HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES ON PATHS SURROUNDING GENERAL CONVEX DOMAINS

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Abstract. In this paper we establish some Hermite-Hadamard type integral inequalities on paths surrounding general convex domains of the plane $\mathbb{R}^2$. Some examples for disks are also provided.

1. Introduction

Let us consider a point $C = (a, b) \in \mathbb{R}^2$ and the disk $D(C, R)$ centered at the point C and having the radius $R > 0$. In [4] we establish between others the following Hermite-Hadamard type inequality for a convex function $f : D(C, R) \to \mathbb{R}$,

$$f(C) \leq \frac{1}{A_{D(C,R)}} \iint_{D(C,R)} f(x, y)\, dxdy \leq \frac{2}{3} \frac{1}{\ell(\mathcal{C}(C, R))} \int_{\mathcal{C}(C,R)} f(\gamma)\, d\ell(\gamma) + \frac{1}{3} f(C) \leq \frac{1}{\ell(\mathcal{C}(C, R))} \int_{\mathcal{C}(C,R)} f(\gamma)\, d\ell(\gamma), \tag{1.1}$$

where $\mathcal{C}(C, R)$ is the circle centered at C and having the radius R and $\int_{\mathcal{C}(C,R)}$ is the path integral with respect to arc length, $A_{D(C,R)} = \pi R^2$ is the area of the disk and $\ell(\mathcal{C}(C, R)) = 2\pi R$ is the length of the circle $\mathcal{C}(C, R)$.

In the following, consider D a closed and bounded convex subset of $\mathbb{R}^2$. Define

$$A_D := \int\int_D dxdy$$

the *area* of D and $(\overline{x_D}, \overline{y_D})$ the *centre of mass* for D, where

$$\overline{x_D} := \frac{1}{A_D} \int\int_D xdxdy, \ \overline{y_D} := \frac{1}{A_D} \int\int_D ydxdy.$$

Consider the function of two variables $f = f(x, y)$ and denote by $\frac{\partial f}{\partial x}$ the partial derivative with respect to the variable x and $\frac{\partial f}{\partial y}$ the partial derivative with respect to the variable y.

In the recent paper [7] we obtained among others the following result:

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Theorem 1. *Let $f : D \to \mathbb{R}$ be a differentiable convex function on D. Then for all $(u, v) \in D$ we have*

$$\frac{\partial f}{\partial x}(u,v)(\overline{x_D}-u)+\frac{\partial f}{\partial y}(u,v)(\overline{y_D}-v) \leq \frac{1}{A_D}\int\int_D f(x,y)\,dxdy - f(u,v) \leq \frac{1}{A_D}\int\int_D \frac{\partial f}{\partial x}(x,y)(x-u)\,dxdy + \frac{1}{A_D}\int\int_D \frac{\partial f}{\partial y}(x,y)(y-v)\,dxdy. \tag{1.2}$$

In particular,

$$0 \leq \frac{1}{A_D}\int\int_D f(x,y)\,dxdy - f(\overline{x_D},\overline{y_D}) \leq \frac{1}{A_D}\int\int_D \frac{\partial f}{\partial x}(x,y)(x-\overline{x_D})\,dxdy + \frac{1}{A_D}\int\int_D \frac{\partial f}{\partial y}(x,y)(y-\overline{y_D})\,dxdy. \tag{1.3}$$

We also have the reverse of Hermite-Hadamard inequality:

Corollary 1. *Let $f : D \to \mathbb{R}$ be a differentiable convex function on D. Let*

$$x_S := \frac{\int\int_D x\frac{\partial f}{\partial x}(x,y)\,dxdy}{\int\int_D \frac{\partial f}{\partial x}(x,y)\,dxdy},\ \ y_S := \frac{\int\int_D y\frac{\partial f}{\partial y}(x,y)\,dxdy}{\int\int_D \frac{\partial f}{\partial y}(x,y)\,dxdy}.$$

If $(x_S, y_S) \in D$, then

$$0 \leq f(x_S,y_S) - \frac{1}{A_D}\int\int_D f(x,y)\,dxdy \leq \frac{\partial f}{\partial x}(x_S,y_S)(x_S-\overline{x_D}) + \frac{\partial f}{\partial y}(x_S,y_S)(y_S-\overline{y_D}). \tag{1.4}$$

For other multivariate Hermite-Hadamard type inequalities, see [1]-[3] and [8]-[14].

Motivated by the above results, in this paper we establish some Hermite-Hadamard type integral inequalities on paths surrounding general convex domains of the plane $\mathbb{R}^2$. Some examples for disks are also provided.

2. The Main Results

Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a region D. Moreover, if the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t \in [a, b]$, with x, y differentiable on (a, b) then the length of the curve ∂D is

$$\ell(\partial D) := \int_{\partial D} d(\ell) = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt.$$

We also consider

$$\overline{x_{\partial D}} := \frac{1}{\ell(\partial D)}\int_{\partial D} x\,d(\ell) = \frac{1}{\ell(\partial D)}\int_a^b x(t)\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt$$

and

$$\overline{y_{\partial D}} := \frac{1}{\ell(\partial D)}\int_{\partial D} y\,d(\ell) = \frac{1}{\ell(\partial D)}\int_a^b y(t)\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt.$$

Consider the function of two variables $f = f(x, y)$ and denote by $\frac{\partial f}{\partial x}$ the partial derivative with respect to the variable x and $\frac{\partial f}{\partial y}$ the partial derivative with respect to the variable y.

Theorem 2. *Let $f : D \to \mathbb{R}$ be a differentiable convex function on D. Then for all $(u, v) \in D$ we have*

$$\begin{aligned} &\frac{\partial f}{\partial x}(u,v)\left(\overline{x_{\partial D}}-u\right)+\frac{\partial f}{\partial y}(u,v)\left(\overline{y_{\partial D}}-v\right) \\ &\qquad\leq \frac{1}{\ell(\partial D)}\int_{\partial D} f(x,y)\,d(\ell)-f(u,v) \\ &\leq \frac{1}{\ell(\partial D)}\int_{\partial D}\frac{\partial f}{\partial x}(x,y)(x-u)\,d(\ell)+\frac{1}{\ell(\partial D)}\int_{\partial D}\frac{\partial f}{\partial y}(x,y)(y-v)\,d(\ell). \end{aligned} \tag{2.1}$$

In particular,

$$\begin{aligned} &0\leq \frac{1}{\ell(\partial D)}\int_{\partial D} f(x,y)\,d(\ell)-f\left(\overline{x_{\partial D}},\overline{y_{\partial D}}\right) \\ &\leq \frac{1}{\ell(\partial D)}\int_{\partial D}\frac{\partial f}{\partial x}(x,y)\left(x-\overline{x_{\partial D}}\right)d(\ell)+\frac{1}{\ell(\partial D)}\int_{\partial D}\frac{\partial f}{\partial y}(x,y)\left(y-\overline{y_{\partial D}}\right)d(\ell). \end{aligned} \tag{2.2}$$

Proof. Since $f : D \to \mathbb{R}$ is a differentiable convex function on D, then for all (x, y), $(u, v) \in D$ we have the gradient inequalities

$$\begin{aligned} &\frac{\partial f}{\partial x}(u,v)(x-u)+\frac{\partial f}{\partial y}(u,v)(y-v)\leq f(x,y)-f(u,v) \\ &\qquad\qquad\leq \frac{\partial f}{\partial x}(x,y)(x-u)+\frac{\partial f}{\partial y}(x,y)(y-v). \end{aligned} \tag{2.3}$$

If the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t \in [a, b]$, with x, y differentiable on (a, b), then by (2.3) we get

$$\begin{aligned} &\frac{\partial f}{\partial x}(u,v)(x(t)-u)+\frac{\partial f}{\partial y}(u,v)(y(t)-v) \\ &\qquad\leq f(x(t),y(t))-f(u,v) \\ &\qquad\leq \frac{\partial f}{\partial x}(x(t),y(t))(x(t)-u)+\frac{\partial f}{\partial y}(x(t),y(t))(y(t)-v) \end{aligned}$$

for all $t \in [a, b]$ and $(u, v) \in D$.

If we multiply this inequality by $\sqrt{[x'(t)]^2+[y'(t)]^2}$ and integrate over t on $[a,b]$, then we get

$$\begin{aligned}(2.4)\quad & \frac{\partial f}{\partial x}(u,v)\int_a^b (x(t)-u)\sqrt{[x'(t)]^2+[y'(t)]^2}dt \\ &\quad + \frac{\partial f}{\partial y}(u,v)\int_a^b (y(t)-v)\sqrt{[x'(t)]^2+[y'(t)]^2}dt \\ &\le \int_a^b f(x(t),y(t))\sqrt{[x'(t)]^2+[y'(t)]^2}dt - f(u,v)\int_a^b \sqrt{[x'(t)]^2+[y'(t)]^2}dt \\ &\le \int_a^b \frac{\partial f}{\partial x}(x(t),y(t))(x(t)-u)\sqrt{[x'(t)]^2+[y'(t)]^2}dt \\ &\quad + \int_a^b \frac{\partial f}{\partial y}(x(t),y(t))(y(t)-v)\sqrt{[x'(t)]^2+[y'(t)]^2}dt\end{aligned}$$

for all $(u,v)\in D$.

Observe that

$$\begin{aligned}&\int_a^b (x(t)-u)\sqrt{[x'(t)]^2+[y'(t)]^2}dt \\ &= \int_a^b x(t)\sqrt{[x'(t)]^2+[y'(t)]^2}dt - u\int_a^b \sqrt{[x'(t)]^2+[y'(t)]^2}dt \\ &= \ell(\partial D)\overline{x_{\partial D}} - \ell(\partial D)u = \ell(\partial D)(\overline{x_{\partial D}}-u)\end{aligned}$$

and

$$\int_a^b (y(t)-v)\sqrt{[x'(t)]^2+[y'(t)]^2}dt = \ell(\partial D)(\overline{y_{\partial D}}-v),$$

then by (2.4) we get the desired result (2.1). □

Corollary 2. *Let $f: D\to\mathbb{R}$ be a differentiable convex function on D. Put*

$$x_{S,\partial D} := \frac{\int_{\partial D} x\frac{\partial f}{\partial x}(x,y)\,d(\ell)}{\int_{\partial D}\frac{\partial f}{\partial x}(x,y)\,d(\ell)},\ \ y_{S,\partial D} := \frac{\int_{\partial D} y\frac{\partial f}{\partial y}(x,y)\,d(\ell)}{\int_{\partial D}\frac{\partial f}{\partial y}(x,y)\,d(\ell)}.$$

If $(x_{S,\partial D}, y_{S,\partial D})\in D$, then

$$\begin{aligned}(2.5)\quad 0 &\le f(x_{S,\partial D}, y_{S,\partial D}) - \frac{1}{\ell(\partial D)}\int_{\partial D} f(x,y)\,d(\ell) \\ &\le \frac{\partial f}{\partial x}(x_{S,\partial D}, y_{S,\partial D})(x_{S,\partial D}-\overline{x_D}) + \frac{\partial f}{\partial y}(x_{S,\partial D}, y_{S,\partial D})(y_{S,\partial D}-\overline{y_D}).\end{aligned}$$

Proof. If we take in (2.1) $(u,v)=(x_{S,\partial D}, y_{S,\partial D})\in D$, then we get

$$\begin{aligned}&\frac{\partial f}{\partial x}(u,v)(\overline{x_{\partial D}}-x_{S,\partial D}) + \frac{\partial f}{\partial y}(u,v)(\overline{y_{\partial D}}-y_{S,\partial D}) \\ &\qquad \le \frac{1}{\ell(\partial D)}\int_{\partial D} f(x,y)\,d(\ell) - f(x_{S,\partial D}, y_{S,\partial D}) \\ &\le \int_{\partial D}\frac{\partial f}{\partial x}(x,y)(x-x_{S,\partial D})\,d(\ell) + \int_{\partial D}\frac{\partial f}{\partial y}(x,y)(y-y_{S,\partial D})\,d(\ell) = 0,\end{aligned}$$

which is equivalent to (2.5). □

We also have:

Corollary 3. *Let $f : D \to \mathbb{R}$ be a differentiable convex function on D. If the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ satisfy the conditions*

$$p_1 \leq \frac{\partial f}{\partial x}(x,y) \leq P_1,\ p_2 \leq \frac{\partial f}{\partial y}(x,y) \leq P_2 \text{ for any } (x,y) \in \partial D \tag{2.6}$$

for some p_1, p_2, P_1 and P_2, then we have

$$0 \leq \frac{1}{\ell(\partial D)} \int_{\partial D} f(x,y)\, d(\ell) - f\left(\overline{x_{\partial D}}, \overline{y_{\partial D}}\right) \tag{2.7}$$
$$\leq \frac{1}{2}(P_1 - p_1) \frac{1}{\ell(\partial D)} \int_{\partial D} |x - \overline{x_{\partial D}}|\, d(\ell) + \frac{1}{2}(P_2 - p_2) \frac{1}{\ell(\partial D)} \int_{\partial D} |y - \overline{y_{\partial D}}|\, d(\ell).$$

Proof. Observe that for all α, β real numbers we have

$$\begin{aligned}
&\frac{1}{\ell(\partial D)} \int_{\partial D} \left[\frac{\partial f}{\partial x}(x,y) - \alpha\right](x - \overline{x_{\partial D}})\, d(\ell) \\
&= \frac{1}{\ell(\partial D)} \int_{\partial D} \frac{\partial f}{\partial x}(x,y)(x - \overline{x_{\partial D}})\, d(\ell) + \alpha \frac{1}{\ell(\partial D)} \int_{\partial D} (x - \overline{x_{\partial D}})\, d(\ell) \\
&= \frac{1}{\ell(\partial D)} \int_{\partial D} \frac{\partial f}{\partial x}(x,y)(x - \overline{x_{\partial D}})\, d(\ell)
\end{aligned}$$

and, similarly

$$\begin{aligned}
&\frac{1}{\ell(\partial D)} \int_{\partial D} \left[\frac{\partial f}{\partial y}(x,y) - \beta\right](y - \overline{y_{\partial D}})\, d(\ell) \\
&= \frac{1}{\ell(\partial D)} \int_{\partial D} \frac{\partial f}{\partial y}(x,y)(y - \overline{y_{\partial D}})\, d(\ell).
\end{aligned}$$

If $f : D \to \mathbb{R}$ is a differentiable function on D, then for all α, β real numbers we have the following equality of interest in itself

$$\begin{aligned}
&\frac{1}{\ell(\partial D)} \int_{\partial D} \left[\frac{\partial f}{\partial x}(x,y) - \alpha\right](x - \overline{x_{\partial D}})\, d(\ell) \\
&\quad + \frac{1}{\ell(\partial D)} \int_{\partial D} \left[\frac{\partial f}{\partial y}(x,y) - \beta\right](y - \overline{y_{\partial D}})\, d(\ell) \\
&\quad = \frac{1}{\ell(\partial D)} \int_{\partial D} \frac{\partial f}{\partial x}(x,y)(x - \overline{x_{\partial D}})\, d(\ell) \\
&\qquad + \frac{1}{\ell(\partial D)} \int_{\partial D} \frac{\partial f}{\partial y}(x,y)(y - \overline{y_{\partial D}})\, d(\ell).
\end{aligned} \tag{2.8}$$

Now, if $f: D \to \mathbb{R}$ is a differentiable convex function on D and the condition (2.6) is satisfied, then

$$
\begin{aligned}
(2.9)\quad 0 &\leq \frac{1}{\ell(\partial D)} \int_{\partial D} \frac{\partial f}{\partial x}(x,y)\left(x-\overline{x_{\partial D}}\right) d(\ell) \\
&\quad + \frac{1}{\ell(\partial D)} \int_{\partial D} \frac{\partial f}{\partial y}(x,y)\left(y-\overline{y_{\partial D}}\right) d(\ell) \\
&= \left| \frac{1}{\ell(\partial D)} \int_{\partial D} \frac{\partial f}{\partial x}(x,y)\left(x-\overline{x_{\partial D}}\right) d(\ell) \right. \\
&\quad \left. + \frac{1}{\ell(\partial D)} \int_{\partial D} \frac{\partial f}{\partial y}(x,y)\left(y-\overline{y_{\partial D}}\right) d(\ell) \right| \\
&= \left| \frac{1}{\ell(\partial D)} \int_{\partial D} \left[\frac{\partial f}{\partial x}(x,y) - \frac{p_1+P_1}{2}\right]\left(x-\overline{x_{\partial D}}\right) d(\ell) \right. \\
&\quad \left. + \frac{1}{\ell(\partial D)} \int_{\partial D} \left[\frac{\partial f}{\partial y}(x,y) - \frac{p_2+P_2}{2}\right]\left(y-\overline{y_{\partial D}}\right) d(\ell) \right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{\ell(\partial D)} \left| \int_{\partial D} \left[\frac{\partial f}{\partial x}(x,y) - \frac{p_1+P_1}{2}\right]\left(x-\overline{x_{\partial D}}\right) d(\ell) \right| \\
&\quad + \frac{1}{\ell(\partial D)} \left| \int_{\partial D} \left[\frac{\partial f}{\partial y}(x,y) - \frac{p_2+P_2}{2}\right]\left(y-\overline{y_{\partial D}}\right) d(\ell) \right| \\
&\leq \frac{1}{\ell(\partial D)} \int_{\partial D} \left|\frac{\partial f}{\partial x}(x,y) - \frac{p_1+P_1}{2}\right| \left|x-\overline{x_{\partial D}}\right| d(\ell) \\
&\quad + \frac{1}{\ell(\partial D)} \int_{\partial D} \left|\frac{\partial f}{\partial y}(x,y) - \frac{p_2+P_2}{2}\right| \left|y-\overline{y_{\partial D}}\right| d(\ell) \\
&\leq \frac{1}{2}(P_1-p_1) \frac{1}{\ell(\partial D)} \int_{\partial D} \left|x-\overline{x_{\partial D}}\right| d(\ell) \\
&\quad + \frac{1}{2}(P_2-p_2) \frac{1}{\ell(\partial D)} \int_{\partial D} \left|y-\overline{y_{\partial D}}\right| d(\ell).
\end{aligned}
$$

By utilising the inequality (2.2) we deduce the desired result (2.5). □

Further, we assume that the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist on D and satisfy the following Lipschitz type conditions

$$
(2.10)\quad \left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(u,v)\right| \leq L_1|x-u| + K_1|y-v| \text{ for all } (x,y), \ (u,v) \in D
$$

and

$$
(2.11)\quad \left|\frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}(u,v)\right| \leq L_2|x-u| + K_2|y-v| \text{ for all } (x,y), \ (u,v) \in D
$$

where L_1, L_2, K_1 and K_2 are positive given numbers.

Theorem 3. *Let $f: G \to \mathbb{R}$ be a differentiable convex function on G. If the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist on G satisfy the conditions (2.10) and (2.11) where L_1,*

L_2, K_1 and K_2 are positive given numbers, then

$$0 \leq \frac{1}{\ell(\partial D)} \int_{\partial D} f(x,y)\, d(\ell) - f\left(\overline{x_{\partial D}}, \overline{y_{\partial D}}\right) \leq L_1 \frac{1}{\ell(\partial D)} \int_{\partial D} \left(x - \overline{x_{\partial D}}\right)^2 d(\ell) + K_2 \frac{1}{\ell(\partial D)} \int_{\partial D} \left(y - \overline{y_{\partial D}}\right)^2 d(\ell) + (K_1 + L_2) \frac{1}{\ell(\partial D)} \int_{\partial D} \left|x - \overline{x_{\partial D}}\right| \left|y - \overline{y_{\partial D}}\right| d(\ell). \tag{2.12}$$

Proof. From (2.8) we get

$$\frac{1}{\ell(\partial D)} \int_{\partial D} \left[\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}\left(\overline{x_{\partial D}}, \overline{y_{\partial D}}\right)\right] \left(x - \overline{x_{\partial D}}\right) d(\ell) + \frac{1}{\ell(\partial D)} \int_{\partial D} \left[\frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}\left(\overline{x_{\partial D}}, \overline{y_{\partial D}}\right)\right] \left(y - \overline{y_{\partial D}}\right) d(\ell) = \frac{1}{\ell(\partial D)} \int_{\partial D} \frac{\partial f}{\partial x}(x,y) \left(x - \overline{x_{\partial D}}\right) d(\ell) + \frac{1}{\ell(\partial D)} \int_{\partial D} \frac{\partial f}{\partial y}(x,y) \left(y - \overline{y_{\partial D}}\right) d(\ell). \tag{2.13}$$

We also have

$$\begin{aligned} 0 &\leq \frac{1}{\ell(\partial D)} \int_{\partial D} \frac{\partial f}{\partial x}(x,y) \left(x - \overline{x_{\partial D}}\right) d(\ell) + \frac{1}{\ell(\partial D)} \int_{\partial D} \frac{\partial f}{\partial y}(x,y) \left(y - \overline{y_{\partial D}}\right) d(\ell) \\ &= \frac{1}{\ell(\partial D)} \int_{\partial D} \left[\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}\left(\overline{x_{\partial D}}, \overline{y_{\partial D}}\right)\right] \left(x - \overline{x_{\partial D}}\right) d(\ell) \\ &\quad + \frac{1}{\ell(\partial D)} \int_{\partial D} \left[\frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}\left(\overline{x_{\partial D}}, \overline{y_{\partial D}}\right)\right] \left(y - \overline{y_{\partial D}}\right) d(\ell) \\ &\leq \frac{1}{\ell(\partial D)} \left|\int_{\partial D} \left[\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}\left(\overline{x_{\partial D}}, \overline{y_{\partial D}}\right)\right] \left(x - \overline{x_{\partial D}}\right) d(\ell)\right| \\ &\quad + \frac{1}{\ell(\partial D)} \left|\int_{\partial D} \left[\frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}\left(\overline{x_{\partial D}}, \overline{y_{\partial D}}\right)\right] \left(y - \overline{y_{\partial D}}\right) d(\ell)\right| \\ &\leq \frac{1}{\ell(\partial D)} \int_{\partial D} \left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}\left(\overline{x_{\partial D}}, \overline{y_{\partial D}}\right)\right| \left|x - \overline{x_{\partial D}}\right| d(\ell) \\ &\quad + \frac{1}{\ell(\partial D)} \int_{\partial D} \left|\frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}\left(\overline{x_{\partial D}}, \overline{y_{\partial D}}\right)\right| \left|y - \overline{y_{\partial D}}\right| d(\ell) \\ &=: M \end{aligned}$$

If $f : G \to \mathbb{R}$ is a differentiable convex function on G and if the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist on G satisfy the conditions (2.10) and (2.11), then

$$\begin{aligned}
M &\leq \frac{1}{\ell(\partial D)} \int_{\partial D} \left[L_1 \left| x - \overline{x_{\partial D}} \right| + K_1 \left| y - \overline{y_{\partial D}} \right| \right] \left| x - \overline{x_{\partial D}} \right| d(\ell) \\
&+ \frac{1}{\ell(\partial D)} \int_{\partial D} \left[L_2 \left| x - \overline{x_{\partial D}} \right| + K_2 \left| y - \overline{y_{\partial D}} \right| \right] \left| y - \overline{y_{\partial D}} \right| d(\ell) \\
&= L_1 \frac{1}{\ell(\partial D)} \int_{\partial D} \left(x - \overline{x_{\partial D}} \right)^2 d(\ell) + K_2 \frac{1}{\ell(\partial D)} \int_{\partial D} \left(y - \overline{y_{\partial D}} \right)^2 d(\ell) \\
&+ (K_1 + L_2) \frac{1}{\ell(\partial D)} \int_{\partial D} \left| x - \overline{x_{\partial D}} \right| \left| y - \overline{y_{\partial D}} \right| d(\ell).
\end{aligned}$$

By utilising the inequality (2.2) we get the desired result (2.12). □

Corollary 4. *Let $f : D \to \mathbb{R}$ be a twice differentiable convex function on D. If the second partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$ exist on D and are bounded, namely*

$$\begin{aligned}
\left\| \frac{\partial^2 f}{\partial x^2} \right\|_{D,\infty} &: = \sup_{(x,y) \in D} \left| \frac{\partial^2 f}{\partial x^2}(x, y) \right| < \infty, \\
\left\| \frac{\partial^2 f}{\partial y^2} \right\|_{D,\infty} &: = \sup_{(x,y) \in D} \left| \frac{\partial^2 f}{\partial y^2}(x, y) \right| < \infty,
\end{aligned}$$

and

$$\left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_{D,\infty} := \sup_{(x,y) \in D} \left| \frac{\partial^2 f}{\partial x \partial y}(x, y) \right| < \infty,$$

then

$$\begin{aligned}
(2.14) \quad 0 &\leq \frac{1}{\ell(\partial D)} \int_{\partial D} f(x, y)\, d(\ell) - f\left(\overline{x_{\partial D}}, \overline{y_{\partial D}} \right) \\
&\leq \left\| \frac{\partial^2 f}{\partial x^2} \right\|_{D,\infty} \frac{1}{\ell(\partial D)} \int_{\partial D} \left(x - \overline{x_{\partial D}} \right)^2 d(\ell) \\
&+ \left\| \frac{\partial^2 f}{\partial y^2} \right\|_{D,\infty} \frac{1}{\ell(\partial D)} \int_{\partial D} \left(y - \overline{y_{\partial D}} \right)^2 d(\ell) \\
&+ 2 \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_{D,\infty} \frac{1}{\ell(\partial D)} \int_{\partial D} \left| x - \overline{x_{\partial D}} \right| \left| y - \overline{y_{\partial D}} \right| d(\ell).
\end{aligned}$$

3. Applications for Disks

Consider the disk centered in $C = (a, b)$ and of radius $R > 0$,

$$D(C, R) := \{(x, y) \mid x = r\cos\theta + a,\ y = r\sin\theta + b,\ r \in [0, R],\ \theta \in [0, 2\pi]\}.$$

We have for $D = D(C, R)$ that $\partial D = \mathcal{C}(C, R)$,

$$\ell(\partial D) = 2\pi R,\ \overline{x_{\partial D}} = a \text{ and } \overline{y_{\partial D}} = b.$$

From (2.2) we then get

$$0 \le \frac{1}{2\pi}\int_0^{2\pi} f(R\cos\theta + a, R\sin\theta + b)\,d\theta - f(a,b) \tag{3.1}$$
$$\le \frac{1}{2\pi}R\int_0^{2\pi} \frac{\partial f}{\partial x}(R\cos\theta + a, R\sin\theta + b)\cos\theta d\theta$$
$$+ \frac{1}{2\pi}R\int_0^{2\pi} \frac{\partial f}{\partial y}(R\cos\theta + a, R\sin\theta + b)\sin\theta d\theta,$$

provided that f is convex on $D(C,R)$.

Moreover, if the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ satisfy the conditions

$$p_1 \le \frac{\partial f}{\partial x}(x,y) \le P_1,\ p_2 \le \frac{\partial f}{\partial y}(x,y) \le P_2 \text{ for any } (x,y) \in \mathcal{C}(C,R)$$

for some p_1, p_2, P_1 and P_2, then we have

$$0 \le \frac{1}{2\pi}\int_0^{2\pi} f(R\cos\theta + a, R\sin\theta + b)\,d\theta - f(a,b) \tag{3.2}$$
$$\le \frac{1}{\pi}R(P_1 - p_1) + \frac{1}{\pi}R(P_2 - p_2).$$

Let $f : D(C,R) \to \mathbb{R}$ be a twice differentiable convex function on $D(C,R)$. If the second partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$ exist on $D(C,R)$ and are bounded, namely

$$\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D(C,R),\infty} := \sup_{(x,y)\in D(C,R)}\left|\frac{\partial^2 f}{\partial x^2}(x,y)\right| < \infty,$$
$$\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D(C,R),\infty} := \sup_{(x,y)\in D(C,R)}\left|\frac{\partial^2 f}{\partial y^2}(x,y)\right| < \infty,$$

and

$$\left\|\frac{\partial^2 f}{\partial x \partial y}\right\|_{D(C,R),\infty} := \sup_{(x,y)\in D(C,R)}\left|\frac{\partial^2 f}{\partial x \partial y}(x,y)\right| < \infty,$$

then by the inequality (2.14) we get

$$0 \le \frac{1}{2\pi}\int_0^{2\pi} f(R\cos\theta + a, R\sin\theta + b)\,d\theta - f(a,b) \tag{3.3}$$
$$\le \frac{1}{2}R^2\left(\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D(C,R),\infty} + \left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D(C,R),\infty}\right) + \frac{R^2}{\pi}\left\|\frac{\partial^2 f}{\partial x \partial y}\right\|_{D(C,R),\infty}.$$

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