SOME INEQUALITIES FOR DOUBLE AND PATH INTEGRALS ON GENERAL DOMAINS VIA GREEN'S IDENTITY

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Abstract. In this paper we establish some upper bounds for the following quatities

$$\left|\int\int_D f(x,y)\,dxdy - \frac{1}{2}\int\int_D \left[(\alpha - x)\frac{\partial f(x,y)}{\partial x} + (\beta - y)\frac{\partial f(x,y)}{\partial y}\right] dxdy\right|$$

and

$$\left|\int\int_D f(x,y)\,dxdy - \frac{1}{2}\oint_{\partial D} \left[(\beta - y) f(x,y)\,dx + (x-\alpha) f(x,y)\,dy\right]\right|$$

in the case that ∂D is a simple, closed counterclockwise curve bounding a region D, f is a complex valued function defined on an open set containing D and having continuous partial derivatives on D while α and β are some complex parameters. Some exmples for rectangles and disks are also given.

1. Introduction

Let us consider a point $C = (a,b) \in \mathbb{R}^2$ and the disk $D(C,R)$ centered at the point C and having the radius $R > 0$. In [4] we establish between others the following Hermite-Hadamard type inequality for a convex function $f : D(C,R) \to \mathbb{R}$,

$$\begin{aligned} (1.1)\quad f(C) &\le \frac{1}{A_{D(C,R)}}\iint_{D(C,R)} f(x,y)\,dxdy \\ &\le \frac{2}{3}\frac{1}{\ell(\mathcal{C}(C,R))}\int_{\mathcal{C}(C,R)} f(\gamma)\,d\ell(\gamma) + \frac{1}{3} f(C) \\ &\le \frac{1}{\ell(\mathcal{C}(C,R))}\int_{\mathcal{C}(C,R)} f(\gamma)\,d\ell(\gamma), \end{aligned}$$

where $\mathcal{C}(C,R)$ is the circle centered at C and having the radius R and $\int_{\mathcal{C}(C,R)}$ is the path integral with respect to arc length, $A_{D(C,R)} = \pi R^2$ is the area of the disk and $\ell(\mathcal{C}(C,R)) = 2\pi R$ is the length of the circle $\mathcal{C}(C,R)$.

In the following, consider D a closed and bounded convex subset of $\mathbb{R}^2$. Define

$$A_D := \int\int_D dxdy$$

the *area* of D and $(\overline{x_D}, \overline{y_D})$ the *centre of mass* for D, where

$$\overline{x_D} := \frac{1}{A_D}\int\int_D x\,dxdy,\ \overline{y_D} := \frac{1}{A_D}\int\int_D y\,dxdy.$$

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Consider the function of two variables $f=f(x,y)$ and denote by $\frac{\partial f}{\partial x}$ the partial derivative with respect to the variable x and $\frac{\partial f}{\partial y}$ the partial derivative with respect to the variable y.

In the recent paper [7] we obtained among others the following result:

Theorem 1. *Let $f:D\to\mathbb{R}$ be a differentiable convex function on D. Then for all $(u,v)\in D$ we have*

$$\begin{aligned}(1.2)\quad &\frac{\partial f}{\partial x}(u,v)(\overline{x_D}-u)+\frac{\partial f}{\partial y}(u,v)(\overline{y_D}-v)\\ &\leq\frac{1}{A_D}\int\int_D f(x,y)\,dxdy-f(u,v)\\ &\leq\frac{1}{A_D}\int\int_D\frac{\partial f}{\partial x}(x,y)(x-u)\,dxdy+\frac{1}{A_D}\int\int_D\frac{\partial f}{\partial y}(x,y)(y-v)\,dxdy.\end{aligned}$$

In particular,

$$\begin{aligned}(1.3)\quad 0&\leq\frac{1}{A_D}\int\int_D f(x,y)\,dxdy-f(\overline{x_D},\overline{y_D})\\ &\leq\frac{1}{A_D}\int\int_D\frac{\partial f}{\partial x}(x,y)(x-\overline{x_D})\,dxdy+\frac{1}{A_D}\int\int_D\frac{\partial f}{\partial y}(x,y)(y-\overline{y_D})\,dxdy.\end{aligned}$$

We also have the reverse of Hermite-Hadamard inequality:

Corollary 1. *Let $f:D\to\mathbb{R}$ be a differentiable convex function on D. Let*

$$x_S:=\frac{\int\int_D x\frac{\partial f}{\partial x}(x,y)\,dxdy}{\int\int_D\frac{\partial f}{\partial x}(x,y)\,dxdy},\ y_S:=\frac{\int\int_D y\frac{\partial f}{\partial y}(x,y)\,dxdy}{\int\int_D\frac{\partial f}{\partial y}(x,y)\,dxdy}.$$

If $(x_S,y_S)\in D$, then

$$\begin{aligned}(1.4)\quad 0&\leq f(x_S,y_S)-\frac{1}{A_D}\int\int_D f(x,y)\,dxdy\\ &\leq\frac{\partial f}{\partial x}(x_S,y_S)(x_S-\overline{x_D})+\frac{\partial f}{\partial y}(x_S,y_S)(y_S-\overline{y_D}).\end{aligned}$$

For other multivariate Hermite-Hadamard type inequalities, see [1]-[3] and [8]-[14].

Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a region D. Let L and M be scalar functions defined at least on an open set containing D. Assume L and M have continuous first partial derivatives. Then the following equality is well known as the Green theorem (see for instance https://en.wikipedia.org/wiki/Green%27s_theorem)

$$\text{(G)}\qquad\int\int_D\left(\frac{\partial M(x,y)}{\partial x}-\frac{\partial L(x,y)}{\partial y}\right)dxdy=\oint_{\partial D}(L(x,y)\,dx+M(x,y)\,dy).$$

By applying this equality for real and imaginary parts, we can also state it for complex valued functions P and Q.

Moreover, if the curve ∂D is described by the function $r(t)=(x(t),y(t))$, $t\in[a,b]$, with x, y differentiable on (a,b) then we can calculate the path integral as

$$\oint_{\partial D} \left(L\left(x,y\right)dx + M\left(x,y\right)dy\right) = \int_a^b \left[L\left(x\left(t\right),y\left(t\right)\right)x'\left(t\right) + M\left(x\left(t\right),y\left(t\right)\right)y'\left(t\right)\right]dt.$$

In this paper we establish some upper bounds for the following quantities

$$\left|\int\int_D f\left(x,y\right)dxdy - \frac{1}{2}\int\int_D \left[\left(\alpha - x\right)\frac{\partial f\left(x,y\right)}{\partial x} + \left(\beta - y\right)\frac{\partial f\left(x,y\right)}{\partial y}\right]dxdy\right|$$

and

$$\left|\int\int_D f\left(x,y\right)dxdy - \frac{1}{2}\oint_{\partial D}\left[\left(\beta - y\right)f\left(x,y\right)dx + \left(x-\alpha\right)f\left(x,y\right)dy\right]\right|$$

in the case that ∂D is a simple, closed counterclockwise curve bounding a region D, f is a complex valued function defined on an open set containing D and having continuous partial derivatives on D while α and β are some complex parameters.

2. Main Results

We start with the following identity that is of interest in itself:

Lemma 1. *Let ∂D be a simple, closed counterclockwise curve bounding a region D and f defined on an open set containing D and having continuous partial derivatives on D. Then for any α, $\beta \in \mathbb{C}$,*

$$\int\int_D f\left(x,y\right)dxdy = \frac{1}{2}\int\int_D \left[\left(\alpha - x\right)\frac{\partial f\left(x,y\right)}{\partial x} + \left(\beta - y\right)\frac{\partial f\left(x,y\right)}{\partial y}\right]dxdy + \frac{1}{2}\oint_{\partial D}\left[\left(\beta - y\right)f\left(x,y\right)dx + \left(x-\alpha\right)f\left(x,y\right)dy\right]. \tag{2.1}$$

In particular, we have

$$\begin{aligned}\int\int_D f\left(x,y\right)dxdy &= \frac{1}{2}\int\int_D \left[\left(\overline{x_D} - x\right)\frac{\partial f\left(x,y\right)}{\partial x} + \left(\overline{y_D} - y\right)\frac{\partial f\left(x,y\right)}{\partial y}\right]dxdy \\ &\quad + \frac{1}{2}\oint_{\partial D}\left[\left(\overline{y_D} - y\right)f\left(x,y\right)dx + \left(x-\overline{x_D}\right)f\left(x,y\right)dy\right].\end{aligned} \tag{2.2}$$

Proof. Observe that

$$\frac{\partial}{\partial x}\left(\left(x-\alpha\right)f\left(x,y\right)\right) = f\left(x,y\right) + \left(x-\alpha\right)\frac{\partial f\left(x,y\right)}{\partial x}$$

and

$$\frac{\partial}{\partial y}\left(\left(y-\beta\right)f\left(x,y\right)\right) = f\left(x,y\right) + \left(y-\beta\right)\frac{\partial f\left(x,y\right)}{\partial y}$$

for all $(x,y) \in D$ and is we add these equalities we get

$$\begin{aligned}&\frac{\partial}{\partial x}\left(\left(x-\alpha\right)f\left(x,y\right)\right) + \frac{\partial}{\partial y}\left(\left(y-\beta\right)f\left(x,y\right)\right) \\ &= 2f\left(x,y\right) + \left(x-\alpha\right)\frac{\partial f\left(x,y\right)}{\partial x} + \left(y-\beta\right)\frac{\partial f\left(x,y\right)}{\partial y}.\end{aligned} \tag{2.3}$$

Further, if we integrate on D the identity (2.3), then we obtain

$$\int\int_D \left[\frac{\partial}{\partial x}\left((x-\alpha) f(x,y)\right)+\frac{\partial}{\partial y}\left((y-\beta) f(x,y)\right)\right] dxdy \tag{2.4}$$
$$= 2\int\int_D f(x,y)\,dxdy + \int\int_D \left[(x-\alpha)\frac{\partial f(x,y)}{\partial x}+(y-\beta)\frac{\partial f(x,y)}{\partial y}\right] dxdy.$$

Now, if we apply Green's identity (G) for the functions $M(x,y)=(x-\alpha) f(x,y)$ and $L(x,y)=(\beta-y) f(x,y)$ then we get

$$\int\int_D \left[\frac{\partial}{\partial x}\left((x-\alpha) f(x,y)\right)+\frac{\partial}{\partial y}\left((y-\beta) f(x,y)\right)\right] dxdy$$
$$= \oint_{\partial D} \left[(\beta-y) f(x,y)\,dx+(x-\alpha) f(x,y)\,dy\right]$$

and by (2.4) we obtain

$$2\int\int_D f(x,y)\,dxdy + \int\int_D \left[(x-\alpha)\frac{\partial f(x,y)}{\partial x}+(y-\beta)\frac{\partial f(x,y)}{\partial y}\right] dxdy$$
$$= \oint_{\partial D} \left[(\beta-y) f(x,y)\,dx+(x-\alpha) f(x,y)\,dy\right]$$

which is equivalent to the desired equality (2.1). □

Corollary 2. *With the assumptions of Lemma 1 and if the curve ∂D is described by the function $r(t)=(x(t),y(t))$, $t\in[a,b]$, with x, y differentiable on (a,b), then*

$$\int\int_D f(x,y)\,dxdy = \frac{1}{2}\int\int_D \left[(\alpha-x)\frac{\partial f(x,y)}{\partial x}+(\beta-y)\frac{\partial f(x,y)}{\partial y}\right] dxdy \tag{2.5}$$
$$+\frac{1}{2}\int_a^b \left[(\beta-y(t))\,x'(t)+(x(t)-\alpha)\,y'(t)\right] f(x(t),y(t))\,dt.$$

In particular,

$$\int\int_D f(x,y)\,dxdy \tag{2.6}$$
$$= \frac{1}{2}\int\int_D \left[(\overline{x_D}-x)\frac{\partial f(x,y)}{\partial x}+(\overline{y_D}-y)\frac{\partial f(x,y)}{\partial y}\right] dxdy$$
$$+\frac{1}{2}\int_a^b \left[(\overline{y_D}-y(t))\,x'(t)+(x(t)-\overline{x_D})\,y'(t)\right] f(x(t),y(t))\,dt.$$

We have the following inequalities:

Theorem 2. *Let ∂D be a simple, closed counterclockwise curve bounding a region D and f defined on an open set containing D and having continuous partial derivatives* *on D. If the curve ∂D is described by the function $r(t)=(x(t),y(t))$, $t\in[a,b]$, with x, y differentiable on (a,b), then*

$$\left|\int\int_D f(x,y)\,dxdy-\frac{1}{2}\int\int_D\left[(\alpha-x)\frac{\partial f(x,y)}{\partial x}+(\beta-y)\frac{\partial f(x,y)}{\partial y}\right]dxdy\right| \tag{2.7}$$
$$\leq\frac{1}{2}\int_a^b\left[|\beta-y(t)|\,|x'(t)|+|x(t)-\alpha|\,|y'(t)|\right]|f(x(t),y(t))|\,dt$$
$$\leq\frac{1}{2}\begin{cases}\int_a^b\max\{|\beta-y(t)|,|x(t)-\alpha|\}\left[|x'(t)|+|y'(t)|\right]|f(x(t),y(t))|\,dt,\\[2ex] \int_a^b\left[|\beta-y(t)|^p+|x(t)-\alpha|^p\right]^{1/p}\left[|x'(t)|^q+|y'(t)|^q\right]^{1/q}|f(x(t),y(t))|\,dt\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \int_a^b\max\{|x'(t)|,|y'(t)|\}\left[|\beta-y(t)|+|x(t)-\alpha|\right]|f(x(t),y(t))|\,dt.\end{cases}$$

In particular,

$$\left|\int\int_D f(x,y)\,dxdy\right. \tag{2.8}$$
$$\left.-\frac{1}{2}\int\int_D\left[(\overline{x_D}-x)\frac{\partial f(x,y)}{\partial x}+(\overline{y_D}-y)\frac{\partial f(x,y)}{\partial y}\right]dxdy\right|$$
$$\leq\frac{1}{2}\int_a^b\left[|\overline{y_D}-y(t)|\,|x'(t)|+|x(t)-\overline{x_D}|\,|y'(t)|\right]|f(x(t),y(t))|\,dt$$
$$\leq\frac{1}{2}\begin{cases}\int_a^b\max\{|\overline{y_D}-y(t)|,|x(t)-\overline{x_D}|\}\left[|x'(t)|+|y'(t)|\right]|f(x(t),y(t))|\,dt,\\[2ex] \int_a^b\left[|\overline{y_D}-y(t)|^p+|x(t)-\overline{x_D}|^p\right]^{1/p}\left[|x'(t)|^q+|y'(t)|^q\right]^{1/q}|f(x(t),y(t))|\,dt\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \int_a^b\max\{|x'(t)|,|y'(t)|\}\left[|\overline{y_D}-y(t)|+|x(t)-\overline{x_D}|\right]|f(x(t),y(t))|\,dt.\end{cases}$$

Proof. From the identity (2.5) we have

$$\left|\int\int_D f(x,y)\,dxdy-\frac{1}{2}\int\int_D\left[(\alpha-x)\frac{\partial f(x,y)}{\partial x}+(\beta-y)\frac{\partial f(x,y)}{\partial y}\right]dxdy\right| \tag{2.9}$$
$$=\frac{1}{2}\left|\int_a^b\left[(\beta-y(t))\,x'(t)+(x(t)-\alpha)\,y'(t)\right]f(x(t),y(t))\,dt\right|$$
$$\leq\frac{1}{2}\int_a^b\left|(\beta-y(t))\,x'(t)+(x(t)-\alpha)\,y'(t)\right||f(x(t),y(t))|\,dt$$
$$\leq\frac{1}{2}\int_a^b\left[|\beta-y(t)|\,|x'(t)|+|x(t)-\alpha|\,|y'(t)|\right]|f(x(t),y(t))|\,dt,$$

which proves the first inequality in (2.8).

Using Hölder's discrete inequality we have

$$
\begin{aligned}
&|\beta - y(t)| \, |x'(t)| + |x(t) - \alpha| \, |y'(t)| \\
&\leq \frac{1}{2} \begin{cases} \max\{|\beta - y(t)|, |x(t) - \alpha|\} \left[|x'(t)| + |y'(t)|\right], \\ \\ \left[|\beta - y(t)|^p + |x(t) - \alpha|^p\right]^{1/p} \left[|x'(t)|^q + |y'(t)|^q\right]^{1/q} \\ p, \ q > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \max\{|x'(t)|, |y'(t)|\} \left[|\beta - y(t)| + |x(t) - \alpha|\right], \end{cases}
\end{aligned}
$$

which, by integration on $[a, b]$ over t, gives the last part of (2.7). □

Corollary 3. *With the assumption of Theorem 2 and if*

$$
x_S := \frac{\int\int_D x \frac{\partial f}{\partial x}(x, y) \, dxdy}{\int\int_D \frac{\partial f}{\partial x}(x, y) \, dxdy}, \ y_S := \frac{\int\int_D y \frac{\partial f}{\partial y}(x, y) \, dxdy}{\int\int_D \frac{\partial f}{\partial y}(x, y) \, dxdy} \tag{2.10}
$$

exist, then

$$
\begin{aligned}
&\left| \int\int_D f(x, y) \, dxdy \right| \\
&\leq \frac{1}{2} \int_a^b \left[|y_S - y(t)| \, |x'(t)| + |x(t) - x_S| \, |y'(t)|\right] |f(x(t), y(t))| \, dt \\
&\leq \frac{1}{2} \begin{cases} \int_a^b \max\{|y_S - y(t)|, |x(t) - x_S|\} \left[|x'(t)| + |y'(t)|\right] |f(x(t), y(t))| \, dt, \\ \\ \int_a^b \left[|y_S - y(t)|^p + |x(t) - x_S|^p\right]^{1/p} \left[|x'(t)|^q + |y'(t)|^q\right]^{1/q} |f(x(t), y(t))| \, dt \\ p, \ q > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \int_a^b \max\{|x'(t)|, |y'(t)|\} \left[|y_S - y(t)| + |x(t) - x_S|\right] |f(x(t), y(t))| \, dt. \end{cases}
\end{aligned} \tag{2.11}
$$

Proof. It follows from the inequality (2.7) by observing that

$$
\int\int_D \left[(x_S - x) \frac{\partial f(x, y)}{\partial x} + (y_S - y) \frac{\partial f(x, y)}{\partial y}\right] dxdy = 0.
$$

□

Corollary 4. *With the assumption of Theorem 2 and if*

$$
\|f\|_{\partial D, \infty} := \sup_{(x,y) \in \partial D} |f(x, y)| = \sup_{t \in [a,b]} |f(x(t), y(t))| < \infty,
$$

then

$$\begin{aligned}
&\left|\int\int_D f(x,y)\,dxdy \right. \\
&\qquad \left. -\frac{1}{2}\int\int_D\left[(\alpha-x)\frac{\partial f(x,y)}{\partial x}+(\beta-y)\frac{\partial f(x,y)}{\partial y}\right]dxdy\right| \\
&\qquad\leq\frac{1}{2}\|f\|_{\partial D,\infty}\int_a^b\left[|\beta-y(t)|\,|x'(t)|+|x(t)-\alpha|\,|y'(t)|\right]dt \\
&\leq\frac{1}{2}\|f\|_{\partial D,\infty}\begin{cases}\int_a^b\max\{|\beta-y(t)|,|x(t)-\alpha|\}\left[|x'(t)|+|y'(t)|\right]dt, \\ \\ \int_a^b\left[|\beta-y(t)|^p+|x(t)-\alpha|^p\right]^{1/p}\left[|x'(t)|^q+|y'(t)|^q\right]^{1/q}dt \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \int_a^b\max\{|x'(t)|,|y'(t)|\}\left[|\beta-y(t)|+|x(t)-\alpha|\right]dt.\end{cases}
\end{aligned}\tag{2.12}$$

In particular,

$$\begin{aligned}
&\left|\int\int_D f(x,y)\,dxdy \right. \\
&\qquad \left. -\frac{1}{2}\int\int_D\left[(\overline{x_D}-x)\frac{\partial f(x,y)}{\partial x}+(\overline{y_D}-y)\frac{\partial f(x,y)}{\partial y}\right]dxdy\right| \\
&\qquad\leq\frac{1}{2}\|f\|_{\partial D,\infty}\int_a^b\left[|\overline{y_D}-y(t)|\,|x'(t)|+|x(t)-\overline{x_D}|\,|y'(t)|\right]dt \\
&\leq\frac{1}{2}\|f\|_{\partial D,\infty}\begin{cases}\int_a^b\max\{|\overline{y_D}-y(t)|,|x(t)-\overline{x_D}|\}\left[|x'(t)|+|y'(t)|\right]dt, \\ \\ \int_a^b\left[|\overline{y_D}-y(t)|^p+|x(t)-\overline{x_D}|^p\right]^{1/p}\left[|x'(t)|^q+|y'(t)|^q\right]^{1/q}dt \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \int_a^b\max\{|x'(t)|,|y'(t)|\}\left[|\overline{y_D}-y(t)|+|x(t)-\overline{x_D}|\right]dt.\end{cases}
\end{aligned}\tag{2.13}$$

Also, we have

$$\begin{aligned}
&\left|\int\int_D f(x,y)\,dxdy\right| \\
&\qquad\leq\frac{1}{2}\|f\|_{\partial D,\infty}\int_a^b\left[|y_S-y(t)|\,|x'(t)|+|x(t)-x_S|\,|y'(t)|\right]dt \\
&\leq\frac{1}{2}\|f\|_{\partial D,\infty}\begin{cases}\int_a^b\max\{|y_S-y(t)|,|x(t)-x_S|\}\left[|x'(t)|+|y'(t)|\right]dt, \\ \\ \int_a^b\left[|y_S-y(t)|^p+|x(t)-x_S|^p\right]^{1/p}\left[|x'(t)|^q+|y'(t)|^q\right]^{1/q}dt \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \int_a^b\max\{|x'(t)|,|y'(t)|\}\left[|y_S-y(t)|+|x(t)-x_S|\right]dt.\end{cases}
\end{aligned}\tag{2.14}$$

Remark 1. *Using the upper bound from (2.7), namely*

$$U(f) := \frac{1}{2}\begin{cases} \int_a^b \max\{|\beta - y(t)|, |x(t)-\alpha|\}[|x'(t)|+|y'(t)|]\,|f(x(t),y(t))|\,dt, \\ \\ \int_a^b [|\beta - y(t)|^p + |x(t)-\alpha|^p]^{1/p}\left[|x'(t)|^q+|y'(t)|^q\right]^{1/q}|f(x(t),y(t))|\,dt \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \int_a^b \max\{|x'(t)|, |y'(t)|\}[|\beta - y(t)|+|x(t)-\alpha|]\,|f(x(t),y(t))|\,dt \end{cases}$$

and Hölder type inequalities we can provide other inequalities where the function f and the parametrization of the curve ∂D are separated.

One of the natural choices is for $p=q=2$, which gives, by applying the weighted Cauchy-Bunyakovsky-Schwarz inequality, that

$$\begin{aligned}
&\int_a^b \left[|\beta - y(t)|^2 + |x(t)-\alpha|^2\right]^{1/2}\left[|x'(t)|^2+|y'(t)|^2\right]^{1/2}|f(x(t),y(t))|\,dt \\
&\leq \left(\int_a^b \left(\left[|\beta - y(t)|^2 + |x(t)-\alpha|^2\right]^{1/2}\right)^2 \left[|x'(t)|^2+|y'(t)|^2\right]^{1/2} dt\right)^{1/2} \\
&\times \left(\int_a^b \left[|x'(t)|^2+|y'(t)|^2\right]^{1/2}|f(x(t),y(t))|^2\,dt\right)^{1/2} \\
&= \left(\int_a^b \left[|\beta - y(t)|^2 + |x(t)-\alpha|^2\right]\left[|x'(t)|^2+|y'(t)|^2\right]^{1/2} dt\right)^{1/2} dt \\
&\times \left(\int_a^b \left[|x'(t)|^2+|y'(t)|^2\right]^{1/2}|f(x(t),y(t))|^2\,dt\right)^{1/2} \\
&= \left(\int_{\partial D}\left[|\beta - y|^2+|x-\alpha|^2\right]d\ell\right)^{1/2}\left(\int_{\partial D}|f(x,y)|^2\,d\ell\right)^{1/2},
\end{aligned}$$

where the latest integrals are arc length integrals.

By utilising the inequality (2.7) we then get the following inequality in terms of the Euclidian norm

$$\begin{aligned}
(2.15)\quad &\left|\int\int_D f(x,y)\,dxdy\right. \\
&\left.-\frac{1}{2}\int\int_D\left[(\alpha - x)\frac{\partial f(x,y)}{\partial x}+(\beta - y)\frac{\partial f(x,y)}{\partial y}\right]dxdy\right| \\
&\leq \frac{1}{2}\left(\int_{\partial D}\left[|\beta - y|^2+|x-\alpha|^2\right]d\ell\right)^{1/2}\left(\int_{\partial D}|f(x,y)|^2\,d\ell\right)^{1/2}
\end{aligned}$$

for all $\alpha,\ \beta \in \mathbb{C}$.

In particular,

$$
\begin{aligned}
(2.16)\quad & \left|\int\int_D f(x,y)\,dxdy \right.\\
& \left. -\frac{1}{2}\int\int_D\left[\left(\overline{x_D}-x\right)\frac{\partial f(x,y)}{\partial x}+\left(\overline{y_D}-y\right)\frac{\partial f(x,y)}{\partial y}\right]dxdy\right|\\
& \leq\frac{1}{2}\left(\int_{\partial D}\left[\left|\overline{y_D}-y\right|^2+\left|x-\overline{x_D}\right|^2\right]d\ell\right)^{1/2}\left(\int_{\partial D}\left|f(x,y)\right|^2 d\ell\right)^{1/2}
\end{aligned}
$$

and

$$
\begin{aligned}
(2.17)\quad & \left|\int\int_D f(x,y)\,dxdy\right|\\
& \leq\frac{1}{2}\left(\int_{\partial D}\left[\left|y_S-y\right|^2+\left|x-x_S\right|^2\right]d\ell\right)^{1/2}\left(\int_{\partial D}\left|f(x,y)\right|^2 d\ell\right)^{1/2},
\end{aligned}
$$

where x_S, y_S are defined by (2.10).

We also have:

Theorem 3. *Let ∂D be a simple, closed counterclockwise curve bounding a region D and f defined on an open set containing D and having continuous partial derivatives on D. If the curve ∂D is described by the function $r(t)=(x(t),y(t))$, $t\in[a,b]$, with x, y differentiable on (a,b), then*

$$
\begin{aligned}
(2.18)\quad & \left|\int\int_D f(x,y)\,dxdy-\frac{1}{2}\oint_{\partial D}\left[(\beta-y)f(x,y)\,dx+(x-\alpha)f(x,y)\,dy\right]\right|\\
& \leq\frac{1}{2}\int\int_D\left[\left|\alpha-x\right|\left|\frac{\partial f(x,y)}{\partial x}\right|+\left|\beta-y\right|\left|\frac{\partial f(x,y)}{\partial y}\right|\right]dxdy\\
& \leq\frac{1}{2}\begin{cases}\int\int_D\max\left\{\left|\alpha-x\right|,\left|\beta-y\right|\right\}\left[\left|\frac{\partial f(x,y)}{\partial x}\right|+\left|\frac{\partial f(x,y)}{\partial y}\right|\right]dxdy,\\[2ex]
\int\int_D\left[\left|\alpha-x\right|^p+\left|\beta-y\right|^p\right]^{1/p}\left[\left|\frac{\partial f(x,y)}{\partial x}\right|^q+\left|\frac{\partial f(x,y)}{\partial y}\right|^q\right]^{1/q}dxdy\\
p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex]
\int\int_D\max\left\{\left|\frac{\partial f(x,y)}{\partial x}\right|,\left|\frac{\partial f(x,y)}{\partial y}\right|\right\}\left[\left|\alpha-x\right|+\left|\beta-y\right|\right]dxdy.\end{cases}
\end{aligned}
$$

In particular,

$$\left|\int\int_D f(x,y)\,dxdy-\frac{1}{2}\oint_{\partial D}\left[\left(\overline{y_D}-y\right)f(x,y)\,dx+\left(x-\overline{x_D}\right)f(x,y)\,dy\right]\right| \tag{2.19}$$
$$\leq\frac{1}{2}\int\int_D\left[\left|\overline{x_D}-x\right|\left|\frac{\partial f(x,y)}{\partial x}\right|+\left|\overline{y_D}-y\right|\left|\frac{\partial f(x,y)}{\partial y}\right|\right]dxdy$$
$$\leq\frac{1}{2}\begin{cases}\int\int_D\max\left\{\left|\overline{x_D}-x\right|,\left|\overline{y_D}-y\right|\right\}\left[\left|\frac{\partial f(x,y)}{\partial x}\right|+\left|\frac{\partial f(x,y)}{\partial y}\right|\right]dxdy,\\ \\ \int\int_D\left[\left|\overline{x_D}-x\right|^p+\left|\overline{y_D}-y\right|^p\right]^{1/p}\left[\left|\frac{\partial f(x,y)}{\partial x}\right|^q+\left|\frac{\partial f(x,y)}{\partial y}\right|^q\right]^{1/q}dxdy\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \\ \int\int_D\max\left\{\left|\frac{\partial f(x,y)}{\partial x}\right|,\left|\frac{\partial f(x,y)}{\partial y}\right|\right\}\left[\left|\overline{x_D}-x\right|+\left|\overline{y_D}-y\right|\right]dxdy.\end{cases}$$

Proof. From the identity (2.1) we have

$$\left|\int\int_D f(x,y)\,dxdy-\frac{1}{2}\oint_{\partial D}\left[(\beta-y)f(x,y)\,dx+(x-\alpha)f(x,y)\,dy\right]\right|$$
$$=\frac{1}{2}\left|\int\int_D\left[(\alpha-x)\frac{\partial f(x,y)}{\partial x}+(\beta-y)\frac{\partial f(x,y)}{\partial y}\right]dxdy\right|$$
$$\leq\int\int_D\left|(\alpha-x)\frac{\partial f(x,y)}{\partial x}+(\beta-y)\frac{\partial f(x,y)}{\partial y}\right|dxdy$$
$$\leq\int\int_D\left[|\alpha-x|\left|\frac{\partial f(x,y)}{\partial x}\right|+|\beta-y|\left|\frac{\partial f(x,y)}{\partial y}\right|\right]dxdy,$$

which proves the first inequality in (2.18).

Using Hölder's discrete inequality we have

$$|\alpha-x|\left|\frac{\partial f(x,y)}{\partial x}\right|+|\beta-y|\left|\frac{\partial f(x,y)}{\partial y}\right|$$
$$\leq\begin{cases}\max\left\{|\alpha-x|,|\beta-y|\right\}\left[\left|\frac{\partial f(x,y)}{\partial x}\right|+\left|\frac{\partial f(x,y)}{\partial y}\right|\right],\\ \\ \left[|\alpha-x|^p+|\beta-y|^p\right]^{1/p}\left[\left|\frac{\partial f(x,y)}{\partial x}\right|^q+\left|\frac{\partial f(x,y)}{\partial y}\right|^q\right]^{1/q}\\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \\ \max\left\{\left|\frac{\partial f(x,y)}{\partial x}\right|,\left|\frac{\partial f(x,y)}{\partial y}\right|\right\}\left[|\alpha-x|+|\beta-y|\right],\end{cases}$$

which, by integration on D, proves the last part of (2.18). $\square$

We define the quantities

$$x_{f,\partial D}:=\frac{\oint_{\partial D}xf(x,y)\,dy}{\oint_{\partial D}f(x,y)\,dy}=\frac{\int_a^b x(t)\,f(x(t),y(t))\,y'(t)\,dt}{\int_a^b f(x(t),y(t))\,y'(t)\,dt}$$

and

$$y_{f,\partial D} := \frac{\oint\limits_{\partial D} y f(x,y)\,dx}{\oint\limits_{\partial D} f(x,y)\,dx} = \frac{\int_a^b y(t)\, f(x(t), y(t))\, x'(t)\,dt}{\int_a^b f(x(t), y(t))\, x'(t)\,dt}$$

provided the denominators are not zero.

Corollary 5. *With the assumptions of Theorem 3 we have*

$$\begin{aligned}
(2.20)\qquad & \left|\int\int_D f(x,y)\,dxdy\right| \\
& \leq \frac{1}{2}\int\int_D \left[\left|x_{f,\partial D} - x\right|\left|\frac{\partial f(x,y)}{\partial x}\right| + \left|y_{f,\partial D} - y\right|\left|\frac{\partial f(x,y)}{\partial y}\right|\right] dxdy
\end{aligned}$$

$$\leq \frac{1}{2}\begin{cases} \int\int_D \max\left\{\left|x_{f,\partial D} - x\right|, \left|y_{f,\partial D} - y\right|\right\}\left[\left|\frac{\partial f(x,y)}{\partial x}\right| + \left|\frac{\partial f(x,y)}{\partial y}\right|\right] dxdy, \\ \\ \int\int_D \left[\left|x_{f,\partial D} - x\right|^p + \left|y_{f,\partial D} - y\right|^p\right]^{1/p}\left[\left|\frac{\partial f(x,y)}{\partial x}\right|^q + \left|\frac{\partial f(x,y)}{\partial y}\right|^q\right]^{1/q} dxdy \\ p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \int\int_D \max\left\{\left|\frac{\partial f(x,y)}{\partial x}\right|, \left|\frac{\partial f(x,y)}{\partial y}\right|\right\}\left[\left|x_{f,\partial D} - x\right| + \left|y_{f,\partial D} - y\right|\right] dxdy. \end{cases}$$

Proof. Observe that

$$\oint\limits_{\partial D} \left[\left(y_{f,\partial D} - y\right) f(x,y)\,dx + \left(x - x_{f,\partial D}\right) f(x,y)\,dy\right] = 0$$

and by (2.18) we get the desired result (2.20). □

Remark 2. *Using the upper bound from (2.18), namely*

$$V(f) := \frac{1}{2}\begin{cases} \int\int_D \max\left\{|\alpha - x|, |\beta - y|\right\}\left[\left|\frac{\partial f(x,y)}{\partial x}\right| + \left|\frac{\partial f(x,y)}{\partial y}\right|\right] dxdy, \\ \\ \int\int_D \left[|\alpha - x|^p + |\beta - y|^p\right]^{1/p}\left[\left|\frac{\partial f(x,y)}{\partial x}\right|^q + \left|\frac{\partial f(x,y)}{\partial y}\right|^q\right]^{1/q} dxdy \\ p, q > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \int\int_D \max\left\{\left|\frac{\partial f(x,y)}{\partial x}\right|, \left|\frac{\partial f(x,y)}{\partial y}\right|\right\}\left[|\alpha - x| + |\beta - y|\right] dxdy \end{cases}$$

and Hölder type inequalities we can provide other inequalities where the factors are separated.

For instance, we have

$$\begin{aligned}
&\int\int_D [|\alpha-x|^p+|\beta-y|^p]^{1/p}\left[\left|\frac{\partial f(x,y)}{\partial x}\right|^q+\left|\frac{\partial f(x,y)}{\partial y}\right|^q\right]^{1/q} dxdy\\
&\le \left[\int\int_D \left([|\alpha-x|^p+|\beta-y|^p]^{1/p}\right)^p dxdy\right]^{1/p}\\
&\times\left[\int\int_D \left(\left[\left|\frac{\partial f(x,y)}{\partial x}\right|^q+\left|\frac{\partial f(x,y)}{\partial y}\right|^q\right]^{1/q}\right)^q dxdy\right]^{1/q}\\
&=\left[\int\int_D [|\alpha-x|^p+|\beta-y|^p]\,dxdy\right]^{1/p}\\
&\times\left[\int\int_D \left[\left|\frac{\partial f(x,y)}{\partial x}\right|^q+\left|\frac{\partial f(x,y)}{\partial y}\right|^q\right] dxdy\right]^{1/q}
\end{aligned}$$

for $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and by (2.18) and (2.19) we get*

$$\begin{aligned}
&\left|\int\int_D f(x,y)\,dxdy-\frac{1}{2}\oint_{\partial D}[(\beta-y)\,f(x,y)\,dx+(x-\alpha)\,f(x,y)\,dy]\right|\\
&\qquad\le\frac{1}{2}\left[\int\int_D [|\alpha-x|^p+|\beta-y|^p]\,dxdy\right]^{1/p}\\
&\qquad\qquad\times\left[\int\int_D \left[\left|\frac{\partial f(x,y)}{\partial x}\right|^q+\left|\frac{\partial f(x,y)}{\partial y}\right|^q\right] dxdy\right]^{1/q}.
\end{aligned}\tag{2.21}$$

In particular,

$$\begin{aligned}
&\left|\int\int_D f(x,y)\,dxdy-\frac{1}{2}\oint_{\partial D}[(\overline{y_D}-y)\,f(x,y)\,dx+(x-\overline{x_D})\,f(x,y)\,dy]\right|\\
&\qquad\le\frac{1}{2}\left[\int\int_D [|\overline{x_D}-x|^p+|\overline{y_D}-y|^p]\,dxdy\right]^{1/p}\\
&\qquad\qquad\times\left[\int\int_D \left[\left|\frac{\partial f(x,y)}{\partial x}\right|^q+\left|\frac{\partial f(x,y)}{\partial y}\right|^q\right] dxdy\right]^{1/q}
\end{aligned}\tag{2.22}$$

and

$$\begin{aligned}
&\left|\int\int_D f(x,y)\,dxdy\right|\\
&\qquad\le\frac{1}{2}\left[\int\int_D [|x_{f,\partial D}-x|^p+|y_{f,\partial D}-y|^p]\,dxdy\right]^{1/p}\\
&\qquad\qquad\times\left[\int\int_D \left[\left|\frac{\partial f(x,y)}{\partial x}\right|^q+\left|\frac{\partial f(x,y)}{\partial y}\right|^q\right] dxdy\right]^{1/q}.
\end{aligned}\tag{2.23}$$

Corollary 6. *With the assumption of Theorem 3 and if*

$$\left\|\frac{\partial f}{\partial x}\right\|_{D,\infty}:=\sup_{(x,y)\in D}\left|\frac{\partial f(x,y)}{\partial x}\right|<\infty$$

and

$$\left\|\frac{\partial f}{\partial y}\right\|_{D,\infty}:=\sup_{(x,y)\in D}\left|\frac{\partial f(x,y)}{\partial y}\right|<\infty,$$

then

$$\begin{aligned}(2.24)\quad &\left|\int\int_D f(x,y)\,dxdy-\frac{1}{2}\oint_{\partial D}[(\beta-y)f(x,y)\,dx+(x-\alpha)f(x,y)\,dy]\right|\\ &\leq\frac{1}{2}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D,\infty}\int\int_D|\alpha-x|\,dxdy+\left\|\frac{\partial f}{\partial y}\right\|_{D,\infty}\int\int_D|\beta-y|\,dxdy\right],\end{aligned}$$

$$\begin{aligned}(2.25)\quad &\left|\int\int_D f(x,y)\,dxdy-\frac{1}{2}\oint_{\partial D}[(\overline{y_D}-y)f(x,y)\,dx+(x-\overline{x_D})f(x,y)\,dy]\right|\\ &\leq\frac{1}{2}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D,\infty}\int\int_D|\overline{x_D}-x|+\left\|\frac{\partial f}{\partial y}\right\|_{D,\infty}\int\int_D|\overline{y_D}-y|\,dxdy\right]\end{aligned}$$

and

$$\begin{aligned}(2.26)\quad &\left|\int\int_D f(x,y)\,dxdy\right|\\ &\leq\frac{1}{2}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D,\infty}\int\int_D|x_{f,\partial D}-x|\,dxdy+\left\|\frac{\partial f}{\partial y}\right\|_{D,\infty}\int\int_D|y_{f,\partial D}-y|\,dxdy\right].\end{aligned}$$

3. Examples for Rectangles

Let $a<b$ and $c<d$. Put $A=(a,c)\,,\ B=(b,c)\,,\ C=(b,d)\,,\ D=(a,d)\in\mathbb{R}^2$ the vertices of the rectangle $ABCD=[a,b]\times[c,d]$. Consider the counterclockwise segments

$$AB:\begin{cases}x=(1-t)\,a+tb\\ y=c\end{cases},\ t\in[0,1]$$

$$BC:\begin{cases}x=b\\ y=(1-t)\,c+td\end{cases},\ t\in[0,1]$$

$$CD:\begin{cases}x=(1-t)\,b+ta\\ y=d\end{cases},\ t\in[0,1]$$

and

$$DA:\begin{cases}x=a\\ y=(1-t)\,d+tc\end{cases},\ t\in[0,1]\,.$$

Therefore $\partial(ABCD)=AB\cup BC\cup CD\cup DA.$

If α, $\beta \in \mathbb{R}$, then

$$\oint_{AB} [(\beta - y) f(x, y) dx + (x - \alpha) f(x, y) dy]$$
$$= (b - a)(\beta - c) \int_0^1 f((1 - t) a + tb, c) dt = (\beta - c) \int_a^b f(x, c) dx,$$

$$\oint_{BC} [(\beta - y) f(x, y) dx + (x - \alpha) f(x, y) dy]$$
$$= (d - c)(b - \alpha) \int_0^1 f(b, (1 - t) c + td) dt = (b - \alpha) \int_c^d f(b, y) dy$$

$$\oint_{CD} [(\beta - y) f(x, y) dx + (x - \alpha) f(x, y) dy]$$
$$= (a - b)(\beta - d) \int_0^1 f((1 - t) b + ta, d) dt = (\beta - d) \int_b^a f(x, d) dx$$
$$= (d - \beta) \int_a^b f(x, d) dx$$

and

$$\oint_{DA} [(\beta - y) f(x, y) dx + (x - \alpha) f(x, y) dy]$$
$$= \int_0^1 (a - \alpha) f(a, (1 - t) d + tc)(c - d) dt = (a - \alpha) \int_d^c f(a, y) dy$$
$$= (\alpha - a) \int_c^d f(a, y) dy.$$

Therefore

$$\begin{aligned} &\oint_{ABCD} [(\beta - y) f(x, y) dx + (x - \alpha) f(x, y) dy] \\ &= (\beta - c) \int_a^b f(x, c) dx + (d - \beta) \int_a^b f(x, d) dx \\ &+ (b - \alpha) \int_c^d f(b, y) dy + (\alpha - a) \int_c^d f(a, y) dy \end{aligned}$$

for all α, $\beta \in \mathbb{R}$.

We also have $\overline{x_D} = \frac{a+b}{2}$ and $\overline{y_D} = \frac{c+d}{2}$, which imply that

$$\oint_{\partial(ABCD)} [(\overline{y_D} - y) f(x, y) dx + (x - \overline{x_D}) f(x, y) dy]$$
$$= (d - c) \int_a^b \left(\frac{f(x, c) + f(x, d)}{2} \right) dx + (b - a) \int_c^d \left(\frac{f(b, y) + f(a, y)}{2} \right) dy.$$

From the identities (2.1) and (2.2) we have for any $\alpha,\ \beta \in \mathbb{C}$ that

$$\int_a^b \int_c^d f(x,y)\, dxdy \tag{3.1}$$
$$= \frac{1}{2} \int_a^b \int_c^d \left[(\alpha - x) \frac{\partial f(x,y)}{\partial x} + (\beta - y) \frac{\partial f(x,y)}{\partial y} \right] dxdy$$
$$+ \int_a^b \left[\frac{(\beta - c) f(x,c) + (d - \beta) f(x,d)}{2} \right] dx$$
$$+ \int_c^d \left[\frac{(b - \alpha) f(b,y) + (\alpha - a) f(a,y)}{2} \right] dy$$

and, in particular,

$$\int_a^b \int_c^d f(x,y)\, dxdy \tag{3.2}$$
$$= \frac{1}{2} \int_a^b \int_c^d \left[\left(\frac{a+b}{2} - x \right) \frac{\partial f(x,y)}{\partial x} + \left(\frac{c+d}{2} - y \right) \frac{\partial f(x,y)}{\partial y} \right] dxdy$$
$$+ \frac{1}{2} \left[(d-c) \int_a^b \left(\frac{f(x,c) + f(x,d)}{2} \right) dx + (b-a) \int_c^d \left(\frac{f(b,y) + f(a,y)}{2} \right) dy \right].$$

From (3.1) we have for $\alpha \in [a,b]$ and $\beta \in [c,d]$ that

$$\left| \int_a^b \int_c^d f(x,y)\, dxdy - \int_a^b \left[\frac{(\beta - c) f(x,c) + (d - \beta) f(x,d)}{2} \right] dx \right. \tag{3.3}$$
$$\left. - \int_c^d \left[\frac{(b - \alpha) f(b,y) + (\alpha - a) f(a,y)}{2} \right] dy \right|$$
$$\leq \frac{1}{2} (d-c)(b-a)^2 \left[\frac{1}{4} + \left(\frac{\alpha - \frac{a+b}{2}}{b-a} \right)^2 \right] \left\| \frac{\partial f}{\partial x} \right\|_{[a,b]\times[c,d],\infty}$$
$$+ \frac{1}{2} (b-a)(d-c)^2 \left[\frac{1}{4} + \left(\frac{\beta - \frac{c+d}{2}}{d-c} \right)^2 \right] \left\| \frac{\partial f}{\partial y} \right\|_{[a,b]\times[c,d],\infty}$$

while from (3.2) we get

$$\left| \int_a^b \int_c^d f(x,y)\, dxdy \right. \tag{3.4}$$
$$\left. - \frac{1}{2} \left[(d-c) \int_a^b \left(\frac{f(x,c) + f(x,d)}{2} \right) dx + (b-a) \int_c^d \left(\frac{f(b,y) + f(a,y)}{2} \right) dy \right] \right|$$
$$\leq \frac{1}{8} (d-c)(b-a) \left[(b-a) \left\| \frac{\partial f}{\partial x} \right\|_{[a,b]\times[c,d],\infty} + (d-c) \left\| \frac{\partial f}{\partial y} \right\|_{[a,b]\times[c,d],\infty} \right],$$

or, equivalently

$$\left|\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)\,dxdy\right. \tag{3.5}$$
$$\left.-\frac{1}{2}\left[\frac{1}{b-a}\int_a^b\left(\frac{f(x,c)+f(x,d)}{2}\right)dx+\frac{1}{d-c}\int_c^d\left(\frac{f(b,y)+f(a,y)}{2}\right)dy\right]\right|$$
$$\leq\frac{1}{8}\left[(b-a)\left\|\frac{\partial f}{\partial x}\right\|_{[a,b]\times[c,d],\infty}+(d-c)\left\|\frac{\partial f}{\partial y}\right\|_{[a,b]\times[c,d],\infty}\right].$$

4. Examples for Disks

We consider the closed disk $D(C,R)$ centered in $C(a,b)$ and of radius $R>0$. This is parametrized by

$$\begin{cases} x=r\cos\theta+a \\ \\ y=r\sin\theta+b\end{cases},\ r\in[0,R],\ \theta\in[0,2\pi]$$

and the circle $\mathcal{C}(C,R)$ parametrized by

$$\begin{cases} x=R\cos\theta+a \\ \\ y=R\sin\theta+b\end{cases},\ \theta\in[0,2\pi].$$

Here $\overline{x_{D(C,R)}}=a$, $\overline{y_{D(C,R)}}=b$ and $A_{D(C,R)}=\pi R^2$.

Then

$$\frac{1}{A_D}\oint_{\partial D}\left[\left(\overline{y_D}-y\right)f(x,y)\,dx+\left(x-\overline{x_D}\right)f(x,y)\,dy\right]$$
$$=\frac{1}{A_D}\int_a^b\left[\left(\overline{y_D}-y(t)\right)x'(t)+\left(x(t)-\overline{x_D}\right)y'(t)\right]f(x(t),y(t))\,dt$$
$$=\frac{1}{\pi R^2}\int_0^{2\pi}\left[\sin^2\theta+\cos^2\theta\right]R^2 f(R\cos\theta+a,R\sin\theta+b)\,d\theta$$
$$=\frac{1}{\pi}\int_0^{2\pi}f(R\cos\theta+a,R\sin\theta+b)\,d\theta.$$

From (2.25) we then get

$$\left|\frac{1}{\pi R^2}\int\int_{D(C,R)}f(x,y)\,dxdy-\frac{1}{2\pi}\int_0^{2\pi}f(R\cos\theta+a,R\sin\theta+b)\,d\theta\right| \tag{4.1}$$
$$\leq\frac{2}{3\pi}R\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(C,R),\infty}+\left\|\frac{\partial f}{\partial y}\right\|_{D(C,R),\infty}\right].$$

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