HERMITE-HADAMARD TYPE INEQUALITIES FOR DOUBLE AND PATH INTEGRALS ON GENERAL DOMAINS VIA GREEN'S IDENTITY

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Abstract. In this paper we establish some Hermite-Hadamard type inequalities for convex functions of two independent variables defined on closed and bounded convex subsets of the plane $\mathbb{R}^2$. The upper bounds are in terms of path integrals and the main tool to obtain these results is the well known Green's identity. Some examples for disks and rectangles are also provided.

1. Introduction

Let us consider a point $C=(a,b)\in\mathbb{R}^2$ and the disk $D(C,R)$ centered at the point C and having the radius $R>0$. In [4] we establish between others the following Hermite-Hadamard type inequality for a convex function $f:D(C,R)\to\mathbb{R}$,

$$f(C)\le\frac{1}{A_{D(C,R)}}\iint_{D(C,R)}f(x,y)\,dxdy \tag{1.1}$$
$$\le\frac{2}{3}\frac{1}{\ell(\mathcal{C}(C,R))}\int_{\mathcal{C}(C,R)}f(\gamma)\,d\ell(\gamma)+\frac{1}{3}f(C)$$
$$\le\frac{1}{\ell(\mathcal{C}(C,R))}\int_{\mathcal{C}(C,R)}f(\gamma)\,d\ell(\gamma),$$

where $\mathcal{C}(C,R)$ is the circle centered at C and having the radius R and $\int_{\mathcal{C}(C,R)}$ is the path integral with respect to arc length, $A_{D(C,R)}=\pi R^2$ is the area of the disk and $\ell(\mathcal{C}(C,R))=2\pi R$ is the length of the circle $\mathcal{C}(C,R)$.

In the following, consider D a closed and bounded convex subset of $\mathbb{R}^2$. Define

$$A_D:=\int\int_D dxdy$$

the *area* of D and $(\overline{x_D},\overline{y_D})$ the *centre of mass* for D, where

$$\overline{x_D}:=\frac{1}{A_D}\int\int_D xdxdy,\ \overline{y_D}:=\frac{1}{A_D}\int\int_D ydxdy.$$

Consider the function of two variables $f=f(x,y)$ and denote by $\frac{\partial f}{\partial x}$ the partial derivative with respect to the variable x and $\frac{\partial f}{\partial y}$ the partial derivative with respect to the variable y.

In the recent paper [7] we obtained among others the following result:

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Theorem 1. *Let $f : D \to \mathbb{R}$ be a differentiable convex function on D. Then for all $(u,v) \in D$ we have*

$$\begin{aligned}(1.2)\quad & \frac{\partial f}{\partial x}(u,v)(\overline{x_D}-u)+\frac{\partial f}{\partial y}(u,v)(\overline{y_D}-v)\\ &\le \frac{1}{A_D}\int\int_D f(x,y)\,dxdy - f(u,v)\\ &\le \frac{1}{A_D}\int\int_D \frac{\partial f}{\partial x}(x,y)(x-u)\,dxdy+\frac{1}{A_D}\int\int_D \frac{\partial f}{\partial y}(x,y)(y-v)\,dxdy.\end{aligned}$$

In particular,

$$\begin{aligned}(1.3)\quad 0 &\le \frac{1}{A_D}\int\int_D f(x,y)\,dxdy - f(\overline{x_D},\overline{y_D})\\ &\le \frac{1}{A_D}\int\int_D \frac{\partial f}{\partial x}(x,y)(x-\overline{x_D})\,dxdy+\frac{1}{A_D}\int\int_D \frac{\partial f}{\partial y}(x,y)(y-\overline{y_D})\,dxdy.\end{aligned}$$

We also have the reverse of Hermite-Hadamard inequality:

Corollary 1. *Let $f : D \to \mathbb{R}$ be a differentiable convex function on D. Put*

$$x_S := \frac{\int\int_D x\frac{\partial f}{\partial x}(x,y)\,dxdy}{\int\int_D \frac{\partial f}{\partial x}(x,y)\,dxdy},\ y_S := \frac{\int\int_D y\frac{\partial f}{\partial y}(x,y)\,dxdy}{\int\int_D \frac{\partial f}{\partial y}(x,y)\,dxdy}.$$

If $(x_S,y_S)\in D$, then

$$\begin{aligned}(1.4)\quad 0 &\le f(x_S,y_S)-\frac{1}{A_D}\int\int_D f(x,y)\,dxdy\\ &\le \frac{\partial f}{\partial x}(x_S,y_S)(x_S-\overline{x_D})+\frac{\partial f}{\partial y}(x_S,y_S)(y_S-\overline{y_D}).\end{aligned}$$

For other multivariate Hermite-Hadamard type inequalities, see [1]-[3] and [8]-[14].

Motivated by the above results, in this paper we establish some Hermite-Hadamard type inequalities for convex functions of two independent variables defined on closed and bounded convex subsets of the plane $\mathbb{R}^2$. The upper bounds are in terms of path integrals. Some examples for disks and rectangles are also provided.

2. The Main Results

Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a region D. Let L and M be scalar functions defined at least on an open set containing D. Assume L and M have continuous first partial derivatives. Then the following equality is well known as the *Green theorem*, see for instance

https : //en.wikipedia.org/wiki/Green%27s_theorem,

$$\text{(G)}\qquad \int\int_D \left(\frac{\partial M(x,y)}{\partial x}-\frac{\partial L(x,y)}{\partial y}\right)dxdy = \oint_{\partial D}(L(x,y)\,dx+M(x,y)\,dy).$$

By applying this equality for real and imaginary parts, we can also state it for complex valued functions L and M.

Moreover, if the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t \in [a,b]$, with x, y differentiable on (a,b) then we can calculate the path integral as

$$\oint_{\partial D} (L(x,y)\,dx + M(x,y)\,dy) = \int_a^b [L(x(t), y(t))\,x'(t) + M(x(t), y(t))\,y'(t)]\,dt.$$

We have:

Theorem 2. *Let $f : D \to \mathbb{R}$ be a differentiable convex function on D, a convex subset of $\mathbb{R}^2$. Then for all $(u,v) \in D$ we have*

$$\begin{aligned}(2.1)\quad \frac{\partial f}{\partial x}(u,v)(\overline{x_D} - u) + \frac{\partial f}{\partial y}(u,v)(\overline{y_D} - v) + f(u,v) \\ \leq \frac{1}{A_D}\int\int_D f(x,y)\,dxdy \\ \leq \frac{1}{3}f(u,v) + \frac{1}{3A_D}\oint_{\partial D}[(v-y)f(x,y)\,dx + (x-u)f(x,y)\,dy].\end{aligned}$$

In particular,

$$\begin{aligned}(2.2)\quad f(\overline{x_D}, \overline{y_D}) \leq \frac{1}{A_D}\int\int_D f(x,y)\,dxdy \\ \leq \frac{1}{3}f(\overline{x_D}, \overline{y_D}) + \frac{1}{3A_D}\oint_{\partial D}[(\overline{y_D} - y)f(x,y)\,dx + (x - \overline{x_D})f(x,y)\,dy].\end{aligned}$$

Proof. Observe that

$$\frac{\partial}{\partial x}((x-u)f(x,y)) = f(x,y) + (x-u)\frac{\partial f(x,y)}{\partial x}$$

and

$$\frac{\partial}{\partial y}((y-v)f(x,y)) = f(x,y) + (y-v)\frac{\partial f(x,y)}{\partial y}$$

for all $(x,y) \in D$ and is we add these equalities we get

$$\begin{aligned}(2.3)\quad \frac{\partial}{\partial x}((x-u)f(x,y)) + \frac{\partial}{\partial y}((y-v)f(x,y)) \\ = 2f(x,y) + (x-u)\frac{\partial f(x,y)}{\partial x} + (y-v)\frac{\partial f(x,y)}{\partial y}.\end{aligned}$$

Further, if we integrate on D the identity (2.3), then we obtain

$$\begin{aligned}(2.4)\quad \int\int_D\left[\frac{\partial}{\partial x}((x-u)f(x,y)) + \frac{\partial}{\partial y}((y-v)f(x,y))\right]dxdy \\ = 2\int\int_D f(x,y)\,dxdy \\ + \int\int_D\left[(x-u)\frac{\partial f(x,y)}{\partial x} + (y-v)\frac{\partial f(x,y)}{\partial y}\right]dxdy.\end{aligned}$$

Now, if we apply Green's identity (G) for the functions $M(x,y) = (x-u) f(x,y)$ and $L(x,y) = (v-y) f(x,y)$, then we get

$$\int\int_D \left[\frac{\partial}{\partial x}((x-u) f(x,y)) + \frac{\partial}{\partial y}((y-v) f(x,y))\right] dxdy$$
$$= \oint_{\partial D} [(\beta - v) f(x,y) dx + (x-u) f(x,y) dy]$$

and by (2.4) we obtain

$$\oint_{\partial D} [(v-y) f(x,y) dx + (x-u) f(x,y) dy] = 2\int\int_D f(x,y) dxdy$$
$$+ \int\int_D \left[(x-u)\frac{\partial f(x,y)}{\partial x} + (y-v)\frac{\partial f(x,y)}{\partial y}\right] dxdy$$

namely

$$\frac{1}{A_D}\int\int_D \left[(x-u)\frac{\partial f(x,y)}{\partial x} + (y-v)\frac{\partial f(x,y)}{\partial y}\right] dxdy \tag{2.5}$$
$$= \frac{1}{A_D}\oint_{\partial D} [(v-y) f(x,y) dx + (x-u) f(x,y) dy] - 2\frac{1}{A_D}\int\int_D f(x,y) dxdy.$$

Using the second inequality in (2.1) and (2.5) we get

$$\frac{1}{A_D}\int\int_D f(x,y) dxdy - f(u,v)$$
$$\le \frac{1}{A_D}\oint_{\partial D} [(v-y) f(x,y) dx + (x-u) f(x,y) dy] - 2\frac{1}{A_D}\int\int_D f(x,y) dxdy$$

namely

$$3\frac{1}{A_D}\int\int_D f(x,y) dxdy - f(u,v)$$
$$\le \frac{1}{A_D}\oint_{\partial D} [(v-y) f(x,y) dx + (x-u) f(x,y) dy],$$

which is equivalent to the second inequality in (2.1). □

Corollary 2. *With the assumptions of Theorem 2 we have*

$$f(\overline{x_D}, \overline{y_D}) \le \frac{1}{A_D}\int\int_D f(x,y) dxdy \tag{2.6}$$
$$\le \frac{1}{2A_D}\oint_{\partial D} [(\overline{y_D} - y) f(x,y) dx + (x - \overline{x_D}) f(x,y) dy].$$

Proof. Since

$$f(\overline{x_D}, \overline{y_D}) \le \frac{1}{A_D}\int\int_D f(x,y) dxdy$$

hence

$$\begin{aligned}
&\frac{1}{A_D}\int\int_D f(x,y)\,dxdy \\
&\leq \frac{1}{3}f(\overline{x_D},\overline{y_D}) + \frac{1}{3A_D}\oint_{\partial D}\left[(x-\overline{x_D})f(x,y)\,dx + (\overline{y_D}-y)f(x,y)\,dy\right] \\
&\leq \frac{1}{3A_D}\int\int_D f(x,y)\,dxdy + \frac{1}{3A_D}\oint_{\partial D}\left[(\overline{y_D}-y)f(x,y)\,dx + (x-\overline{x_D})f(x,y)\,dy\right],
\end{aligned}$$

which implies that

$$\frac{2}{3A_D}\int\int_D f(x,y)\,dxdy \leq \frac{1}{3A_D}\oint_{\partial D}\left[(\overline{y_D}-y)f(x,y)\,dx + (x-\overline{x_D})f(x,y)\,dy\right]$$

namely

$$\begin{aligned}
&\frac{1}{A_D}\int\int_D f(x,y)\,dxdy \\
&\qquad\leq \frac{1}{2A_D}\oint_{\partial D}\left[(\overline{y_D}-y)f(x,y)\,dx + (x-\overline{x_D})f(x,y)\,dy\right].
\end{aligned}$$

□

If the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t \in [a,b]$, with x, y differentiable on (a,b) then

$$\begin{aligned}
&\oint_{\partial D}\left[(v-y)f(x,y)\,dx + (x-u)f(x,y)\,dy\right] \\
&= \int_a^b \left[(v-y(t))x'(t) + (x(t)-u)y'(t)\right]f(x(t),y(t))\,dt
\end{aligned}$$

and by (2.1) we get

$$\begin{aligned}
\frac{1}{A_D}\int\int_D f(x,y)\,dxdy &\leq \frac{1}{3}f(u,v) \\
&+ \frac{1}{3A_D}\int_a^b \left[(v-y(t))x'(t) + (x(t)-u)y'(t)\right]f(x(t),y(t))\,dt,
\end{aligned}\tag{2.7}$$

by (2.2) we get

$$\begin{aligned}
\frac{1}{A_D}\int\int_D f(x,y)\,dxdy &\leq \frac{1}{3}f(\overline{x_D},\overline{y_D}) \\
&+ \frac{1}{3A_D}\int_a^b \left[(\overline{y_D}-y(t))x'(t) + (x(t)-\overline{x_D})y'(t)\right]f(x(t),y(t))\,dt,
\end{aligned}\tag{2.8}$$

while from (2.6) we get

$$\begin{aligned}
&\frac{1}{A_D}\int\int_D f(x,y)\,dxdy \\
&\leq \frac{1}{2A_D}\int_a^b \left[(\overline{y_D}-y(t))x'(t) + (x(t)-\overline{x_D})y'(t)\right]f(x(t),y(t))\,dt.
\end{aligned}\tag{2.9}$$

We define the quantities

$$x_{f,\partial D} := \frac{\oint_{\partial D} xf(x,y)\,dy}{\oint_{\partial D} f(x,y)\,dy} = \frac{\int_a^b x(t)\,f(x(t),y(t))\,y'(t)\,dt}{\int_a^b f(x(t),y(t))\,y'(t)\,dt}$$

and

$$y_{f,\partial D} := \frac{\oint_{\partial D} yf(x,y)\,dx}{\oint_{\partial D} f(x,y)\,dx} = \frac{\int_a^b y(t)\,f(x(t),y(t))\,x'(t)\,dt}{\int_a^b f(x(t),y(t))\,x'(t)\,dt}$$

provided the denominators are not zero.

Corollary 3. *With the assumptions of Theorem 2 and if* $(x_{f,\partial D}, y_{f,\partial D}) \in D$, *then we have*

$$\begin{aligned}(2.10)\quad \frac{\partial f}{\partial x}(x_{f,\partial D}, y_{f,\partial D})(\overline{x_D} - x_{f,\partial D}) + \frac{\partial f}{\partial y}(x_{f,\partial D}, y_{f,\partial D})(\overline{y_D} - y_{f,\partial D}) \\ + f(x_{f,\partial D}, y_{f,\partial D}) \\ \le \frac{1}{A_D}\int\int_D f(x,y)\,dxdy \le \frac{1}{3}f(x_{f,\partial D}, y_{f,\partial D}).\end{aligned}$$

The proof follows by (2.1) observing that

$$\oint_{\partial D}\left[(y_{f,\partial D} - y)\,f(x,y)\,dx + (x - x_{f,\partial D})\,f(x,y)\,dy\right] = 0.$$

Theorem 3. *Let* $f : D \to \mathbb{R}$ *be a differentiable convex function on* D, *a convex subset of* $\mathbb{R}^2$. *If* ∂D *is described by the function* $r(t) = (x(t), y(t))$, $t \in [a,b]$, *with* x, y *differentiable on* (a,b). *Define*

$$\overline{x_{\partial D}} := \frac{1}{\ell(\partial D)}\int_{\partial D} x\,d(\ell) = \frac{1}{\ell(\partial D)}\int_a^b x(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

and

$$\overline{y_{\partial D}} := \frac{1}{\ell(\partial D)}\int_{\partial D} y\,d(\ell) = \frac{1}{\ell(\partial D)}\int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt.$$

Then

$$\begin{aligned}(2.11)\quad \frac{1}{\ell(\partial D)}\int_{\partial D}\frac{\partial f}{\partial x}(x,y)(\overline{x_D} - x)\,d(\ell) + \frac{1}{\ell(\partial D)}\int_{\partial D}\frac{\partial f}{\partial y}(x,y)(\overline{y_D} - y)\,d(\ell) \\ \le \frac{1}{A_D}\int\int_D f(x,y)\,dxdy - \frac{1}{\ell(\partial D)}\int_{\partial D} f(x,y)\,d(\ell) \\ \le \frac{1}{A_D}\int\int_D \frac{\partial f}{\partial x}(x,y)(x - \overline{x_{\partial D}})\,dxdy + \frac{1}{A_D}\int\int_D \frac{\partial f}{\partial y}(x,y)(y - \overline{y_{\partial D}})\,dxdy.\end{aligned}$$

Proof. From the inequality (1.2) we get

$$\begin{aligned}(2.12)\quad & \frac{\partial f}{\partial x}\left(x\left(t\right),y\left(t\right)\right)\left(\overline{x_D}-x\left(t\right)\right)+\frac{\partial f}{\partial y}\left(x\left(t\right),y\left(t\right)\right)\left(\overline{y_D}-y\left(t\right)\right)\\ & \leq\frac{1}{A_D}\int\int_D f\left(x,y\right)dxdy-f\left(x\left(t\right),y\left(t\right)\right)\\ & \leq\frac{1}{A_D}\int\int_D\frac{\partial f}{\partial x}\left(x,y\right)\left(x-x\left(t\right)\right)dxdy+\frac{1}{A_D}\int\int_D\frac{\partial f}{\partial y}\left(x,y\right)\left(y-y\left(t\right)\right)dxdy\end{aligned}$$

for all $t\in\left[a,b\right]$.

If we multiply (2.12) by $\sqrt{\left(x'\left(t\right)\right)^2+\left(y'\left(t\right)\right)^2}$, $t\in\left(a,b\right)$ and integrate on $\left[a,b\right]$ we get

$$\begin{aligned}(2.13)\quad & \int_a^b\frac{\partial f}{\partial x}\left(x\left(t\right),y\left(t\right)\right)\left(\overline{x_D}-x\left(t\right)\right)\sqrt{\left(x'\left(t\right)\right)^2+\left(y'\left(t\right)\right)^2}dt\\ & +\int_a^b\frac{\partial f}{\partial y}\left(x\left(t\right),y\left(t\right)\right)\left(\overline{y_D}-y\left(t\right)\right)\sqrt{\left(x'\left(t\right)\right)^2+\left(y'\left(t\right)\right)^2}dt\\ & \leq\frac{1}{A_D}\int\int_D f\left(x,y\right)dxdy\int_a^b\sqrt{\left(x'\left(t\right)\right)^2+\left(y'\left(t\right)\right)^2}dt\\ & -\int_a^b f\left(x\left(t\right),y\left(t\right)\right)\sqrt{\left(x'\left(t\right)\right)^2+\left(y'\left(t\right)\right)^2}dt\\ & \leq\frac{1}{A_D}\int_a^b\left(\int\int_D\frac{\partial f}{\partial x}\left(x,y\right)\left(x-x\left(t\right)\right)\sqrt{\left(x'\left(t\right)\right)^2+\left(y'\left(t\right)\right)^2}dxdy\right)dt\\ & +\frac{1}{A_D}\int_a^b\left(\int\int_D\frac{\partial f}{\partial y}\left(x,y\right)\left(y-y\left(t\right)\right)\sqrt{\left(x'\left(t\right)\right)^2+\left(y'\left(t\right)\right)^2}dxdy\right)dt.\end{aligned}$$

Since

$$\int_a^b\sqrt{\left(x'\left(t\right)\right)^2+\left(y'\left(t\right)\right)^2}dt=\ell\left(\partial D\right),$$

and by Fubini's theorem

$$\begin{aligned}& \int_a^b\left(\int\int_D\frac{\partial f}{\partial x}\left(x,y\right)\left(x-x\left(t\right)\right)\sqrt{\left(x'\left(t\right)\right)^2+\left(y'\left(t\right)\right)^2}dxdy\right)dt\\ & =\int\int_D\frac{\partial f}{\partial x}\left(x,y\right)\left(\int_a^b\left(x-x\left(t\right)\right)\sqrt{\left(x'\left(t\right)\right)^2+\left(y'\left(t\right)\right)^2}dt\right)dxdy\\ & =\int\int_D\frac{\partial f}{\partial x}\left(x,y\right)\left(x\ell\left(\partial D\right)-\int_a^b x\left(t\right)\sqrt{\left(x'\left(t\right)\right)^2+\left(y'\left(t\right)\right)^2}dt\right)dxdy\\ & =\int\int_D\frac{\partial f}{\partial x}\left(x,y\right)\left(x\ell\left(\partial D\right)-\int_a^b x\left(t\right)\sqrt{\left(x'\left(t\right)\right)^2+\left(y'\left(t\right)\right)^2}dt\right)dxdy\end{aligned}$$

then

$$\frac{1}{\ell(\partial D)}\int_a^b\left(\int\int_D\frac{\partial f}{\partial x}(x,y)(x-x(t))\sqrt{(x'(t))^2+(y'(t))^2}dxdy\right)dt$$
$$=\int\int_D\frac{\partial f}{\partial x}(x,y)\left(x-\frac{1}{\ell(\partial D)}\int_a^b x(t)\sqrt{(x'(t))^2+(y'(t))^2}dt\right)dxdy$$
$$=\int\int_D\frac{\partial f}{\partial x}(x,y)\left(x-\overline{x_{\partial D}}\right)dxdy.$$

Similarly,

$$\frac{1}{\ell(\partial D)}\int_a^b\left(\int\int_D\frac{\partial f}{\partial y}(x,y)(y-y(t))\sqrt{(x'(t))^2+(y'(t))^2}dxdy\right)dt$$
$$=\int\int_D\frac{\partial f}{\partial y}(x,y)\left(y-\overline{y_{\partial D}}\right)dxdy.$$

By dividing with $\ell(\partial D)$ in the inequality (2.13) we get the desired result (2.11). □

Corollary 4. *With the assumptions of Theorem 3 we have the inequality*

$$\frac{1}{A_D}\int\int_D f(x,y)\,dxdy\le\frac{1}{3\ell(\partial D)}\int_{\partial D}f(x,y)\,d(\ell)$$
$$+\frac{1}{3A_D}\oint_{\partial D}\left[\left(\overline{y_{\partial D}}-y\right)f(x,y)\,dx+\left(x-\overline{x_{\partial D}}\right)f(x,y)\,dy\right]. \tag{2.14}$$

Proof. As in the proof of Theorem 2, we obtain, by employing Green's identity (G), that

$$\frac{1}{A_D}\int\int_D\left[\frac{\partial f}{\partial x}(x,y)\left(x-\overline{x_{\partial D}}\right)dxdy+\frac{\partial f}{\partial y}(x,y)\left(y-\overline{y_{\partial D}}\right)\right]dxdy$$
$$=\frac{1}{A_D}\oint_{\partial D}\left[\left(\overline{y_{\partial D}}-y\right)f(x,y)\,dx+\left(x-\overline{x_{\partial D}}\right)f(x,y)\,dy\right]-2\frac{1}{A_D}\int\int_D f(x,y)\,dxdy.$$

By the second inequality in (2.11) we derive

$$\frac{1}{A_D}\int\int_D f(x,y)\,dxdy-\frac{1}{\ell(\partial D)}\int_{\partial D}f(x,y)\,d(\ell)$$
$$\le\frac{1}{A_D}\oint_{\partial D}\left[\left(\overline{y_{\partial D}}-y\right)f(x,y)\,dx+\left(x-\overline{x_{\partial D}}\right)f(x,y)\,dy\right]$$
$$-2\frac{1}{A_D}\int\int_D f(x,y)\,dxdy,$$

namely

$$\frac{3}{A_D}\int\int_D f(x,y)\,dxdy\le\frac{1}{\ell(\partial D)}\int_{\partial D}f(x,y)\,d(\ell)$$
$$+\frac{1}{A_D}\oint_{\partial D}\left[\left(\overline{y_{\partial D}}-y\right)f(x,y)\,dx+\left(x-\overline{x_{\partial D}}\right)f(x,y)\,dy\right],$$

which is equivalent to (2.14). □

If the curve ∂D is described by the function $r(t)=(x(t),y(t))$, $t\in[a,b]$, with x, y differentiable on (a,b) then by (2.11) and(2.14) we get the following inequality useful for applications

$$
\begin{aligned}
&\frac{1}{\ell(\partial D)}\int_a^b \frac{\partial f}{\partial x}(x(t),y(t))\left(\overline{x_D}-x(t)\right)\sqrt{(x'(t))^2+(y'(t))^2}dt \\
&\quad+\frac{1}{\ell(\partial D)}\int_a^b \frac{\partial f}{\partial y}(x(t),y(t))\left(\overline{y_D}-y(t)\right)\sqrt{(x'(t))^2+(y'(t))^2}dt \\
&\qquad+\frac{1}{\ell(\partial D)}\int_a^b f(x(t),y(t))\sqrt{(x'(t))^2+(y'(t))^2}dt \\
&\leq\frac{1}{A_D}\int\int_D f(x,y)\,dxdy\leq\frac{1}{3\ell(\partial D)}\int_a^b f(x(t),y(t))\sqrt{(x'(t))^2+(y'(t))^2}dt \\
&\quad+\frac{1}{3A_D}\int_a^b\left[\left(\overline{y_{\partial D}}-y(y)\right)x'(t)+\left(x(t)-\overline{x_{\partial D}}\right)y'(t)\right]f(x(t),y(t))\,dt.
\end{aligned}
\tag{2.15}
$$

3. Examples for Disks

We consider the closed disk $D(C,R)$ centered in $C(a,b)$ and of radius $R>0$. This is parametrized by

$$
\begin{cases} x=r\cos\theta+a \\ \\ y=r\sin\theta+b \end{cases},\ r\in[0,R],\ \theta\in[0,2\pi]
$$

and the circle $\mathcal{C}(C,R)$ is parametrized by

$$
\begin{cases} x=R\cos\theta+a \\ \\ y=R\sin\theta+b \end{cases},\ \theta\in[0,2\pi].
$$

Here $\overline{x_{D(C,R)}}=a$, $\overline{y_{D(C,R)}}=b$ and $A_{D(C,R)}=\pi R^2$.

Then

$$
\begin{aligned}
&\frac{1}{A_D}\int_a^b\left[\left(\overline{y_D}-y(t)\right)x'(t)+\left(x(t)-\overline{x_D}\right)y'(t)\right]f(x(t),y(t))\,dt \\
&=\frac{1}{\pi R^2}\int_0^{2\pi}\left[\sin^2\theta+\cos^2\theta\right]R^2 f(R\cos\theta+a,R\sin\theta+b)\,d\theta \\
&=\frac{1}{\pi}\int_0^{2\pi} f(R\cos\theta+a,R\sin\theta+b)\,d\theta
\end{aligned}
$$

and by (2.2) we get

$$
\begin{aligned}
f(a,b)&\leq\frac{1}{\pi R^2}\int\int_{D(C,R)} f(x,y)\,dxdy \\
&\leq\frac{1}{3}f(a,b)+\frac{1}{3\pi}\int_0^{2\pi} f(R\cos\theta+a,R\sin\theta+b)\,d\theta,
\end{aligned}
\tag{3.1}
$$

provided that f is convex on an open set containing the disk $D(C,R)$ and has continuous partial derivatives on $D(C,R)$.

Since

$$\frac{1}{3\pi}\int_{0}^{2\pi} f\left(R\cos\theta + a, R\sin\theta + b\right)dt = \frac{2}{3}\frac{1}{\ell\left(\mathcal{C}\left(C,R\right)\right)}\int_{\mathcal{C}(C,R)} f\left(\gamma\right)d\ell\left(\gamma\right),$$

hence we get the inequality (1.1) with a different proof than the original one in [4].

We also have

$$\overline{x_{\mathcal{C}(C,R)}} := \frac{1}{2\pi R}\int_{0}^{2\pi}\left(R\cos\theta + a\right)Rd\theta = a$$

and

$$\overline{x_{\mathcal{C}(C,R)}} := \frac{1}{2\pi R}\int_{a}^{b}\left(r\sin\theta + b\right)Rd\theta = b$$

and by (2.15) we get

$$\begin{aligned}
&-\frac{1}{2\pi}R\int_{a}^{b}\frac{\partial f}{\partial x}\left(R\cos\theta + a, R\sin\theta + b\right)\cos\theta d\theta \\
&\qquad -\frac{1}{2\pi}R\int_{a}^{b}\frac{\partial f}{\partial y}\left(R\cos\theta + a, R\sin\theta + b\right)\sin\theta d\theta \\
&\qquad +\frac{1}{2\pi}\int_{a}^{b} f\left(R\cos\theta + a, R\sin\theta + b\right)d\theta \\
&\leq \frac{1}{\pi R^2}\int\int_{D(C,R)} f\left(x,y\right)dxdy \leq \frac{1}{6\pi}\int_{0}^{2\pi} f\left(R\cos\theta + a, R\sin\theta + b\right)d\theta \\
&+\frac{1}{3\pi}\int_{a}^{b} f\left(R\cos\theta + a, R\sin\theta + b\right)d\theta = \frac{1}{2\pi}\int_{a}^{b} f\left(R\cos\theta + a, R\sin\theta + b\right)d\theta
\end{aligned}$$

namely

$$\begin{aligned}
(3.2)\quad 0 &\leq \frac{1}{2\pi}\int_{a}^{b} f\left(R\cos\theta + a, R\sin\theta + b\right)d\theta - \frac{1}{\pi R^2}\int\int_{D(C,R)} f\left(x,y\right)dxdy \\
&\leq \frac{1}{2\pi}R\int_{a}^{b}\frac{\partial f}{\partial x}\left(R\cos\theta + a, R\sin\theta + b\right)\cos\theta d\theta \\
&\qquad +\frac{1}{2\pi}R\int_{a}^{b}\frac{\partial f}{\partial y}\left(R\cos\theta + a, R\sin\theta + b\right)\sin\theta d\theta,
\end{aligned}$$

provided that f is convex on an open set containing the disk $D\left(C,R\right)$ and has continuous partial derivatives on $D\left(C,R\right)$.

This result provides a reverse inequality for the last part of (1.1).

4. Examples for Rectangles

Let $a < b$ and $c < d$. Put $A = (a,c)$, $B = (b,c)$, $C = (b,d)$, $D = (a,d) \in \mathbb{R}^2$ the vertices of the rectangle $ABCD = [a,b] \times [c,d]$. Consider the counterclockwise segments

$$AB:\begin{cases} x=(1-t)\,a+tb \\ \\ y=c \end{cases},\ t\in[0,1]$$

$$BC:\begin{cases} x=b \\ \\ y=(1-t)\,c+td \end{cases},\ t\in[0,1]$$

$$CD:\begin{cases} x=(1-t)\,b+ta \\ \\ y=d \end{cases},\ t\in[0,1]$$

and

$$DA:\begin{cases} x=a \\ \\ y=(1-t)\,d+tc \end{cases},\ t\in[0,1]\,.$$

Therefore $\partial\,(ABCD)=AB\cup BC\cup CD\cup DA$.

If α, $\beta\in\mathbb{R}$, then

$$\begin{aligned}&\oint_{AB}\left[(\beta-y)\,f\,(x,y)\,dx+(x-\alpha)\,f\,(x,y)\,dy\right]\\&=(b-a)\,(\beta-c)\int_0^1 f\,((1-t)\,a+tb,c)\,dt=(\beta-c)\int_a^b f\,(x,c)\,dx,\end{aligned}$$

$$\begin{aligned}&\oint_{BC}\left[(\beta-y)\,f\,(x,y)\,dx+(x-\alpha)\,f\,(x,y)\,dy\right]\\&=(d-c)\,(b-\alpha)\int_0^1 f\,(b,(1-t)\,c+td)\,dt=(b-\alpha)\int_c^d f\,(b,y)\,dy\end{aligned}$$

$$\begin{aligned}&\oint_{CD}\left[(\beta-y)\,f\,(x,y)\,dx+(x-\alpha)\,f\,(x,y)\,dy\right]\\&=(a-b)\,(\beta-d)\int_0^1 f\,((1-t)\,b+ta,d)\,dt=(\beta-d)\int_b^a f\,(x,d)\,dx\\&=(d-\beta)\int_a^b f\,(x,d)\,dx\end{aligned}$$

and

$$\begin{aligned}&\oint_{DA}\left[(\beta-y)\,f\,(x,y)\,dx+(x-\alpha)\,f\,(x,y)\,dy\right]\\&=\int_0^1(a-\alpha)\,f\,(a,(1-t)\,d+tc)\,(c-d)\,dt=(a-\alpha)\int_d^c f\,(a,y)\,dy\\&=(\alpha-a)\int_c^d f\,(a,y)\,dy.\end{aligned}$$

Therefore

$$\oint_{ABCD} [(\beta - y) f(x,y)\,dx + (x-\alpha) f(x,y)\,dy]$$
$$= (\beta - c)\int_a^b f(x,c)\,dx + (d-\beta)\int_a^b f(x,d)\,dx$$
$$+ (b-\alpha)\int_c^d f(b,y)\,dy + (\alpha - a)\int_c^d f(a,y)\,dy$$

for all $\alpha,\ \beta \in \mathbb{R}$.

We also have $\overline{x_D} = \frac{a+b}{2}$ and $\overline{y_D} = \frac{c+d}{2}$, which imply that

$$\oint_{\partial(ABCD)} [(\overline{y_D} - y) f(x,y)\,dx + (x - \overline{x_D}) f(x,y)\,dy]$$
$$= (d-c)\int_a^b \left(\frac{f(x,c)+f(x,d)}{2}\right) dx + (b-a)\int_c^d \left(\frac{f(b,y)+f(a,y)}{2}\right) dy.$$

If we write the inequalities (2.1) and (2.2) for the rectangle $ABCD = [a,b]\times[c,d]$ and the convex function f defined on $[a,b]\times[c,d]$ we get the inequalities

$$\frac{\partial f}{\partial x}(u,v)\left(\frac{a+b}{2} - u\right) + \frac{\partial f}{\partial y}(u,v)\left(\frac{c+d}{2} - v\right) + f(u,v) \tag{4.1}$$
$$\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy \le \frac{1}{3} f(u,v)$$
$$+ \frac{1}{3(b-a)(d-c)}\left[(v-c)\int_a^b f(x,c)\,dx + (d-v)\int_a^b f(x,d)\,dx\right]$$
$$+ \frac{1}{3(b-a)(d-c)}\left[(b-u)\int_c^d f(b,y)\,dy + (u-a)\int_c^d f(a,y)\,dy\right]$$

for all $(u,v) \in [a,b]\times[c,d]$.

In particular,

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy \tag{4.2}$$
$$\le \frac{1}{3} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$
$$+ \frac{1}{3}\left[\frac{1}{b-a}\int_a^b \left(\frac{f(x,c)+f(x,d)}{2}\right) dx + \frac{1}{d-c}\int_c^d \left(\frac{f(b,y)+f(a,y)}{2}\right) dy\right].$$

From the inequality (2.6) we also get

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy \tag{4.3}$$
$$\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b \left(\frac{f(x,c)+f(x,d)}{2}\right) dx + \frac{1}{d-c}\int_c^d \left(\frac{f(b,y)+f(a,y)}{2}\right) dy\right].$$

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