If $f : D(C, R) \to \mathbb{R}$ has continuous partial derivatives on $D(C, R)$, the disk centered in the point $C = (a, b)$ with the radius $R > 0$, and*

$$\left\|\frac{\partial f}{\partial x}\right\|_{D(C,R),\infty} := \sup_{(x,y)\in D(C,R)}\left|\frac{\partial f(x,y)}{\partial x}\right| < \infty,$$
$$\left\|\frac{\partial f}{\partial y}\right\|_{D(C,R),\infty} := \sup_{(x,y)\in D(C,R)}\left|\frac{\partial f(x,y)}{\partial y}\right| < \infty;$$

then

$$\left|f(C) - \frac{1}{\pi R^2}\iint_{D(C,R)} f(x,y)\,dx\,dy\right| \leq \frac{4}{3\pi}R\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(C,R),\infty} + \left\|\frac{\partial f}{\partial y}\right\|_{D(C,R),\infty}\right]. \tag{1.1}$$

The constant $\frac{4}{3\pi}$ is sharp.

We also have

$$\left|\frac{1}{\pi R^2}\iint_{D(C,R)} f(x,y)\,dx\,dy - \frac{1}{2\pi R}\int_{\sigma(C,R)} f(\gamma)\,dl(\gamma)\right| \leq \frac{2R}{3\pi}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(C,R),\infty} + \left\|\frac{\partial f}{\partial y}\right\|_{D(C,R),\infty}\right], \tag{1.2}$$

1991 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Ostrowski inequality, Hermite-Hadamard inequality, Double integral inequalities.

RGMIA Res. Rep. Coll. 22 (2019), Art. 52, 17 pp. Received 10/05/19

where $\sigma(C,R)$ *is the circle centered in* $C=(a,b)$ *with the radius* $R>0$ *and*

$$\left|f(C)-\frac{1}{2\pi R}\int_{\sigma(C,R)}f(\gamma)\,dl(\gamma)\right| \leq \frac{2R}{\pi}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(C,R),\infty}+\left\|\frac{\partial f}{\partial y}\right\|_{D(C,R),\infty}\right]. \tag{1.3}$$

In the same paper [1] the authors also established the following Ostrowski type inequality:

Theorem 2. *If* f *has bounded partial derivatives on* $D(0,1)$*, the unity disk, then*

$$\begin{aligned}\left|f(u,v)-\frac{1}{\pi}\iint_{D(0,1)}f(x,y)\,dx\,dy\right| &\leq \frac{2}{\pi}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(0,1),\infty}\left(u\arcsin u+\frac{1}{3}\sqrt{1-u^2}\,(2+u^2)\right)\right.\\ &\quad\left.+\left\|\frac{\partial f}{\partial y}\right\|_{D(0,1),\infty}\left(v\arcsin v+\frac{1}{3}\sqrt{1-v^2}\,(2+v^2)\right)\right]\end{aligned} \tag{1.4}$$

for any $(u,v)\in D(0,1)$*.*

For other Ostrowski type integral inequalities for multiple integrals see [2]-[14].

In the following, consider G_n a closed and bounded convex subset of $\mathbb{R}^n$. Define

$$V_{G_n}:=\int\cdots\int_{G_n}dx_1...dx_n$$

the *n-volume* of G_n and $(\overline{x_{1G_n}},...,\overline{x_{nG_n}})$ the *n-centre of gravity* for G_n, where

$$\overline{x_{i,G_n}}:=\frac{1}{V_{G_n}}\int\cdots\int_{G_n}x_i dx_1...dx_n,\ \text{for } i\in\{1,...,n\}.$$

Consider the function of n variables $f=f(x_1,...,x_n)$ and denote by $\frac{\partial f}{\partial x_i}$ the partial derivative with respect to the variable x_i for $i\in\{1,...,n\}$.

As examples, we can consider the *n-hyper box* $R_n:=[a_1,b_1]\times...\times[a_n,b_n]\subset\mathbb{R}^n$, for which

$$V_{R_n}=\prod_{i=1}^{n}(b_i-a_i)\ \text{and}\ \overline{x_{i,R_n}}=\frac{b_i+a_i}{2}\ \text{for } i\in\{1,...,n\}.$$

Also, if we consider the *n-hyper ball* centered in $C=(c_1,...,c_n)$ and with radius $R>0$ defined by

$$B_n(C,R):=\left\{(x_1,...,x_n)\mid \sum_{i=1}^{n}(x_i-c_i)^2\leq R^2\right\}\subset\mathbb{R}^n,$$

then the *n-volume of* $B_n(C,R)$ is

$$V_{B_n}=\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}R^n,\ n\geq 2$$

where Γ is Euler's *gamma function.*

Using explicit formulas for particular values of the gamma function at the integers and half integers gives formulas for the n-volume of the Euclidean ball as

$$V_{B_{2k}} = \frac{\pi^k}{k!} R^{2k} \text{ and } V_{B_{2k+1}} = \frac{2\,(k!)\,(4\pi)^k}{(2k+1)!} R^{2k+1},\ k \geq 1.$$

We also have $\overline{x_{i,B_n}} = c_i$ for all $i \in \{1, ..., n\}$.

Motivated by the above results, in this paper we establish some Ostrowski type inequalities for functions of n-variables defined on closed and bounded convex bodies of the Euclidean space $\mathbb{R}^n$ whose partial derivatives are Lipschitzian. Some examples for *n-hyper boxes* $R_n := [a_1, b_1] \times ... \times [a_n, b_n] \subset \mathbb{R}^n$, $n \geq 2$ and 3-dimensional balls are also provided.

2. The Main Results

We have:

Lemma 1. *If $f : G_n \to \mathbb{C}$ is differentiable on G_n, then for all $(x_1, ..., x_n)$, $(u_1, ..., u_n) \in G_n$ and $\lambda_i \in \mathbb{C}$, $i \in \{1, ..., n\}$ we have the equality*

$$f(x_1, ..., x_n) = f(u_1, ..., u_n) + \sum_{i=1}^n (x_i - u_i)\,\lambda_i + \sum_{i=1}^n (x_i - u_i) \int_0^1 \left(\frac{\partial f}{\partial x_i} [t(x_1, ..., x_n) + (1-t)(u_1, ..., u_n)] - \lambda_i \right) dt. \tag{2.1}$$

Proof. By Taylor's multivariate theorem with integral remainder, we have

$$f(x_1, ..., x_n) = f(u_1, ..., u_n) + \sum_{i=1}^n (x_i - u_i) \int_0^1 \frac{\partial f}{\partial x_i} [t(x_1, ..., x_n) + (1-t)(u_1, ..., u_n)]\, dt \tag{2.2}$$

for all $(x_1, ..., x_n)$, $(u_1, ..., u_n) \in G_n$.

If $\lambda_i \in \mathbb{C}$, $i \in \{1, ..., n\}$, then

$$\begin{aligned} &(x_i - u_i) \int_0^1 \left(\frac{\partial f}{\partial x_i} [t(x_1, ..., x_n) + (1-t)(u_1, ..., u_n)] - \lambda_i \right) dt \\ &= (x_i - u_i) \int_0^1 \frac{\partial f}{\partial x_i} [t(x_1, ..., x_n) + (1-t)(u_1, ..., u_n)]\, dt - (x_i - u_i)\,\lambda_i \end{aligned}$$

and by (2.2) we get the desired result (2.1). □

We assume that the partial derivatives $\frac{\partial f}{\partial x_i}$, $i \in \{1, ..., n\}$ satisfy the Lipschitz type conditions

$$\left| \frac{\partial f}{\partial x_i}(x_1, ..., x_n) - \frac{\partial f}{\partial x_i}(u_1, ..., u_n) \right| \leq \sum_{j=1}^n L_{ij} |x_j - u_j| \tag{2.3}$$

for all $(x_1, ..., x_n)$, $(u_1, ..., u_n) \in G_n$ where $L_{ij} > 0$, $i, j \in \{1, ..., n\}$ are given positive constants.

Theorem 3. *If $f : G \to \mathbb{C}$ is differentiable on G and the partial derivatives $\frac{\partial f}{\partial x_i}$, $i \in \{1, ..., n\}$ satisfy the Lipschitz type conditions (2.3), then for all $(u_1, ..., u_n) \in G_n$ we have*

$$\left| \frac{1}{V_{G_n}} \int \cdots \int_{G_n} f(x_1, ..., x_n) \, dx_1 ... dx_n - f(u_1, ..., u_n) - \sum_{i=1}^{n} \left(\overline{x_{i,G_n}} - u_i \right) \frac{\partial f}{\partial x_i}(u_1, ..., u_n) \right| \leq \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} L_{ij} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} |x_i - u_i| \, |x_j - u_j| \, dx_1 ... dx_n, \tag{2.4}$$

In particular,

$$\left| \frac{1}{V_{G_n}} \int \cdots \int_{G_n} f(x_1, ..., x_n) \, dx_1 ... dx_n - f\left(\overline{x_{1,G_n}}, ..., \overline{x_{n,G_n}} \right) \right| \leq \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} L_{ij} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left| x_i - \overline{x_{i,G_n}} \right| \left| x_j - \overline{x_{j,G_n}} \right| dx_1 ... dx_n. \tag{2.5}$$

Proof. From Lemma 1 we have for $\lambda_i = \frac{\partial f}{\partial x_i}(u_1, ..., u_n)$, $i \in \{1, ..., n\}$ that

$$f(x_1, ..., x_n) = f(u_1, ..., u_n) + \sum_{i=1}^{n} (x_i - u_i) \frac{\partial f}{\partial x_i}(u_1, ..., u_n) + \sum_{i=1}^{n} (x_i - u_i) \int_0^1 \left(\frac{\partial f}{\partial x_i}[t(x_1, ..., x_n) + (1-t)(u_1, ..., u_n)] - \frac{\partial f}{\partial x_i}(u_1, ..., u_n) \right) dt \tag{2.6}$$

for all $(x_1, ..., x_n), (u_1, ..., u_n) \in G_n$.

By taking the integral mean $\frac{1}{V_{G_n}} \int \cdots \int_{G_n} dx_1 ... dx_n$ over $(x_1, ..., x_n)$ in (2.6) we get

$$\frac{1}{V_{G_n}} \int \cdots \int_{G_n} f(x_1, ..., x_n) \, dx_1 ... dx_n = f(u_1, ..., u_n) + \sum_{i=1}^{n} \left(\frac{1}{V_{G_n}} \int \cdots \int_{G_n} x_i dx_1 ... dx_n - u_i \right) \frac{\partial f}{\partial x_i}(u_1, ..., u_n) + \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \sum_{i=1}^{n} (x_i - u_i) \times \left(\int_0^1 \left(\frac{\partial f}{\partial x_i}[t(x_1, ..., x_n) + (1-t)(u_1, ..., u_n)] - \frac{\partial f}{\partial x_i}(u_1, ..., u_n) \right) dt \right) dx_1 ... dx_n \tag{2.7}$$

for all $(u_1, ..., u_n) \in G_n$.

By using the equality (2.7) we get

$$\left| \frac{1}{V_{G_n}} \int \cdots \int_{G_n} f(x_1, ..., x_n) \, dx_1 ... dx_n - f(u_1, ..., u_n) - \sum_{i=1}^{n} \left(\overline{x_{i,G_n}} - u_i \right) \frac{\partial f}{\partial x_i}(u_1, ..., u_n) \right| \tag{2.8}$$

$$= \left| \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \sum_{i=1}^{n} (x_i - u_i) \right.$$
$$\left. \times \left(\int_0^1 \left(\frac{\partial f}{\partial x_i} [t(x_1, ..., x_n) + (1-t)(u_1, ..., u_n)] - \frac{\partial f}{\partial x_i}(u_1, ..., u_n) \right) dt \right) dx_1 ... dx_n \right|$$
$$\leq \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left| \sum_{i=1}^{n} (x_i - u_i) \right.$$
$$\left. \times \left(\int_0^1 \left(\frac{\partial f}{\partial x_i} [t(x_1, ..., x_n) + (1-t)(u_1, ..., u_n)] - \frac{\partial f}{\partial x_i}(u_1, ..., u_n) \right) dt \right) \right| dx_1 ... dx_n$$

$$\leq \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \sum_{i=1}^{n} |x_i - u_i|$$
$$\times \left| \int_0^1 \left(\frac{\partial f}{\partial x_i} [t(x_1, ..., x_n) + (1-t)(u_1, ..., u_n)] - \frac{\partial f}{\partial x_i}(u_1, ..., u_n) \right) dt \right| dx_1 ... dx_n$$
$$\leq \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \sum_{i=1}^{n} |x_i - u_i|$$
$$\times \left(\int_0^1 \left| \frac{\partial f}{\partial x_i} [t(x_1, ..., x_n) + (1-t)(u_1, ..., u_n)] - \frac{\partial f}{\partial x_i}(u_1, ..., u_n) \right| dt \right) dx_1 ... dx_n$$
$$\leq \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \sum_{i=1}^{n} |x_i - u_i| \left(\sum_{j=1}^{n} L_{ij} |x_j - u_j| \int_0^1 t dt \right) dx_1 ... dx_n$$
$$= \frac{1}{2} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \sum_{i=1}^{n} \sum_{j=1}^{n} L_{ij} |x_i - u_i| |x_j - u_j| dx_1 ... dx_n$$
$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} L_{ij} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} |x_i - u_i| |x_j - u_j| dx_1 ... dx_n,$$

which proves the inequality (2.4). □

We observe that if we put $L := \max_{i,j \in \{1,...,n\}} L_{ij}$ then

$$\sum_{i=1}^{n} \sum_{j=1}^{n} L_{ij} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} |x_i - u_i| |x_j - u_j| dx_1 ... dx_n$$
$$\leq L \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - u_i| |x_j - u_j| dx_1 ... dx_n$$
$$= L \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left(\sum_{i=1}^{n} |x_i - u_i| \right)^2 dx_1 ... dx_n$$
$$\leq nL \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \sum_{i=1}^{n} (x_i - u_i)^2 dx_1 ... dx_n$$
$$= nL \sum_{i=1}^{n} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} (x_i - u_i)^2 dx_1 ... dx_n,$$

where for the last inequality we used the discrete Cauchy-Bunyakovsky-Schwarz inequality.

By (2.4) we get

$$\left|\frac{1}{V_{G_n}}\int\cdots\int_{G_n} f(x_1,...,x_n)\,dx_1...dx_n - f(u_1,...,u_n) - \sum_{i=1}^{n}\left(\overline{x_{i,G_n}}-u_i\right)\frac{\partial f}{\partial x_i}(u_1,...,u_n)\right| \tag{2.9}$$

$$\leq \frac{1}{2}L\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left(\sum_{i=1}^{n}|x_i-u_i|\right)^2 dx_1...dx_n \leq \frac{1}{2}nL\sum_{i=1}^{n}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}(x_i-u_i)^2\,dx_1...dx_n$$

for all $(u_1,...,u_n)\in G_n$.

In particular, we have

$$\left|\frac{1}{V_{G_n}}\int\cdots\int_{G_n} f(x_1,...,x_n)\,dx_1...dx_n - f\left(\overline{x_{1,G_n}},...,\overline{x_{n,G_n}}\right)\right| \tag{2.10}$$

$$\leq \frac{1}{2}L\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left(\sum_{i=1}^{n}\left|x_i-\overline{x_{i,G_n}}\right|\right)^2 dx_1...dx_n \leq \frac{1}{2}nL\sum_{i=1}^{n}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}(x_i-u_i)^2\,dx_1...dx_n.$$

From Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\frac{1}{V_{G_n}}\int\cdots\int_{G_n}|x_i-u_i|\,|x_j-u_j|\,dx_1...dx_n \leq \left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}(x_i-u_i)^2\,dx_1...dx_n\right)^{1/2} \times\left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}(x_j-u_j)^2\,dx_1...dx_n\right)^{1/2}$$

therefore

$$\sum_{i=1}^{n}\sum_{j=1}^{n}L_{ij}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}|x_i-u_i|\,|x_j-u_j|\,dx_1...dx_n$$

$$\leq L\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}|x_i-u_i|\,|x_j-u_j|\,dx_1...dx_n$$

$$\leq L\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}(x_i-u_i)^2\,dx_1...dx_n\right)^{1/2} \times\left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}(x_j-u_j)^2\,dx_1...dx_n\right)^{1/2}$$

$$= L\left[\sum_{i=1}^{n}\left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}(x_i-u_i)^2\,dx_1...dx_n\right)^{1/2}\right]^2$$
$$\leq nL\sum_{i=1}^{n}\left[\left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}(x_i-u_i)^2\,dx_1...dx_n\right)^{1/2}\right]^2$$
$$= nL\sum_{i=1}^{n}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}(x_i-u_i)^2\,dx_1...dx_n,$$

where for the last inequality we used the discrete CBS inequality.

By (2.4) we get

$$\left|\frac{1}{V_{G_n}}\int\cdots\int_{G_n}f(x_1,...,x_n)\,dx_1...dx_n-f(u_1,...,u_n)\right.$$
$$\left.-\sum_{i=1}^{n}\left(\overline{x_{i,G_n}}-u_i\right)\frac{\partial f}{\partial x_i}(u_1,...,u_n)\right|$$
$$\leq\frac{1}{2}L\left[\sum_{i=1}^{n}\left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}(x_i-u_i)^2\,dx_1...dx_n\right)^{1/2}\right]^2$$
$$\leq\frac{1}{2}nL\sum_{i=1}^{n}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}(x_i-u_i)^2\,dx_1...dx_n \tag{2.11}$$

for all $(u_1,...,u_n)\in G_n$.

In particular,

$$\left|\frac{1}{V_{G_n}}\int\cdots\int_{G_n}f(x_1,...,x_n)\,dx_1...dx_n-f\left(\overline{x_{1,G_n}},...,\overline{x_{n,G_n}}\right)\right|$$
$$\leq\frac{1}{2}L\left[\sum_{i=1}^{n}\left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left(x_i-\overline{x_{i,G_n}}\right)^2dx_1...dx_n\right)^{1/2}\right]^2$$
$$\leq\frac{1}{2}nL\sum_{i=1}^{n}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left(x_i-\overline{x_{i,G_n}}\right)^2dx_1...dx_n. \tag{2.12}$$

We have:

Corollary 1. *Assume that $f:G_n\to\mathbb{C}$ is twice differentiable on G_n and the second partial derivatives $\frac{\partial^2 f}{\partial x_i\partial x_j}$ for $i,\ j\in\{1,...,n\}$ are bounded on G_n (for $j=i$ we denote $\frac{\partial^2 f}{\partial x_i\partial x_j}=\frac{\partial^2 f}{\partial x_i^2}$). Put*

$$\left\|\frac{\partial^2 f}{\partial x_i\partial x_j}\right\|_{G_n,\infty}:=\sup_{(x_1,...,x_n)\in G_n}\left|\frac{\partial^2 f}{\partial x_i\partial x_j}(x_1,...,x_n)\right|<\infty,$$

then

$$\left|\frac{1}{V_{G_n}}\int\cdots\int_{G_n} f(x_1,...,x_n)\,dx_1...dx_n - f(u_1,...,u_n) - \sum_{i=1}^{n}\left(\overline{x_{i,G_n}} - u_i\right)\frac{\partial f}{\partial x_i}(u_1,...,u_n)\right| \leq \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\|\frac{\partial^2 f}{\partial x_i \partial x_j}\right\|_{G_n,\infty}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}|x_i-u_i|\,|x_j-u_j|\,dx_1...dx_n, \tag{2.13}$$

for all $(u_1,...,u_n) \in G_n$.

In particular,

$$\left|\frac{1}{V_{G_n}}\int\cdots\int_{G_n} f(x_1,...,x_n)\,dx_1...dx_n - f\left(\overline{x_{1,G_n}},...,\overline{x_{n,G_n}}\right)\right| \leq \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\|\frac{\partial^2 f}{\partial x_i \partial x_j}\right\|_{G_n,\infty}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_i-\overline{x_{i,G_n}}\right|\left|x_j-\overline{x_{j,G_n}}\right|dx_1...dx_n. \tag{2.14}$$

The proof follows by the fact that if $\frac{\partial^2 f}{\partial x_i \partial x_j}$, for $i,\ j \in \{1,...,n\}$, are bounded on G_n then the partial derivatives $\frac{\partial f}{\partial x_i}$, $i \in \{1,...,n\}$ are Lipschitzian with the constants $L_{ij} = \left\|\frac{\partial^2 f}{\partial x_i \partial x_j}\right\|_{G_n,\infty}$, for $i,\ j \in \{1,...,n\}$.

We also have:

Theorem 4. *If* $f : G \to \mathbb{C}$ *is differentiable on* G *and the partial derivatives* $\frac{\partial f}{\partial x_i}$, $i \in \{1,...,n\}$ *satisfy the Lipschitz type conditions (2.3), then for all* $(u_1,...,u_n) \in G_n$ *we have*

$$\left|\frac{1}{V_{G_n}}\int\cdots\int_{G_n} f(x_1,...,x_n)\,dx_1...dx_n - f(u_1,...,u_n) - \frac{1}{V_{G_n}}\int\cdots\int_{G_n}\sum_{i=1}^{n}(x_i-u_i)\frac{\partial f}{\partial x_i}(x_1,...,x_n)\,dx_1...dx_n\right| \leq \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}L_{ij}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}|x_i-u_i|\,|x_j-u_j|\,dx_1...dx_n. \tag{2.15}$$

In particular,

$$\left|\frac{1}{V_{G_n}}\int\cdots\int_{G_n} f(x_1,...,x_n)\,dx_1...dx_n - f\left(\overline{x_{1,G_n}},...,\overline{x_{n,G_n}}\right) - \frac{1}{V_{G_n}}\int\cdots\int_{G_n}\sum_{i=1}^{n}\left(x_i-\overline{x_{i,G_n}}\right)\frac{\partial f}{\partial x_i}(x_1,...,x_n)\,dx_1...dx_n\right| \leq \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}L_{ij}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_i-\overline{x_{i,G_n}}\right|\left|x_j-\overline{x_{j,G_n}}\right|dx_1...dx_n. \tag{2.16}$$

Proof. From Lemma 1 we have for $\lambda_i = \frac{\partial f}{\partial x_i}(x_1,...,x_n)$, $i \in \{1,...,n\}$ that

$$f(x_1,...,x_n) = f(u_1,...,u_n) + \sum_{i=1}^{n}(x_i - u_i)\frac{\partial f}{\partial x_i}(x_1,...,x_n) \tag{2.17}$$
$$+\sum_{i=1}^{n}(x_i - u_i)\int_0^1\left(\frac{\partial f}{\partial x_i}[t(x_1,...,x_n) + (1-t)(u_1,...,u_n)] - \frac{\partial f}{\partial x_i}(x_1,...,x_n)\right)dt$$

for all $(x_1,...,x_n)$, $(u_1,...,u_n) \in G_n$.

By taking the integral mean $\frac{1}{V_{G_n}}\int\cdots\int_{G_n} dx_1...dx_n$ over $(x_1,...,x_n)$ in (2.17) we get

$$\frac{1}{V_{G_n}}\int\cdots\int_{G_n} f(x_1,...,x_n)\,dx_1...dx_n = f(u_1,...,u_n) \tag{2.18}$$
$$+\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\sum_{i=1}^{n}(x_i - u_i)\frac{\partial f}{\partial x_i}(x_1,...,x_n)\,dx_1...dx_n$$
$$+\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\sum_{i=1}^{n}(x_i - u_i)$$
$$\times\left(\int_0^1\left(\frac{\partial f}{\partial x_i}[t(x_1,...,x_n) + (1-t)(u_1,...,u_n)] - \frac{\partial f}{\partial x_i}(x_1,...,x_n)\right)dt\right)dx_1...dx_n$$

for all $(u_1,...,u_n) \in G_n$.

By the equality (2.18) we get

$$\left|\frac{1}{V_{G_n}}\int\cdots\int_{G_n} f(x_1,...,x_n)\,dx_1...dx_n - f(u_1,...,u_n)\right. \tag{2.19}$$
$$\left.-\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\sum_{i=1}^{n}(x_i - u_i)\frac{\partial f}{\partial x_i}(x_1,...,x_n)\,dx_1...dx_n\right|$$
$$=\left|\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\sum_{i=1}^{n}(x_i - u_i)\right.$$
$$\left.\times\left(\int_0^1\left(\frac{\partial f}{\partial x_i}[t(x_1,...,x_n) + (1-t)(u_1,...,u_n)] - \frac{\partial f}{\partial x_i}(x_1,...,x_n)\right)dt\right)dx_1...dx_n\right|$$
$$\le\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|\sum_{i=1}^{n}(x_i - u_i)\right.$$
$$\left.\times\left(\int_0^1\left(\frac{\partial f}{\partial x_i}[t(x_1,...,x_n) + (1-t)(u_1,...,u_n)] - \frac{\partial f}{\partial x_i}(x_1,...,x_n)\right)dt\right)\right|dx_1...dx_n$$

$$\leq \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \sum_{i=1}^{n} |x_i - u_i|$$

$$\times \left(\int_0^1 \left| \frac{\partial f}{\partial x_i} \left[t (x_1, ..., x_n) + (1-t) (u_1, ..., u_n) \right] - \frac{\partial f}{\partial x_i} (x_1, ..., x_n) \right| dt \right) dx_1 ... dx_n$$

$$\leq \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \sum_{i=1}^{n} |x_i - u_i| \left(\sum_{j=1}^{n} L_{ij} |x_j - u_j| \int_0^1 (1-t)\, dt \right) dx_1 ... dx_n$$

$$= \frac{1}{2} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \sum_{i=1}^{n} \sum_{j=1}^{n} L_{ij} |x_i - u_i| |x_j - u_j| dx_1 ... dx_n$$

$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} L_{ij} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} |x_i - u_i| |x_j - u_j| dx_1 ... dx_n,$$

which proves the desired result (2.15). □

We define

$$x_{i,S} := \frac{\int \cdots \int_{G_n} x_i \frac{\partial f}{\partial x_i} (x_1, ..., x_n)\, dx_1 ... dx_n}{\int \cdots \int_{G_n} \frac{\partial f}{\partial x_i} (x_1, ..., x_n)\, dx_1 ... dx_n}, \ i \in \{1, ..., n\}, \tag{2.20}$$

provided that the integrals from the denominators are not zero.

Corollary 2. *With the assumptions of Theorem 4 and if* $(x_{1,S}, ..., x_{n,S}) \in G_n$*, then*

$$\left| \frac{1}{V_{G_n}} \int \cdots \int_{G_n} f (x_1, ..., x_n)\, dx_1 ... dx_n - f (x_{1,S}, ..., x_{n,S}) \right| \leq \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} L_{ij} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} |x_i - x_{i,S}| |x_j - x_{j,S}| dx_1 ... dx_n. \tag{2.21}$$

Remark 1. *Assume that* $f : G_n \to \mathbb{C}$ *is twice differentiable on* G_n *and the second partial derivatives* $\frac{\partial^2 f}{\partial x_i \partial x_j}$ *for* $i, j \in \{1, ..., n\}$ *are bounded on* G_n*, then*

$$\left| \frac{1}{V_{G_n}} \int \cdots \int_{G_n} f (x_1, ..., x_n)\, dx_1 ... dx_n - f (u_1, ..., u_n) - \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \sum_{i=1}^{n} (x_i - u_i) \frac{\partial f}{\partial x_i} (x_1, ..., x_n)\, dx_1 ... dx_n \right| \leq \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_{G_n, \infty} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} |x_i - u_i| |x_j - u_j| dx_1 ... dx_n. \tag{2.22}$$

In particular,

$$\left| \frac{1}{V_{G_n}} \int \cdots \int_{G_n} f (x_1, ..., x_n)\, dx_1 ... dx_n - f \left(\overline{x_{1,G_n}}, ..., \overline{x_{n,G_n}} \right) - \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \sum_{i=1}^{n} \left(x_i - \overline{x_{i,G_n}} \right) \frac{\partial f}{\partial x_i} (x_1, ..., x_n)\, dx_1 ... dx_n \right| \leq \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_{G_n, \infty} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left| x_i - \overline{x_{i,G_n}} \right| \left| x_j - \overline{x_{j,G_n}} \right| dx_1 ... dx_n. \tag{2.23}$$

If $(x_{1,S},...,x_{n,S}) \in G_n$, *then also*

$$\left|\frac{1}{V_{G_n}}\int\cdots\int_{G_n} f(x_1,...,x_n)\,dx_1...dx_n - f(x_{1,S},...,x_{n,S})\right| \tag{2.24}$$

$$\leq \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\|\frac{\partial^2 f}{\partial x_i \partial x_j}\right\|_{G_n,\infty}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}|x_i - x_{i,S}|\,|x_j - x_{j,S}|\,dx_1...dx_n.$$

3. Examples for n-Dimensional Boxes

We can consider the *n-hyper box* $R_n := [a_1,b_1]\times...\times[a_n,b_n] \subset \mathbb{R}^n$, $n \geq 2$ and assume that $f : R_n \to \mathbb{R}$ is a differentiable convex function on R_n. We have

$$V_{R_n} = \prod_{i=1}^{n}(b_i - a_i) \text{ and } \overline{x_{i,R_n}} = \frac{b_i + a_i}{2} \text{ for } i \in \{1,...,n\}.$$

Also for $i \in \{1,...,n\}$ we have

$$\frac{1}{V_{R_n}}\int\cdots\int_{R_n}(x_i - u_i)^2\,dx_1...dx_n$$
$$= \frac{1}{\prod_{i=1}^{n}(b_i - a_i)}\int_{a_1}^{b_1}...\int_{a_i}^{b_i}...\int_{a_n}^{b_n}(x_i - u_i)^2\,dx_1...dx_i...dx_n$$
$$= \frac{1}{b_i - a_i}\int_{a_i}^{b_i}(x_i - u_i)^2\,dx_i = \frac{(u_i - a_i)^3 + (b_i - u_i)^3}{3(b_i - a_i)}.$$

Using the elementary identity

$$X^3 + Y^3 = (X+Y)\left[\left(\frac{X+Y}{2}\right)^2 + 3\left(\frac{X-Y}{2}\right)^2\right]$$

we get

$$(u_i - a_i)^3 + (b_i - u_i)^3 = (b_i - a_i)\left[\left(\frac{b_i - a_i}{2}\right)^2 + 3\left(u_i - \frac{b_i + a_i}{2}\right)^2\right].$$

Therefore

$$\frac{(u_i - a_i)^3 + (b_i - u_i)^3}{3(b_i - a_i)} = \left[\frac{1}{12} + \left(\frac{u_i - \frac{b_i + a_i}{2}}{b_i - a_i}\right)^2\right](b_i - a_i)^2$$

and we get

$$\frac{1}{\prod_{i=1}^{n}(b_i - a_i)}\int_{a_1}^{b_1}...\int_{a_n}^{b_n}(x_i - u_i)^2\,dx_1...dx_n$$
$$= \left[\frac{1}{12} + \left(\frac{u_i - \frac{b_i + a_i}{2}}{b_i - a_i}\right)^2\right](b_i - a_i)^2$$

for all $i \in \{1,...,n\}$ and $(u_1,...,u_n) \in [a_1,b_1]\times...\times[a_n,b_n]$.

For $i,\, j \in \{1, ..., n\}$ with $i \neq j$ we have

$$\begin{aligned}
&\frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} |x_i - u_i| \, |x_j - u_j| \, dx_1...dx_n \\
&\qquad = \frac{1}{(b_i - a_i)(b_j - a_j)} \int_{a_i}^{b_i} \int_{a_j}^{b_j} |x_i - u_i| \, |x_j - u_j| \, dx_i dx_j \\
&\qquad = \left[\frac{1}{4} + \left(\frac{u_i - \frac{a_i+b_i}{2}}{b_i - a_i}\right)^2\right] \left[\frac{1}{4} + \left(\frac{u_j - \frac{a_j+b_j}{2}}{b_j - a_j}\right)^2\right] (b_i - a_i)(b_j - a_j)
\end{aligned}$$

for all $(u_1, ..., u_n) \in [a_1, b_1] \times ... \times [a_n, b_n]$.

Observe that, in general

$$\begin{aligned}
&\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_{G_n,\infty} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} |x_i - u_i| \, |x_j - u_j| \, dx_1...dx_n \\
&= \frac{1}{2} \sum_{i=1}^{n} \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_{G_n,\infty} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} (x_i - u_i)^2 \, dx_1...dx_n \\
&+ \sum_{1 \le i < j \le n}^{n} \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_{G_n,\infty} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} |x_i - u_i| \, |x_j - u_j| \, dx_1...dx_n
\end{aligned}$$

for all $(u_1, ..., u_n) \in G_n$.

Using the inequality (2.13) we get

$$\begin{aligned}
&\left| \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} f(x_1, ..., x_n) \, dx_1...dx_n - f(u_1, ..., u_n) \right. \\
&\qquad \left. - \sum_{i=1}^{n} \left(\frac{a_i + b_i}{2} - u_i\right) \frac{\partial f}{\partial x_i}(u_1, ..., u_n) \right| \\
&\qquad \le \frac{1}{2} \sum_{i=1}^{n} \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_{R_n,\infty} \left[\frac{1}{12} + \left(\frac{u_i - \frac{b_i+a_i}{2}}{b_i - a_i}\right)^2\right] (b_i - a_i)^2 \\
&+ \sum_{1 \le i < j \le n}^{n} \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_{R_n,\infty} \left[\frac{1}{4} + \left(\frac{u_i - \frac{a_i+b_i}{2}}{b_i - a_i}\right)^2\right] \left[\frac{1}{4} + \left(\frac{u_j - \frac{a_j+b_j}{2}}{b_j - a_j}\right)^2\right] \\
&\qquad \times (b_i - a_i)(b_j - a_j)
\end{aligned} \tag{3.1}$$

for all $(u_1, ..., u_n) \in [a_1, b_1] \times ... \times [a_n, b_n]$.

In particular, we have

$$\begin{aligned}
&\left| f\left(\frac{a_1 + b_1}{2}, ..., \frac{a_n + b_n}{2}\right) \right. \\
&\qquad \left. - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} f(x_1, ..., x_n) \, dx_1...dx_n \right| \\
&\le \frac{1}{24} \sum_{i=1}^{n} \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_{R_n,\infty} (b_i - a_i)^2 + \frac{1}{16} \sum_{1 \le i < j \le n}^{n} \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_{R_n,\infty} (b_i - a_i)(b_j - a_j).
\end{aligned} \tag{3.2}$$

If the function f is convex on the box $[a_1,b_1]\times ...\times [a_n,b_n]$, then we have the Hermite-Hadamard type inequality

$$0\le \frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_n}^{b_n} f(x_1,...,x_n)\,dx_1...dx_n - f\left(\frac{a_1+b_1}{2},...,\frac{a_n+b_n}{2}\right) \le \frac{1}{24}\sum_{i=1}^{n}\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_{R_n,\infty}(b_i-a_i)^2+\frac{1}{16}\sum_{1\le i<j\le n}\left\|\frac{\partial^2 f}{\partial x_i\partial x_j}\right\|_{R_n,\infty}(b_i-a_i)(b_j-a_j), \tag{3.3}$$

see also [5] for other results.

From the inequality (2.22) we get

$$\left|\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_n}^{b_n} f(x_1,...,x_n)\,dx_1...dx_n - f(u_1,...,u_n) - \frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_n}^{b_n}\sum_{i=1}^{n}(x_i-u_i)\frac{\partial f}{\partial x_i}(x_1,...,x_n)\,dx_1...dx_n\right| \le \frac{1}{2}\sum_{i=1}^{n}\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_{R_n,\infty}\left[\frac{1}{12}+\left(\frac{u_i-\frac{b_i+a_i}{2}}{b_i-a_i}\right)^2\right](b_i-a_i)^2 + \sum_{1\le i<j\le n}\left\|\frac{\partial^2 f}{\partial x_i\partial x_j}\right\|_{R_n,\infty}\left[\frac{1}{4}+\left(\frac{u_i-\frac{a_i+b_i}{2}}{b_i-a_i}\right)^2\right]\left[\frac{1}{4}+\left(\frac{u_j-\frac{a_j+b_j}{2}}{b_j-a_j}\right)^2\right]\times(b_i-a_i)(b_j-a_j) \tag{3.4}$$

for all $(u_1,...,u_n)\in[a_1,b_1]\times...\times[a_n,b_n]$.

In particular,

$$\left|\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_n}^{b_n} f(x_1,...,x_n)\,dx_1...dx_n - f\left(\frac{a_1+b_1}{2},...,\frac{a_n+b_n}{2}\right) - \frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_n}^{b_n}\sum_{i=1}^{n}\left(x_i-\frac{a_i+b_i}{2}\right)\frac{\partial f}{\partial x_i}(x_1,...,x_n)\,dx_1...dx_n\right| \le \frac{1}{24}\sum_{i=1}^{n}\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_{R_n,\infty}(b_i-a_i)^2+\frac{1}{16}\sum_{1\le i<j\le n}\left\|\frac{\partial^2 f}{\partial x_i\partial x_j}\right\|_{R_n,\infty}(b_i-a_i)(b_j-a_j). \tag{3.5}$$

Define

$$x_{i,S}:=\frac{\int_{a_1}^{b_1}...\int_{a_n}^{b_n} x_i\frac{\partial f}{\partial x_i}(x_1,...,x_n)\,dx_1...dx_n}{\int_{a_1}^{b_1}...\int_{a_n}^{b_n}\frac{\partial f}{\partial x_i}(x_1,...,x_n)\,dx_1...dx_n},\ i\in\{1,...,n\}$$

and assume that $x_{i,S}\in[a_i,b_i]$, $i\in\{1,...,n\}$. For this condition to happen, it is enough to assume that $\frac{\partial f}{\partial x_i}(x_1,...,x_n)>0$, $i\in\{1,...,n\}$ on R_n.

From the inequality (2.24) we get

$$\left|\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_n}^{b_n}f(x_1,...,x_n)\,dx_1...dx_n-f(x_{1,S},...,x_{n,S})\right| \tag{3.6}$$
$$\leq\frac{1}{2}\sum_{i=1}^{n}\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_{R_n,\infty}\left[\frac{1}{12}+\left(\frac{x_{i,S}-\frac{b_i+a_i}{2}}{b_i-a_i}\right)^2\right](b_i-a_i)^2$$
$$+\sum_{1\leq i<j\leq n}^{n}\left\|\frac{\partial^2 f}{\partial x_i\partial x_j}\right\|_{R_n,\infty}\left[\frac{1}{4}+\left(\frac{x_{i,S}-\frac{a_i+b_i}{2}}{b_i-a_i}\right)^2\right]\left[\frac{1}{4}+\left(\frac{x_{j,S}-\frac{a_j+b_j}{2}}{b_j-a_j}\right)^2\right]$$
$$\times(b_i-a_i)(b_j-a_j).$$

If the function f is convex on the box $[a_1,b_1]\times...\times[a_n,b_n]$, then we have the reverse Hermite-Hadamard type inequality

$$0\leq f(x_{1,S},...,x_{n,S})-\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_n}^{b_n}f(x_1,...,x_n)\,dx_1...dx_n \tag{3.7}$$
$$\leq\frac{1}{2}\sum_{i=1}^{n}\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_{R_n,\infty}\left[\frac{1}{12}+\left(\frac{x_{i,S}-\frac{b_i+a_i}{2}}{b_i-a_i}\right)^2\right](b_i-a_i)^2$$
$$+\sum_{1\leq i<j\leq n}^{n}\left\|\frac{\partial^2 f}{\partial x_i\partial x_j}\right\|_{R_n,\infty}\left[\frac{1}{4}+\left(\frac{x_{i,S}-\frac{a_i+b_i}{2}}{b_i-a_i}\right)^2\right]\left[\frac{1}{4}+\left(\frac{x_{j,S}-\frac{a_j+b_j}{2}}{b_j-a_j}\right)^2\right]$$
$$\times(b_i-a_i)(b_j-a_j),$$

see also [5] for other results.

4. Examples for 3-Dimensional Balls

In this section we point out some inequalities of Hermite-Hadamard's type for convex functions defined on a ball $B(C,R)$ where $C=(a,b,c)\in\mathbb{R}^3$, $R>0$ and

$$B(C,R):=\left\{(x,y,z)\in\mathbb{R}^3\,\middle|\,(x-a)^2+(y-b)^2+(z-c)^2\leq R^2\right\}.$$

Let us consider the transformation $T_2:\mathbb{R}^3\to\mathbb{R}^3$ given by:

$$T_2(r,\psi,\varphi):=(r\cos\psi\cos\varphi+a,r\cos\psi\sin\varphi+b,r\sin\psi+c).$$

It is well known that the Jacobian of T_2 is $J(T_2)=r^2\cos\psi$ and T_2 is a one-to-one mapping defined on the interval of $\mathbb{R}^3$, $[0,R]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi]$, with values in the ball $B(C,R)$ from $\mathbb{R}^3$. Thus we have the change of variable:

$$\iiint_{B(C,R)}f(x,y,z)\,dxdydz$$
$$=\int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}f(r\cos\psi\cos\varphi+a,r\cos\psi\sin\varphi+b,r\sin\psi+c)\,r^2\cos\psi drd\psi d\varphi.$$

We have

$$\overline{x_{B(C,R)}}=a,\ \overline{y_{B(C,R)}}=b\text{ and }\overline{z_{B(C,R)}}=c,$$
$$V_{B(C,R)}=\frac{4\pi}{3}R^3,$$

$$\begin{aligned}
&\iiint_{B(C,R)} \left|z-\overline{z_{B(C,R)}}\right|^2 dxdydz \\
&= \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} r^2 \sin^2\psi r^2 \cos\psi dr d\psi d\varphi \\
&= \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} r^4 \sin^2\psi \cos\psi dr d\psi d\varphi = \frac{4}{15}\pi R^5
\end{aligned}$$

and, similarly

$$\iiint_{B(C,R)} \left|x-\overline{x_{B(C,R)}}\right|^2 dxdydz = \iiint_{B(C,R)} \left|y-\overline{y_{B(C,R)}}\right|^2 dxdydz = \frac{4}{15}\pi R^5.$$

Also

$$\begin{aligned}
&\iiint_{B(C,R)} \left|x-\overline{x_{B(C,R)}}\right| \left|y-\overline{y_{B(C,R)}}\right| dxdydz \\
&= \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \left|r\cos\psi\cos\varphi\right| \left|r\cos\psi\sin\varphi\right| r^2 \cos\psi dr d\psi d\varphi \\
&= \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} r^4 \cos^3\psi \left|\sin\varphi\cos\varphi\right| dr d\psi d\varphi = \frac{8}{15} R^5
\end{aligned}$$

and, similarly

$$\begin{aligned}
&\iiint_{B(C,R)} \left|x-\overline{x_{B(C,R)}}\right| \left|z-\overline{z_{B(C,R)}}\right| dxdydz \\
&= \iiint_{B(C,R)} \left|y-\overline{y_{B(C,R)}}\right| \left|z-\overline{z_{B(C,R)}}\right| dxdydz = \frac{8}{15} R^5.
\end{aligned}$$

By making use of the inequality (2.14) we have

$$\begin{aligned}
(4.1)\quad &\left|\frac{1}{\frac{4\pi}{3}R^3}\iiint_{B(C,R)} f(x,y,z)\,dxdydz - f(a,b,c)\right| \\
&\leq \frac{1}{10}R^2\left[\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{B(C,R),\infty} + \left\|\frac{\partial^2 f}{\partial y^2}\right\|_{B(C,R),\infty} + \left\|\frac{\partial^2 f}{\partial x^2}\right\|_{B(C,R),\infty}\right] \\
&\quad + \frac{2}{5\pi}R^2\left[\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{B(C,R),\infty} + \left\|\frac{\partial^2 f}{\partial x\partial z}\right\|_{B(C,R),\infty} + \left\|\frac{\partial^2 f}{\partial y\partial z}\right\|_{B(C,R),\infty}\right].
\end{aligned}$$

If f is convex on $B(C,R)$, then we get the following reverse of Hermite-Hadamard inequality

$$\begin{aligned}
(4.2)\quad 0 &\leq \frac{1}{\frac{4\pi}{3}R^3}\iiint_{B(C,R)} f(x,y,z)\,dxdydz - f(a,b,c) \\
&\leq \frac{1}{10}R^2\left[\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{B(C,R),\infty} + \left\|\frac{\partial^2 f}{\partial y^2}\right\|_{B(C,R),\infty} + \left\|\frac{\partial^2 f}{\partial x^2}\right\|_{B(C,R),\infty}\right] \\
&\quad + \frac{2}{5\pi}R^2\left[\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{B(C,R),\infty} + \left\|\frac{\partial^2 f}{\partial x\partial z}\right\|_{B(C,R),\infty} + \left\|\frac{\partial^2 f}{\partial y\partial z}\right\|_{B(C,R),\infty}\right],
\end{aligned}$$

which is twice better than the corresponding inequality from [5].

From the inequality (2.23) we also have

$$\begin{aligned}
(4.3)\quad & \Bigg| \frac{1}{\frac{4\pi}{3}R^3} \iiint_{B(C,R)} f(x,y,z)\, dxdydz \\
& - f(a,b,c) - \frac{1}{\frac{4\pi}{3}R^3} \iiint_{B(C,R)} \left[(x-a) \frac{\partial f}{\partial x}(x,y,z) \right. \\
& \left. + (y-b) \frac{\partial f}{\partial y}(x,y,z) + (z-c) \frac{\partial f}{\partial z}(x,y,z) \right] dxdydz \Bigg| \\
& \leq \frac{1}{10} R^2 \left[\left\| \frac{\partial^2 f}{\partial x^2} \right\|_{B(C,R),\infty} + \left\| \frac{\partial^2 f}{\partial y^2} \right\|_{B(C,R),\infty} + \left\| \frac{\partial^2 f}{\partial x^2} \right\|_{B(C,R),\infty} \right] \\
& + \frac{2}{5\pi} R^2 \left[\left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_{B(C,R),\infty} + \left\| \frac{\partial^2 f}{\partial x \partial z} \right\|_{B(C,R),\infty} + \left\| \frac{\partial^2 f}{\partial y \partial z} \right\|_{B(C,R),\infty} \right].
\end{aligned}$$

References

[1] Barnett, N. S.; Cîrstea, F. C. and Dragomir, S. S. Some inequalities for the integral mean of Hölder continuous functions defined on disks in a plane, in *Inequality Theory and Applications*, Vol. **2** (Chinju/Masan, 2001), 7–18, Nova Sci. Publ., Hauppauge, NY, 2003. Preprint *RGMIA Res. Rep. Coll.* **5** (2002), Nr. 1, Art. 7, 10 pp. [Online https://rgmia.org/papers/v5n1/BCD.pdf].

[2] Barnett, N. S.; Dragomir, S. S. An Ostrowski type inequality for double integrals and applications for cubature formulae. Soochow J. Math. 27 (2001), no. 1, 1–10.

[3] Barnett, N. S.; Dragomir, S. S.; Pearce, C. E. M. A quasi-trapezoid inequality for double integrals. ANZIAM J. 44 (2003), no. 3, 355–364.

[4] Budak, Hüseyin; Sarıkaya, Mehmet Zeki An inequality of Ostrowski-Grüss type for double integrals. Stud. Univ. Babeş-Bolyai Math. 62 (2017), no. 2, 163–173.

[5] Dragomir, S. S. Hermite-Hadamard type tntegral tnequalities for multiple integrals on convex bodies, Preprint *RGMIA Res. Rep. Coll.* **22** (2019), Art. 49, 16 pp. [Online http://rgmia.org/papers/v22/v22a49.pdf].

[6] Dragomir, S. S.; Cerone, P.; Barnett, N. S.; Roumeliotis, J. An inequality of the Ostrowski type for double integrals and applications for cubature formulae. Tamsui Oxf. J. Math. Sci. 16 (2000), no. 1, 1–16.

[7] Erden, Samet; Sarikaya, Mehmet Zeki On exponential Pompeiu's type inequalities for double integrals with applications to Ostrowski's inequality. New Trends Math. Sci. 4 (2016), no. 1, 256–267.

[8] Hanna, George Some results for double integrals based on an Ostrowski type inequality. Ostrowski type inequalities and applications in numerical integration, 331–371, Kluwer Acad. Publ., Dordrecht, 2002.

[9] Hanna, G.; Dragomir, S. S.; Cerone, P. A general Ostrowski type inequality for double integrals. Tamkang J. Math. 33 (2002), no. 4, 319–333.

[10] Liu, Zheng A sharp general Ostrowski type inequality for double integrals. Tamsui Oxf. J. Inf. Math. Sci. 28 (2012), no. 2, 217–226.

[11] Özdemir, M. Emin; Akdemir, Ahmet Ocak; Set, Erhan A new Ostrowski-type inequality for double integrals. J. Inequal. Spec. Funct. 2 (2011), no. 1, 27–34.

[12] Pachpatte, B. G. A new Ostrowski type inequality for double integrals. Soochow J. Math. 32 (2006), no. 2, 317–322.

[13] Sarikaya, Mehmet Zeki On the Ostrowski type integral inequality for double integrals. Demonstratio Math. 45 (2012), no. 3, 533–540.

[14] Sarikaya, Mehmet Zeki; Ogunmez, Hasan On the weighted Ostrowski-type integral inequality for double integrals. Arab. J. Sci. Eng. 36 (2011), no. 6, 1153–1160.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa