OSTROWSKI TYPE INEQUALITIES FOR DOUBLE INTEGRAL ON GENERAL DOMAINS VIA GREEN'S IDENTITY

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Abstract. In this paper, by the use of the celebrated Green's identity for double integral, we establish some Ostrowski type inequalities for functions of two independent variables defined on closed and bounded convex subsets of the plane $\mathbb{R}^2$. Some examples for rectangles and disks are also provided.

1. Introduction

In paper [1], the authors obtained among others the following results concerning the difference between the double integral on the disk and the values in the center or the path integral on the circle:

Theorem 1. *If $f : D(C,R) \to \mathbb{R}$ has continuous partial derivatives on $D(C,R)$, the disk centered in the point $C = (a,b)$ with the radius $R > 0$, and*

$$\left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} := \sup_{(x,y)\in D(C,R)} \left| \frac{\partial f(x,y)}{\partial x} \right| < \infty,$$
$$\left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} := \sup_{(x,y)\in D(C,R)} \left| \frac{\partial f(x,y)}{\partial y} \right| < \infty;$$

then

$$\left| f(C) - \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\,dx\,dy \right| \le \frac{4}{3\pi} R \left[\left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} \right]. \tag{1.1}$$

The constant $\frac{4}{3\pi}$ is sharp.

We also have

$$\left| \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\,dx\,dy - \frac{1}{2\pi R} \int_{\sigma(C,R)} f(\gamma)\,dl(\gamma) \right| \le \frac{2R}{3\pi} \left[\left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} \right], \tag{1.2}$$

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where $\sigma(C,R)$ is the circle centered in $C=(a,b)$ with the radius $R>0$ and

$$\left|f(C)-\frac{1}{2\pi R}\int_{\sigma(C,R)}f(\gamma)\,dl(\gamma)\right| \leq \frac{2R}{\pi}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(C,R),\infty}+\left\|\frac{\partial f}{\partial y}\right\|_{D(C,R),\infty}\right]. \tag{1.3}$$

In the same paper [1] the authors also established the following Ostrowski type inequality:

Theorem 2. *If f has bounded partial derivatives on $D(0,1)$, then*

$$\begin{aligned}\left|f(u,v)-\frac{1}{\pi}\iint_{D(0,1)}f(x,y)\,dx\,dy\right| &\leq \frac{2}{\pi}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(0,1),\infty}\left(u\arcsin u+\frac{1}{3}\sqrt{1-u^2}\left(2+u^2\right)\right)\right.\\ &\quad\left.+\left\|\frac{\partial f}{\partial y}\right\|_{D(0,1),\infty}\left(v\arcsin v+\frac{1}{3}\sqrt{1-v^2}\left(2+v^2\right)\right)\right]\end{aligned} \tag{1.4}$$

for any $(u,v)\in D(0,1)$.

For other Ostrowski type integral inequalities for double integrals see [2]-[13].

Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a region D. Let L and M be scalar functions defined at least on an open set containing D. Assume L and M have continuous first partial derivatives. Then the following equality is well known as the Green theorem (see for instance https://en.wikipedia.org/wiki/Green%27s_theorem)

$$\int\int_D\left(\frac{\partial M(x,y)}{\partial x}-\frac{\partial L(x,y)}{\partial y}\right)dxdy=\oint_{\partial D}\left(L(x,y)\,dx+M(x,y)\,dy\right). \tag{G}$$

By applying this equality for real and imaginary parts, we can also state it for complex valued functions P and Q.

Moreover, if the curve ∂D is described by the function $r(t)=(x(t),y(t))$, $t\in[a,b]$, with x, y differentiable on (a,b) then we can calculate the path integral as

$$\oint_{\partial D}\left(L(x,y)\,dx+M(x,y)\,dy\right)=\int_a^b\left[L(x(t),y(t))\,x'(t)+M(x(t),y(t))\,y'(t)\right]dt.$$

In the following, consider D a closed and bounded convex subset of $\mathbb{R}^2$. Define

$$A_D:=\int\int_D dxdy$$

the *area* of D and $(\overline{x_D},\overline{y_D})$ the *centre of mass* for D, where

$$\overline{x_D}:=\frac{1}{A_D}\int\int_D xdxdy,\ \overline{y_D}:=\frac{1}{A_D}\int\int_D ydxdy.$$

Consider the function of two variables $f = f(x, y)$ and denote by $\frac{\partial f}{\partial x}$ the partial derivative with respect to the variable x and $\frac{\partial f}{\partial y}$ the partial derivative with respect to the variable y.

Motivated by the above results, by the use of Green's identity (G), in this paper we establish some bounds for the absolute value of the difference

$$\frac{1}{A_D}\iint_D f(x,y)\,dxdy - \lambda - \frac{1}{2A_D}\int\int_D \left[(\alpha - x)\frac{\partial f(x,y)}{\partial x} + (\beta - y)\frac{\partial f(x,y)}{\partial y}\right] dxdy$$

for λ, α and β complex numbers, and, in particular, the Ostrowski perturbed difference

$$\frac{1}{A_D}\iint_D f(x,y)\,dxdy - f(u,v) - \frac{1}{2A_D}\int\int_D \left[(u - x)\frac{\partial f(x,y)}{\partial x} + (v - y)\frac{\partial f(x,y)}{\partial y}\right] dxdy$$

and the centre of mass perturbed difference

$$\frac{1}{A_D}\iint_D f(x,y)\,dxdy - f(\overline{x_D},\overline{y_D}) - \frac{1}{2A_D}\int\int_D \left[(\overline{x_D} - x)\frac{\partial f(x,y)}{\partial x} + (\overline{y_D} - y)\frac{\partial f(x,y)}{\partial y}\right] dxdy$$

in the general case of closed and bounded subset of $\mathbb{R}^2$ and f is defined on an open set containing D and having continuous partial derivatives on D. Some examples for rectangles and disks are also provided.

2. Some Identities of Interest

We start with the following identity of interest:

Lemma 1. *Let ∂D be a simple, closed counterclockwise curve bounding a region D and f defined on an open set containing D and having continuous partial derivatives on D. Then for any α, $\beta \in \mathbb{C}$,*

$$\int\int_D f(x,y)\,dxdy = \frac{1}{2}\int\int_D \left[(\alpha - x)\frac{\partial f(x,y)}{\partial x} + (\beta - y)\frac{\partial f(x,y)}{\partial y}\right] dxdy + \frac{1}{2}\oint_{\partial D}\left[(\beta - y) f(x,y)\,dx + (x-\alpha) f(x,y)\,dy\right]. \tag{2.1}$$

In particular, we have

$$\int\int_D f(x,y)\,dxdy = \frac{1}{2}\int\int_D \left[(\overline{x_D} - x)\frac{\partial f(x,y)}{\partial x} + (\overline{y_D} - y)\frac{\partial f(x,y)}{\partial y}\right] dxdy + \frac{1}{2}\oint_{\partial D}\left[(\overline{y_D} - y) f(x,y)\,dx + (x-\overline{x_D}) f(x,y)\,dy\right]. \tag{2.2}$$

Proof. Observe that

$$\frac{\partial}{\partial x}\left((x-\alpha) f(x,y)\right) = f(x,y) + (x-\alpha)\frac{\partial f(x,y)}{\partial x}$$

and

$$\frac{\partial}{\partial y}\left((y-\beta) f(x,y)\right) = f(x,y) + (y-\beta)\frac{\partial f(x,y)}{\partial y}$$

for all $(x,y) \in D$ and is we add these equalities we get

$$\begin{aligned} &\frac{\partial}{\partial x}\left((x-\alpha) f(x,y)\right) + \frac{\partial}{\partial y}\left((y-\beta) f(x,y)\right) \\ &= 2f(x,y) + (x-\alpha)\frac{\partial f(x,y)}{\partial x} + (y-\beta)\frac{\partial f(x,y)}{\partial y}. \end{aligned} \tag{2.3}$$

Further, if we integrate on D the identity (2.3), then we obtain

$$\begin{aligned} &\int\int_D \left[\frac{\partial}{\partial x}\left((x-\alpha) f(x,y)\right) + \frac{\partial}{\partial y}\left((y-\beta) f(x,y)\right)\right] dxdy \\ &= 2\int\int_D f(x,y)\,dxdy \\ &+ \int\int_D \left[(x-\alpha)\frac{\partial f(x,y)}{\partial x} + (y-\beta)\frac{\partial f(x,y)}{\partial y}\right] dxdy. \end{aligned} \tag{2.4}$$

Now, if we apply Green's identity (G) for the functions $M(x,y) = (x-\alpha) f(x,y)$ and $L(x,y) = (\beta - y) f(x,y)$ then we get

$$\begin{aligned} &\int\int_D \left[\frac{\partial}{\partial x}\left((x-\alpha) f(x,y)\right) + \frac{\partial}{\partial y}\left((y-\beta) f(x,y)\right)\right] dxdy \\ &= \oint_{\partial D} \left[(\beta - y) f(x,y)\,dx + (x-\alpha) f(x,y)\,dy\right] \end{aligned}$$

and by (2.4) we obtain

$$\begin{aligned} &2\int\int_D f(x,y)\,dxdy + \int\int_D \left[(x-\alpha)\frac{\partial f(x,y)}{\partial x} + (y-\beta)\frac{\partial f(x,y)}{\partial y}\right] dxdy \\ &= \oint_{\partial D} \left[(\beta - y) f(x,y)\,dx + (x-\alpha) f(x,y)\,dy\right] \end{aligned}$$

which is equivalent to the desired equality (2.1). □

Corollary 1. *With the assumptions of Lemma 1 and if the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t \in [a,b]$, with x, y differentiable on (a,b), then*

$$\begin{aligned} \int\int_D f(x,y)\,dxdy &= \frac{1}{2}\int\int_D \left[(\alpha - x)\frac{\partial f(x,y)}{\partial x} + (\beta - y)\frac{\partial f(x,y)}{\partial y}\right] dxdy \\ &+ \frac{1}{2}\int_a^b \left[(\beta - y(t))\,x'(t) + (x(t) - \alpha)\,y'(t)\right] f(x(t), y(t))\,dt. \end{aligned} \tag{2.5}$$

In particular,

$$\int\int_D f(x,y)\,dxdy = \frac{1}{2}\int\int_D \left[(\overline{x_D}-x)\frac{\partial f(x,y)}{\partial x} + (\overline{y_D}-y)\frac{\partial f(x,y)}{\partial y}\right] dxdy + \frac{1}{2}\int_a^b [(\overline{y_D}-y(t))\,x'(t) + (x(t)-\overline{x_D})\,y'(t)]\,f(x(t),y(t))\,dt. \tag{2.6}$$

We also have:

Corollary 2. *With the assumptions of Corollary 1 we have for all $\lambda \in \mathbb{C}$ that*

$$\frac{1}{A_D}\int\int_D f(x,y)\,dxdy - \lambda = \frac{1}{2A_D}\int\int_D \left[(\alpha-x)\frac{\partial f(x,y)}{\partial x} + (\beta-y)\frac{\partial f(x,y)}{\partial y}\right] dxdy + \frac{1}{2A_D}\int_a^b [(\beta-y(t))\,x'(t) + (x(t)-\alpha)\,y'(t)]\,[f(x(t),y(t))-\lambda]\,dt. \tag{2.7}$$

In particular, for all $(u,v)\in D$ we have

$$\frac{1}{A_D}\int\int_D f(x,y)\,dxdy - f(u,v) = \frac{1}{2A_D}\int\int_D \left[(\alpha-x)\frac{\partial f(x,y)}{\partial x} + (\beta-y)\frac{\partial f(x,y)}{\partial y}\right] dxdy + \frac{1}{2A_D}\int_a^b [(\beta-y(t))\,x'(t) + (x(t)-\alpha)\,y'(t)]\,[f(x(t),y(t))-f(u,v)]\,dt \tag{2.8}$$

and for $\alpha = u$, $\beta = v$, that

$$\frac{1}{A_D}\int\int_D f(x,y)\,dxdy - f(u,v) = \frac{1}{2A_D}\int\int_D \left[(u-x)\frac{\partial f(x,y)}{\partial x} + (v-y)\frac{\partial f(x,y)}{\partial y}\right] dxdy + \frac{1}{2A_D}\int_a^b [(v-y(t))\,x'(t) + (x(t)-u)\,y'(t)]\,[f(x(t),y(t))-f(u,v)]\,dt \tag{2.9}$$

Remark 1. *If we take in (2.7) $\alpha = \overline{x_D}$ and $\beta = \overline{y_D}$ then we get*

$$\frac{1}{A_D}\int\int_D f(x,y)\,dxdy - \lambda = \frac{1}{2A_D}\int\int_D \left[(\overline{x_D}-x)\frac{\partial f(x,y)}{\partial x} + (\overline{y_D}-y)\frac{\partial f(x,y)}{\partial y}\right] dxdy + \frac{1}{2A_D}\int_a^b [(\overline{y_D}-y(t))\,x'(t) + (x(t)-\overline{x_D})\,y'(t)]\,[f(x(t),y(t))-\lambda]\,dt \tag{2.10}$$

for all $\lambda \in \mathbb{C}$, while from (2.8) we obtain

$$\begin{aligned}(2.11)\quad & \frac{1}{A_D}\int\int_D f(x,y)\,dxdy - f(u,v) \\ &= \frac{1}{2A_D}\int\int_D \left[(\overline{x_D}-x)\frac{\partial f(x,y)}{\partial x} + (\overline{y_D}-y)\frac{\partial f(x,y)}{\partial y}\right] dxdy \\ &+ \frac{1}{2A_D}\int_a^b \left[(\overline{y_D}-y(t))\,x'(t) + (x(t)-\overline{x_D})\,y'(t)\right]\left[f(x(t),y(t)) - f(u,v)\right] dt\end{aligned}$$

for all $(u,v) \in D$.

Moreover, if we assume that $(\overline{x_D},\overline{y_D}) \in D$ (which happens if, for instance, D is a convex subset in $\mathbb{R}^2$) then we get from (2.11) the centre of mass identity

$$\begin{aligned}(2.12)\quad & \frac{1}{A_D}\int\int_D f(x,y)\,dxdy - f(\overline{x_D},\overline{y_D}) \\ &= \frac{1}{2A_D}\int\int_D \left[(\overline{x_D}-x)\frac{\partial f(x,y)}{\partial x} + (\overline{y_D}-y)\frac{\partial f(x,y)}{\partial y}\right] dxdy \\ &+ \frac{1}{2A_D}\int_a^b \left[(\overline{y_D}-y(t))\,x'(t) + (x(t)-\overline{x_D})\,y'(t)\right]\left[f(x(t),y(t)) - f(\overline{x_D},\overline{y_D})\right] dt.\end{aligned}$$

Define

$$x_{S,\partial f} := \frac{\int\int_D x\frac{\partial f(x,y)}{\partial x}dxdy}{\int\int_D \frac{\partial f(x,y)}{\partial x}dxdy} \text{ and } y_{S,\partial f} := \frac{\int\int_D y\frac{\partial f(x,y)}{\partial y}dxdy}{\int\int_D \frac{\partial f(x,y)}{\partial y}dxdy}$$

provided the integrals from the denominators are not zero.

Corollary 3. *With the assumptions of Corollary 1 we have for all $\lambda \in \mathbb{C}$ that*

$$\begin{aligned}(2.13)\quad & \frac{1}{A_D}\int\int_D f(x,y)\,dxdy - \lambda \\ &= \frac{1}{2A_D}\int_a^b \left[(y_{S,\partial f}-y(t))\,x'(t) + (x(t)-x_{S,\partial f})\,y'(t)\right]\left[f(x(t),y(t)) - \lambda\right] dt.\end{aligned}$$

In particular, for all $(u,v) \in D$ we have

$$\begin{aligned}(2.14)\quad & \frac{1}{A_D}\int\int_D f(x,y)\,dxdy - f(u,v) \\ &= \frac{1}{2A_D}\int_a^b \left[(y_{S,\partial f}-y(t))\,x'(t) + (x(t)-x_{S,\partial f})\,y'(t)\right]\left[f(x(t),y(t)) - f(u,v)\right] dt\end{aligned}$$

and if $(x_{S,\partial f}, y_{S,\partial f}) \in D$, then

$$\begin{aligned}(2.15)\quad & \frac{1}{A_D}\int\int_D f(x,y)\,dxdy - f(x_{S,\partial f}, y_{S,\partial f}) \\ &= \frac{1}{2A_D}\int_a^b \left[(y_{S,\partial f}-y(t))\,x'(t) + (x(t)-x_{S,\partial f})\,y'(t)\right] \\ &\qquad \times \left[f(x(t),y(t)) - f(x_{S,\partial f}, y_{S,\partial f})\right] dt.\end{aligned}$$

The proof follows by Corollary 2 by observing that

$$\int\int_D \left[(x_{S,\partial f}-x)\frac{\partial f(x,y)}{\partial x} + (y_{S,\partial f}-y)\frac{\partial f(x,y)}{\partial y}\right] dxdy = 0.$$

We define the quantities

$$x_{f,\partial D} := \frac{\oint_{\partial D} x f(x,y)\,dy}{\oint_{\partial D} f(x,y)\,dy} = \frac{\int_a^b x(t) f(x(t), y(t))\, y'(t)\,dt}{\int_a^b f(x(t), y(t))\, y'(t)\,dt}$$

and

$$y_{f,\partial D} := \frac{\oint_{\partial D} y f(x,y)\,dx}{\oint_{\partial D} f(x,y)\,dx} = \frac{\int_a^b y(t) f(x(t), y(t))\, x'(t)\,dt}{\int_a^b f(x(t), y(t))\, x'(t)\,dt}$$

provided the denominators are not zero.

Corollary 4. *With the assumptions of Corollary 1 we have that*

$$\begin{aligned}(2.16)\quad & \frac{1}{A_D}\int\int_D f(x,y)\,dxdy \\ &= \frac{1}{2A_D}\int\int_D \left[(x_{f,\partial D} - x)\frac{\partial f(x,y)}{\partial x} + (y_{f,\partial D} - y)\frac{\partial f(x,y)}{\partial y}\right] dxdy.\end{aligned}$$

The proof follows by (2.1) observing that

$$\oint_{\partial D} \left[(y_{f,\partial D} - y) f(x,y)\,dx + (x - x_{f,\partial D}) f(x,y)\,dy \right] = 0.$$

3. Some General Inequalities

We have:

Theorem 3. *Let ∂D be a simple, closed counterclockwise curve bounding a region D and f defined on an open set containing D and having continuous partial derivatives on D. If the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t \in [a,b]$, with x, y differentiable on (a,b), then for any λ, α, $\beta \in \mathbb{C}$,*

$$\begin{aligned}(3.1)\quad & \left| \frac{1}{A_D}\int\int_D f(x,y)\,dxdy - \lambda \right. \\ & \left. - \frac{1}{2A_D}\int\int_D \left[(\alpha - x)\frac{\partial f(x,y)}{\partial x} + (\beta - y)\frac{\partial f(x,y)}{\partial y}\right] dxdy \right| \le B(\lambda, \alpha, \beta)\end{aligned}$$

where

$$\begin{aligned} & B(\lambda, \alpha, \beta) \\ & := \frac{1}{2A_D}\int_a^b \left[|\beta - y(t)|\,|x'(t)| + |x(t) - \alpha|\,|y'(t)| \right] |f(x(t), y(t)) - \lambda|\,dt.\end{aligned}$$

Moreover, we have the bounds

$$B(\lambda,\alpha,\beta) \leq \frac{1}{2A_D}\begin{cases} \int_a^b \max\{|\beta-y(t)|,|x(t)-\alpha|\}[|x'(t)|+|y'(t)|] \\ \times|f(x(t),y(t))-\lambda|\,dt \\ \\ \int_a^b (|\beta-y(t)|^p+|x(t)-\alpha|^p)^{1/p}\left[|x'(t)|^q+|y'(t)|^q\right]^{1/q} \\ \times|f(x(t),y(t))-\lambda|\,dt \text{ for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \int_a^b [|\beta-y(t)|+|x(t)-\alpha|]\max\{|x'(t)|,|y'(t)|\} \\ \times|f(x(t),y(t))-\lambda|\,dt. \end{cases} \tag{3.2}$$

Proof. Using the identity (2.7), we get

$$\begin{aligned} &\left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy-\lambda\right. \\ &\qquad\left.-\frac{1}{2A_D}\int\int_D\left[(\alpha-x)\frac{\partial f(x,y)}{\partial x}+(\beta-y)\frac{\partial f(x,y)}{\partial y}\right]dxdy\right| \\ &=\frac{1}{2A_D}\left|\int_a^b[(\beta-y(t))x'(t)+(x(t)-\alpha)y'(t)][f(x(t),y(t))-\lambda]\,dt\right| \\ &\leq\frac{1}{2A_D}\int_a^b|[(\beta-y(t))x'(t)+(x(t)-\alpha)y'(t)][f(x(t),y(t))-\lambda]|\,dt \\ &=\frac{1}{2A_D}\int_a^b|(\beta-y(t))x'(t)+(x(t)-\alpha)y'(t)||f(x(t),y(t))-\lambda|\,dt \\ &\leq\frac{1}{2A_D}\int_a^b[|\beta-y(t)||x'(t)|+|x(t)-\alpha||y'(t)|]|f(x(t),y(t))-\lambda|\,dt \\ &\qquad=B(\lambda,\alpha,\beta), \end{aligned} \tag{3.3}$$

which proves the first inequality in (3.1).

By utilising Hölder's discrete inequality we have

$$\begin{aligned} &[|\beta-y(t)||x'(t)|+|x(t)-\alpha||y'(t)|] \\ &\leq\begin{cases} \max\{|\beta-y(t)|,|x(t)-\alpha|\}[|x'(t)|+|y'(t)|] \\ \\ (|\beta-y(t)|^p+|x(t)-\alpha|^p)^{1/p}\left[|x'(t)|^q+|y'(t)|^q\right]^{1/q} \\ \text{for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ [|\beta-y(t)|+|x(t)-\alpha|]\max\{|x'(t)|,|y'(t)|\} \end{cases} \end{aligned}$$

for $t\in[a,b]$.

By multiplying this inequality with $|f(x(t),y(t))-\lambda|$ and integrating over t in $[a,b]$ we get the estimate (3.2). □

Remark 2. *If we take in Theorem 3* $\alpha = \overline{x_D}$ *and* $\beta = \overline{y_D}$, *then we have the inequalities*

$$\left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy - \lambda - \frac{1}{2A_D}\int\int_D\left[(\overline{x_D}-x)\frac{\partial f(x,y)}{\partial x} + (\overline{y_D}-y)\frac{\partial f(x,y)}{\partial y}\right]dxdy\right| \leq B(\lambda,\overline{x_D},\overline{y_D}) \tag{3.4}$$

where

$$B(\lambda,\overline{x_D},\overline{y_D}) = \frac{1}{2A_D}\int_a^b \left[\left|\overline{y_D}-y(t)\right|\left|x'(t)\right| + \left|x(t)-\overline{x_D}\right|\left|y'(t)\right|\right]\left|f(x(t),y(t))-\lambda\right|dt.$$

Moreover, we have the bounds

$$B(\lambda,\overline{x_D},\overline{y_D}) \leq \frac{1}{2A_D}\begin{cases} \int_a^b \max\left\{\left|\overline{y_D}-y(t)\right|,\left|x(t)-\overline{x_D}\right|\right\}\left[\left|x'(t)\right|+\left|y'(t)\right|\right]\left|f(x(t),y(t))-\lambda\right|dt \\ \int_a^b \left(\left|\overline{y_D}-y(t)\right|^p + \left|x(t)-\overline{x_D}\right|^p\right)^{1/p}\left[\left|x'(t)\right|^q+\left|y'(t)\right|^q\right]^{1/q} \\ \times\left|f(x(t),y(t))-\lambda\right|dt \text{ for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \int_a^b \left[\left|\overline{y_D}-y(t)\right|+\left|x(t)-\overline{x_D}\right|\right]\max\left\{\left|x'(t)\right|,\left|y'(t)\right|\right\}\left|f(x(t),y(t))-\lambda\right|dt. \end{cases} \tag{3.5}$$

Also, if we take in Theorem 3 $\alpha = x_{S,\partial f}$ *and* $\beta = y_{S,\partial f}$, *then we have the inequalities*

$$\left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy - \lambda\right| \leq B(\lambda, x_{S,\partial f}, y_{S,\partial f}) \tag{3.6}$$

where

$$B(\lambda, x_{S,\partial f}, y_{S,\partial f}) = \frac{1}{2A_D}\int_a^b \left[\left|y_{S,\partial f}-y(t)\right|\left|x'(t)\right| + \left|x(t)-x_{S,\partial f}\right|\left|y'(t)\right|\right]\left|f(x(t),y(t))-\lambda\right|dt.$$

Moreover, we have the bounds

$$B(\lambda, x_{S,\partial f}, y_{S,\partial f}) \leq \frac{1}{2A_D}\begin{cases} \int_a^b \max\left\{\left|y_{S,\partial f}-y(t)\right|,\left|x(t)-x_{S,\partial f}\right|\right\}\left[\left|x'(t)\right|+\left|y'(t)\right|\right] \\ \times\left|f(x(t),y(t))-\lambda\right|dt \\ \int_a^b \left(\left|y_{S,\partial f}-y(t)\right|^p + \left|x(t)-x_{S,\partial f}\right|^p\right)^{1/p}\left[\left|x'(t)\right|^q+\left|y'(t)\right|^q\right]^{1/q} \\ \times\left|f(x(t),y(t))-\lambda\right|dt \text{ for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \int_a^b \left[\left|y_{S,\partial f}-y(t)\right|+\left|x(t)-x_{S,\partial f}\right|\right]\max\left\{\left|x'(t)\right|,\left|y'(t)\right|\right\} \\ \times\left|f(x(t),y(t))-\lambda\right|dt. \end{cases} \tag{3.7}$$

By utilising Hölder's type integral inequalities one can separate the factors in the bounds above. One of the most natural such inequality is incorporated in the following corollary in which the bounds are in terms of the well known arc length integrals.

Corollary 5. *With the assumptions of Theorem 3 we have*

$$\begin{aligned}(3.8)\quad & \left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy-\lambda\right.\\ &\left.-\frac{1}{2A_D}\int\int_D\left[(\alpha-x)\frac{\partial f(x,y)}{\partial x}+(\beta-y)\frac{\partial f(x,y)}{\partial y}\right]dxdy\right|\\ &\leq\left(\int_{\partial D}\left(|x-\alpha|^2+|y-\beta|^2\right)d(\ell)\right)^{1/2}\left(\int_{\partial D}|f(x,y)-\lambda|^2\,d(\ell)\right)^{1/2}\end{aligned}$$

where the latest two integrals are arc length integrals.

In particular,

$$\begin{aligned}(3.9)\quad & \left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy-\lambda\right.\\ &\left.-\frac{1}{2A_D}\int\int_D\left[(\overline{x_D}-x)\frac{\partial f(x,y)}{\partial x}+(\overline{y_D}-y)\frac{\partial f(x,y)}{\partial y}\right]dxdy\right|\\ &\leq\left(\int_{\partial D}\left(|x-\overline{x_D}|^2+|y-\overline{y_D}|^2\right)d(\ell)\right)^{1/2}\left(\int_{\partial D}|f(x,y)-\lambda|^2\,d(\ell)\right)^{1/2}\end{aligned}$$

and

$$\begin{aligned}(3.10)\quad & \left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy-\lambda\right|\\ &\leq\left(\int_{\partial D}\left(|x-x_{S,\partial f}|^2+|y-y_{S,\partial f}|^2\right)d(\ell)\right)^{1/2}\left(\int_{\partial D}|f(x,y)-\lambda|^2\,d(\ell)\right)^{1/2}.\end{aligned}$$

Proof. For $p=q=2$ we have from (3.2) that

$$\begin{aligned}B(\lambda,\alpha,\beta)\leq{}&\frac{1}{2A_D}\int_a^b\left(|\beta-y(t)|^2+|x(t)-\alpha|^2\right)^{1/2}\left[|x'(t)|^2+|y'(t)|^2\right]^{1/2}\\ &\times|f(x(t),y(t))-\lambda|\,dt.\end{aligned}$$

By making use of the weighted Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\begin{aligned}
&\int_a^b \left(|\beta - y(t)|^2 + |x(t) - \alpha|^2\right)^{1/2} \left[|x'(t)|^2 + |y'(t)|^2\right]^{1/2} |f(x(t), y(t)) - \lambda|\, dt \\
&\le \left(\int_a^b \left[\left(|\beta - y(t)|^2 + |x(t) - \alpha|^2\right)^{1/2}\right]^2 \left[|x'(t)|^2 + |y'(t)|^2\right]^{1/2} dt\right)^{1/2} \\
&\times \left(\int_a^b |f(x(t), y(t)) - \lambda|^2 \left[|x'(t)|^2 + |y'(t)|^2\right]^{1/2} dt\right)^{1/2} \\
&= \left(\int_a^b \left(|\beta - y(t)|^2 + |x(t) - \alpha|^2\right) \left[|x'(t)|^2 + |y'(t)|^2\right]^{1/2} dt\right)^{1/2} \\
&\times \left(\int_a^b |f(x(t), y(t)) - \lambda|^2 \left[|x'(t)|^2 + |y'(t)|^2\right]^{1/2} dt\right)^{1/2} \\
&= \left(\int_{\partial D} \left(|x - \alpha|^2 + |y - \beta|^2\right) d(\ell)\right)^{1/2} \left(\int_{\partial D} |f(x, y) - \lambda|^2\, d(\ell)\right)^{1/2},
\end{aligned}$$

which proves the desired result (3.8). □

Remark 3. *If we take $\lambda = f(u, v)$ with $(u, v) \in D$, then we get*

$$\begin{aligned}
(3.11)\quad &\left|\frac{1}{A_D}\int\int_D f(x, y)\, dxdy - f(u, v)\right. \\
&\left. -\frac{1}{2A_D}\int\int_D \left[(u - x)\frac{\partial f(x, y)}{\partial x} + (v - y)\frac{\partial f(x, y)}{\partial y}\right] dxdy\right| \\
&\le \left(\int_{\partial D} \left(|x - u|^2 + |y - v|^2\right) d(\ell)\right)^{1/2} \left(\int_{\partial D} |f(x, y) - f(u, v)|^2\, d(\ell)\right)^{1/2}
\end{aligned}$$

where the latest two integrals are arc length integrals.

Corollary 6. *In particular,*

$$\begin{aligned}
(3.12)\quad &\left|\frac{1}{A_D}\int\int_D f(x, y)\, dxdy - f(u, v)\right. \\
&\left. -\frac{1}{2A_D}\int\int_D \left[(\overline{x_D} - x)\frac{\partial f(x, y)}{\partial x} + (\overline{y_D} - y)\frac{\partial f(x, y)}{\partial y}\right] dxdy\right| \\
&\le \left(\int_{\partial D} \left(|x - \overline{x_D}|^2 + |y - \overline{y_D}|^2\right) d(\ell)\right)^{1/2} \left(\int_{\partial D} |f(x, y) - f(u, v)|^2\, d(\ell)\right)^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
(3.13)\quad &\left|\frac{1}{A_D}\int\int_D f(x, y)\, dxdy - f(u, v)\right| \\
&\le \left(\int_{\partial D} \left(|x - x_{S,\partial f}|^2 + |y - y_{S,\partial f}|^2\right) d(\ell)\right)^{1/2} \left(\int_{\partial D} |f(x, y) - f(u, v)|^2\, d(\ell)\right)^{1/2}.
\end{aligned}$$

4. Inequalities for Bounded Functions

Let ∂D be a simple, closed counterclockwise curve bounding a region D. Now, for ϕ, $\Phi \in \mathbb{C}$, define the sets of complex-valued functions

$$\begin{aligned}&\bar{U}_{\partial D}(\phi, \Phi)\\ &:= \left\{ f : \partial D \to \mathbb{C} | \operatorname{Re}\left[(\Phi - f(x,y))\left(\overline{f(x,y)} - \overline{\phi}\right)\right] \geq 0 \text{ for each } (x,y) \in \partial D \right\}\end{aligned}$$

and

$$\bar{\Delta}_{\partial D}(\phi, \Phi) := \left\{ f : \partial D \to \mathbb{C} | \left| f(x,y) - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi| \text{ for each } (x,y) \in \partial D \right\}.$$

The following representation result may be stated.

Proposition 1. *For any ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that $\bar{U}_{\partial D}(\phi, \Phi)$ and $\bar{\Delta}_{\partial D}(\phi, \Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_{\partial D}(\phi, \Phi) = \bar{\Delta}_{\partial D}(\phi, \Phi). \tag{4.1}$$

Proof. We observe that for any $w \in \mathbb{C}$ we have the equivalence

$$\left| w - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi|$$

if and only if

$$\operatorname{Re}\left[(\Phi - w)\left(\overline{w} - \overline{\phi}\right)\right] \geq 0.$$

This follows by the equality

$$\frac{1}{4} |\Phi - \phi|^2 - \left| w - \frac{\phi + \Phi}{2} \right|^2 = \operatorname{Re}\left[(\Phi - w)\left(\overline{w} - \overline{\phi}\right)\right]$$

that holds for any $w \in \mathbb{C}$.

The equality (3.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 7. *For any ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that*

$$\begin{aligned}\bar{U}_{\partial D}(\phi, \Phi) = \{ f : \partial D \to \mathbb{C} \mid\ & (\operatorname{Re}\Phi - \operatorname{Re} f(x,y))(\operatorname{Re} f(x,y) - \operatorname{Re}\phi) \\ & + (\operatorname{Im}\Phi - \operatorname{Im} f(x,y))(\operatorname{Im} f(x,y) - \operatorname{Im}\phi) \geq 0 \textit{ for each } (x,y) \in \partial D \}.\end{aligned} \tag{4.2}$$

Now, if we assume that $\operatorname{Re}(\Phi) \geq \operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi) \geq \operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_{\partial D}(\phi, \Phi) := \{ f : \partial D \to \mathbb{C} \mid\ & \operatorname{Re}(\Phi) \geq \operatorname{Re} f(x,y) \geq \operatorname{Re}(\phi) \\ & \text{and } \operatorname{Im}(\Phi) \geq \operatorname{Im} f(x,y) \geq \operatorname{Im}(\phi) \text{ for each } (x,y) \in \partial D \}.\end{aligned} \tag{4.3}$$

One can easily observe that $\bar{S}_{\partial D}(\phi, \Phi)$ is closed, convex and

$$\emptyset \neq \bar{S}_{\partial D}(\phi, \Phi) \subseteq \bar{U}_{\partial D}(\phi, \Phi). \tag{4.4}$$

We have

Theorem 4. *Let ∂D be a simple, closed counterclockwise curve bounding a region D and f defined on an open set containing D and having continuous partial derivatives on D. If the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t \in [a,b]$, with x, y differentiable on (a,b), $f \in \bar{\Delta}_{\partial D}(\phi, \Phi)$ for some distinct ϕ, $\Phi \in \mathbb{C}$ then for any α, $\beta \in \mathbb{C}$, we have*

$$\begin{aligned}(4.5)\quad & \left| \frac{1}{A_D} \int\int_D f(x,y)\,dxdy - \frac{\phi+\Phi}{2} \right. \\ & \left. -\frac{1}{2A_D} \int\int_D \left[(\alpha - x) \frac{\partial f(x,y)}{\partial x} + (\beta - y) \frac{\partial f(x,y)}{\partial y} \right] dxdy \right| \\ & \leq B\left(\frac{\phi+\Phi}{2}, \alpha, \beta \right),\end{aligned}$$

where

$$\begin{aligned}& B\left(\frac{\phi+\Phi}{2}, \alpha, \beta \right) \\ & := \frac{1}{4} |\Phi - \phi| \frac{1}{A_D} \int_a^b \left[|\beta - y(t)| |x'(t)| + |x(t) - \alpha| |y'(t)| \right] dt.\end{aligned}$$

Moreover, we have the bounds

$$\begin{aligned}(4.6)\quad & B\left(\frac{\phi+\Phi}{2}, \alpha, \beta \right) \\ & \leq \frac{1}{4} |\Phi - \phi| \frac{1}{A_D} \begin{cases} \int_a^b \max\{|\beta - y(t)|, |x(t) - \alpha|\} [|x'(t)| + |y'(t)|]\,dt \\ \\ \int_a^b \left(|\beta - y(t)|^p + |x(t) - \alpha|^p \right)^{1/p} \left[|x'(t)|^q + |y'(t)|^q \right]^{1/q} dt \\ \text{for } p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \int_a^b [|\beta - y(t)| + |x(t) - \alpha|] \max\{|x'(t)|, |y'(t)|\}\,dt. \end{cases}\end{aligned}$$

The proof follows by Theorem 3 for $\lambda = \frac{\phi+\Phi}{2}$ and taking into account that, if $f \in \bar{\Delta}_{\partial D}(\phi, \Phi)$, then

$$\left| f(x(t), y(t)) - \frac{\phi+\Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi|$$

for all $(x(t), y(t)) \in \partial D$ and $t \in [a,b]$.

If we take in Theorem 4 $\alpha = \overline{x_D}$ and $\beta = \overline{y_D}$, then we have the inequalities

$$\begin{aligned}(4.7)\quad & \left| \frac{1}{A_D} \int\int_D f(x,y)\,dxdy - \frac{\phi+\Phi}{2} \right. \\ & \left. -\frac{1}{2A_D} \int\int_D \left[(\overline{x_D} - x) \frac{\partial f(x,y)}{\partial x} + (\overline{y_D} - y) \frac{\partial f(x,y)}{\partial y} \right] dxdy \right| \\ & \leq B\left(\frac{\phi+\Phi}{2}, \overline{x_D}, \overline{y_D} \right),\end{aligned}$$

where

$$B\left(\frac{\phi+\Phi}{2},\overline{x_D},\overline{y_D}\right) = \frac{1}{4}\left|\Phi-\phi\right|\frac{1}{A_D}\int_a^b\left[\left|\overline{y_D}-y\left(t\right)\right|\left|x'\left(t\right)\right|+\left|x\left(t\right)-\overline{x_D}\right|\left|y'\left(t\right)\right|\right]dt.$$

Moreover, we have the bounds

$$B\left(\frac{\phi+\Phi}{2},\overline{x_D},\overline{y_D}\right) \leq \frac{1}{4}\left|\Phi-\phi\right|\frac{1}{A_D}\begin{cases}\int_a^b\max\left\{\left|\overline{y_D}-y\left(t\right)\right|,\left|x\left(t\right)-\overline{x_D}\right|\right\}\left[\left|x'\left(t\right)\right|+\left|y'\left(t\right)\right|\right]dt\\ \\ \int_a^b\left(\left|\overline{y_D}-y\left(t\right)\right|^p+\left|x\left(t\right)-\overline{x_D}\right|^p\right)^{1/p}\left[\left|x'\left(t\right)\right|^q+\left|y'\left(t\right)\right|^q\right]^{1/q}dt\\ \text{for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,\\ \\ \int_a^b\left[\left|\overline{y_D}-y\left(t\right)\right|+\left|x\left(t\right)-\overline{x_D}\right|\right]\max\left\{\left|x'\left(t\right)\right|,\left|y'\left(t\right)\right|\right\}dt.\end{cases} \tag{4.8}$$

Also, if we take in Theorem 4 $\alpha=x_{S,\partial f}$ and $\beta=y_{S,\partial f}$, then we have the inequalities

$$\left|\frac{1}{A_D}\int\int_D f\left(x,y\right)dxdy-\frac{\phi+\Phi}{2}\right|\leq B\left(\lambda,x_{S,\partial f},y_{S,\partial f}\right) \tag{4.9}$$

where

$$B\left(\frac{\phi+\Phi}{2},x_{S,\partial f},y_{S,\partial f}\right) = \frac{1}{4}\left|\Phi-\phi\right|\frac{1}{A_D}\int_a^b\left[\left|y_{S,\partial f}-y\left(t\right)\right|\left|x'\left(t\right)\right|+\left|x\left(t\right)-x_{S,\partial f}\right|\left|y'\left(t\right)\right|\right]dt.$$

Moreover, we have the bounds

$$B\left(\frac{\phi+\Phi}{2},x_{S,\partial f},y_{S,\partial f}\right) \leq \frac{1}{4}\left|\Phi-\phi\right|\frac{1}{A_D}\begin{cases}\int_a^b\max\left\{\left|y_{S,\partial f}-y\left(t\right)\right|,\left|x\left(t\right)-x_{S,\partial f}\right|\right\}\left[\left|x'\left(t\right)\right|+\left|y'\left(t\right)\right|\right]dt\\ \\ \int_a^b\left(\left|y_{S,\partial f}-y\left(t\right)\right|^p+\left|x\left(t\right)-x_{S,\partial f}\right|^p\right)^{1/p}\\ \times\left[\left|x'\left(t\right)\right|^q+\left|y'\left(t\right)\right|^q\right]^{1/q}dt\\ \text{for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,\\ \\ \int_a^b\left[\left|y_{S,\partial f}-y\left(t\right)\right|+\left|x\left(t\right)-x_{S,\partial f}\right|\right]\max\left\{\left|x'\left(t\right)\right|,\left|y'\left(t\right)\right|\right\}dt.\end{cases} \tag{4.10}$$

For $p = q = 2$ we get

$$\begin{aligned}(4.11)\quad & \left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy - \frac{\phi+\Phi}{2}\right. \\ & \left. -\frac{1}{2A_D}\int\int_D\left[(\alpha - x)\frac{\partial f(x,y)}{\partial x} + (\beta - y)\frac{\partial f(x,y)}{\partial y}\right]dxdy\right| \\ & \leq \frac{1}{4}|\Phi-\phi|\frac{1}{A_D}\int_{\partial D}\left(|\beta-y|^2+|x-\alpha|^2\right)^{1/2}d(\ell) \\ & \leq \frac{1}{4}|\Phi-\phi|\frac{\ell(\partial D)}{A_D}\max_{(x,y)\in\partial D}\left(|\beta-y|^2+|x-\alpha|^2\right)^{1/2}\end{aligned}$$

for all $\alpha,\ \beta\in\mathbb{C}$.

In particular, we have

$$\begin{aligned}(4.12)\quad & \left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy - \frac{\phi+\Phi}{2}\right. \\ & \left. -\frac{1}{2A_D}\int\int_D\left[(\overline{x_D} - x)\frac{\partial f(x,y)}{\partial x} + (\overline{y_D} - y)\frac{\partial f(x,y)}{\partial y}\right]dxdy\right| \\ & \leq \frac{1}{4}|\Phi-\phi|\frac{1}{A_D}\int_{\partial D}\left(|x-\overline{x_D}|^2+|\overline{y_D}-y|^2\right)^{1/2}d(\ell) \\ & \leq \frac{1}{4}|\Phi-\phi|\frac{\ell(\partial D)}{A_D}\max_{(x,y)\in\partial D}\left(|x-\overline{x_D}|^2+|\overline{y_D}-y|^2\right)^{1/2}\end{aligned}$$

and

$$\begin{aligned}(4.13)\quad & \left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy - \frac{\phi+\Phi}{2}\right| \\ & \leq \frac{1}{4}|\Phi-\phi|\frac{1}{A_D}\int_{\partial D}\left(|y_{S,\partial f}-y|^2+|x-x_{S,\partial f}|^2\right)^{1/2}d(\ell) \\ & \leq \frac{1}{4}|\Phi-\phi|\frac{\ell(\partial D)}{A_D}\max_{(x,y)\in\partial D}\left(|y_{S,\partial f}-y|^2+|x-x_{S,\partial f}|^2\right)^{1/2}.\end{aligned}$$

5. Inequalities for Lipschitzian Functions

Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a region D. Let (u,v) be fixed in D and assume that there exists $L_u,\ K_v > 0$ such that

$$(5.1)\qquad |f(x,y) - f(u,v)| \leq L_u|x-u| + K_v|y-v|$$

for all $(x,y)\in\partial D$.

If $f : D\to\mathbb{C}$ is Lipschitzian in the usual sense, namely there exists $L,\ K > 0$ such that

$$(5.2)\qquad |f(x,y) - f(u,v)| \leq L|x-u| + K|y-v|$$

for all $(x,y),(u,v)\in D$, then for each fixed (u,v) in D we have the condition (5.1) for $L_u = L$ and $K_v = K$.

Also, if f has bounded partial derivatives on D then we can take in (5.2)

$$L = \left\|\frac{\partial f}{\partial x}\right\|_{D,\infty} := \sup_{(x,y)\in D}\left|\frac{\partial f}{\partial x}(x,y)\right| < \infty$$

and

$$K=\left\|\frac{\partial f}{\partial y}\right\|_{D,\infty}:=\sup_{(x,y)\in D}\left|\frac{\partial f}{\partial y}(x,y)\right|<\infty.$$

We have

Theorem 5. *Let ∂D be a simple, closed counterclockwise curve bounding a region D and f defined on an open set containing D and having continuous partial derivatives on D. If the curve ∂D is described by the function $r(t)=(x(t),y(t))$, $t\in[a,b]$, with x, y differentiable on (a,b), $(u,v)\in D$ and the function f satisfies the condition (5.1), then we have the following perturbed Ostrowsky type inequality*

$$\begin{aligned}(5.3)\quad &\left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy-f(u,v)\right.\\ &\left.-\frac{1}{2A_D}\int\int_D\left[(u-x)\frac{\partial f(x,y)}{\partial x}+(v-y)\frac{\partial f(x,y)}{\partial y}\right]dxdy\right|\\ &\leq B(L_u,K_v,u,v),\end{aligned}$$

where

$$\begin{aligned}(5.4)\quad &B(L_u,K_v,u,v)\\ &:=\frac{1}{2A_D}L_u\int_a^b\left[|x(t)-u|\left(|v-y(t)|\left|x'(t)\right|+|x(t)-u|\left|y'(t)\right|\right)\right]dt\\ &+\frac{1}{2A_D}K_v\int_a^b\left[|y(t)-v|\left(|v-y(t)|\left|x'(t)\right|+|x(t)-u|\left|y'(t)\right|\right)\right]dt.\end{aligned}$$

Proof. From the identity (2.9) we get

$$\begin{aligned}&\left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy-f(u,v)\right.\\ &\left.-\frac{1}{2A_D}\int\int_D\left[(u-x)\frac{\partial f(x,y)}{\partial x}+(v-y)\frac{\partial f(x,y)}{\partial y}\right]dxdy\right|\\ &=\frac{1}{2A_D}\left|\int_a^b\left[(v-y(t))x'(t)+(x(t)-u)y'(t)\right]\left[f(x(t),y(t))-f(u,v)\right]dt\right|\\ &\leq\frac{1}{2A_D}\int_a^b\left|\left[(v-y(t))x'(t)+(x(t)-u)y'(t)\right]\left[f(x(t),y(t))-f(u,v)\right]\right|dt\\ &=\frac{1}{2A_D}\int_a^b\left|\left[(v-y(t))x'(t)+(x(t)-u)y'(t)\right]\right|\left|f(x(t),y(t))-f(u,v)\right|dt\\ &\leq\frac{1}{2A_D}\int_a^b\left|\left[(v-y(t))x'(t)+(x(t)-u)y'(t)\right]\right|\left[L_u|x(t)-u|+K_v|y(t)-v|\right]dt\\ &\leq\frac{1}{2A_D}\int_a^b\left[|v-y(t)|\left|x'(t)\right|+|x(t)-u|\left|y'(t)\right|\right]\left[L_u|x(t)-u|+K_v|y(t)-v|\right]dt\\ &=\frac{1}{2A_D}L_u\int_a^b\left[|x(t)-u||v-y(t)|\left|x'(t)\right|+|x(t)-u|^2\left|y'(t)\right|\right]dt\\ &+\frac{1}{2A_D}K_v\int_a^b\left[|v-y(t)|^2\left|x'(t)\right|+|x(t)-u||y(t)-v|\left|y'(t)\right|\right]dt,\end{aligned}$$

which proves the desired inequality (5.3). $\square$

Remark 4. *Observe that*

$$\begin{aligned}&|v-y(t)|\,|x'(t)|+|x(t)-u|\,|y'(t)|\\&\leq\left(|v-y(t)|^2+|x(t)-u|^2\right)^{1/2}\left(|x'(t)|^2+|y'(t)|^2\right)^{1/2},\end{aligned}$$

for all $t\in[a,b]$, *which implies that*

$$\begin{aligned}&\int_a^b\left[|x(t)-u|\left(|v-y(t)|\,|x'(t)|+|x(t)-u|\,|y'(t)|\right)\right]dt\\&\leq\int_a^b|x(t)-u|\left(|v-y(t)|^2+|x(t)-u|^2\right)^{1/2}\left(|x'(t)|^2+|y'(t)|^2\right)^{1/2}dt\\&\leq\left(\int_a^b|x(t)-u|^2\left(|x'(t)|^2+|y'(t)|^2\right)^{1/2}dt\right)^{1/2}\\&\times\left(\int_a^b\left(|v-y(t)|^2+|x(t)-u|^2\right)\left(|x'(t)|^2+|y'(t)|^2\right)^{1/2}dt\right)^{1/2}\\&=\left(\int_{\partial D}|x-u|^2\,d(\ell)\right)^{1/2}\left(\int_{\partial D}\left(|v-y|^2+|x-u|^2\right)d(\ell)\right)^{1/2}\end{aligned}$$

and

$$\begin{aligned}&\int_a^b\left[|y(t)-v|\left(|v-y(t)|\,|x'(t)|+|x(t)-u|\,|y'(t)|\right)\right]dt\\&\leq\left(\int_{\partial D}|y-u|^2\,d(\ell)\right)^{1/2}\left(\int_{\partial D}\left(|v-y|^2+|x-u|^2\right)d(\ell)\right)^{1/2},\end{aligned}$$

therefore

$$\begin{aligned}&B(L_u,K_v,u,v)\\&\leq\frac{1}{2A_D}L_u\left(\int_{\partial D}|x-u|^2\,d(\ell)\right)^{1/2}\left(\int_{\partial D}\left(|v-y|^2+|x-u|^2\right)d(\ell)\right)^{1/2}\\&\quad+\frac{1}{2A_D}K_v\left(\int_{\partial D}|y-u|^2\,d(\ell)\right)^{1/2}\left(\int_{\partial D}\left(|v-y|^2+|x-u|^2\right)d(\ell)\right)^{1/2},\end{aligned}$$

which implies the following inequality of interest

$$\begin{aligned}B(L_u,K_v,u,v)&\leq\frac{1}{2A_D}\left(\int_{\partial D}\left(|v-y|^2+|x-u|^2\right)d(\ell)\right)^{1/2}\\&\times\left[L_u\left(\int_{\partial D}|x-u|^2\,d(\ell)\right)^{1/2}+K_v\left(\int_{\partial D}|y-u|^2\,d(\ell)\right)^{1/2}\right].\end{aligned}\tag{5.5}$$

If f has bounded partial derivatives on D then we can take in the above inequalities (5.3)-(5.5)

$$L_u=\left\|\frac{\partial f}{\partial x}\right\|_{D,\infty}\ \text{and}\ K_v=\left\|\frac{\partial f}{\partial y}\right\|_{D,\infty}.$$

If the function f satisfies the condition (5.1) for the point $(\overline{x_D}, \overline{y_D})$ that is assumed to belong to D, then

$$
\begin{aligned}
(5.6)\quad & \left| \frac{1}{A_D} \int\int_D f(x,y)\,dxdy - f(\overline{x_D}, \overline{y_D}) \right. \\
& \left. - \frac{1}{2A_D} \int\int_D \left[(\overline{x_D} - x) \frac{\partial f(x,y)}{\partial x} + (\overline{y_D} - y) \frac{\partial f(x,y)}{\partial y} \right] dxdy \right| \\
& \leq B\left(L_{\overline{x_D}}, K_{K_{\overline{y_D}}}, \overline{x_D}, \overline{y_D} \right),
\end{aligned}
$$

where

$$
\begin{aligned}
(5.7)\quad & B\left(L_{\overline{x_D}}, K_{K_{\overline{y_D}}}, \overline{x_D}, \overline{y_D} \right) \\
:= & \frac{1}{2A_D} L_{\overline{x_D}} \int_a^b \left[|x(t) - \overline{x_D}| \left(|\overline{y_D} - y(t)| \, |x'(t)| + |x(t) - \overline{x_D}| \, |y'(t)| \right) \right] dt \\
& + \frac{1}{2A_D} K_{\overline{y_D}} \int_a^b \left[|y(t) - \overline{y_D}| \left(|\overline{y_D} - y(t)| \, |x'(t)| + |x(t) - \overline{x_D}| \, |y'(t)| \right) \right] dt.
\end{aligned}
$$

We also have

$$
\begin{aligned}
(5.8)\quad B(L_u, K_v, \overline{x_D}, \overline{y_D}) \leq & \frac{1}{2A_D} \left(\int_{\partial D} \left(|\overline{y_D} - y|^2 + |x - \overline{x_D}|^2 \right) d(\ell) \right)^{1/2} \\
& \times \left[L_{\overline{x_D}} \left(\int_{\partial D} |x - \overline{x_D}|^2 \, d(\ell) \right)^{1/2} + K_{\overline{y_D}} \left(\int_{\partial D} |\overline{y_D} - y|^2 \, d(\ell) \right)^{1/2} \right].
\end{aligned}
$$

6. Examples for Rectangles

Let $a < b$ and $c < d$. Put $A = (a,c)$, $B = (b,c)$, $C = (b,d)$, $D = (a,d) \in \mathbb{R}^2$ the vertices of the rectangle $ABCD = [a,b] \times [c,d]$. Consider the counterclockwise segments

$$
AB : \begin{cases} x = (1-t)a + tb \\ \\ y = c \end{cases}, \; t \in [0,1]
$$

$$
BC : \begin{cases} x = b \\ \\ y = (1-t)c + td \end{cases}, \; t \in [0,1]
$$

$$
CD : \begin{cases} x = (1-t)b + ta \\ \\ y = d \end{cases}, \; t \in [0,1]
$$

and

$$
DA : \begin{cases} x = a \\ \\ y = (1-t)d + tc \end{cases}, \; t \in [0,1].
$$

Therefore $\partial(ABCD) = AB \cup BC \cup CD \cup DA$.

If $\alpha,\,\beta\in\mathbb{R}$, then

$$\begin{aligned}&\oint_{AB}\left[(\beta-y)\,f(x,y)\,dx+(x-\alpha)\,f(x,y)\,dy\right]\\&=(b-a)(\beta-c)\int_0^1 f((1-t)\,a+tb,c)\,dt=(\beta-c)\int_a^b f(x,c)\,dx,\end{aligned}$$

$$\begin{aligned}&\oint_{BC}\left[(\beta-y)\,f(x,y)\,dx+(x-\alpha)\,f(x,y)\,dy\right]\\&=(d-c)(b-\alpha)\int_0^1 f(b,(1-t)\,c+td)\,dt=(b-\alpha)\int_c^d f(b,y)\,dy\end{aligned}$$

$$\begin{aligned}&\oint_{CD}\left[(\beta-y)\,f(x,y)\,dx+(x-\alpha)\,f(x,y)\,dy\right]\\&=(a-b)(\beta-d)\int_0^1 f((1-t)\,b+ta,d)\,dt=(\beta-d)\int_b^a f(x,d)\,dx\\&=(d-\beta)\int_a^b f(x,d)\,dx\end{aligned}$$

and

$$\begin{aligned}&\oint_{DA}\left[(\beta-y)\,f(x,y)\,dx+(x-\alpha)\,f(x,y)\,dy\right]\\&=\int_0^1(a-\alpha)\,f(a,(1-t)\,d+tc)(c-d)\,dt=(a-\alpha)\int_d^c f(a,y)\,dy\\&=(\alpha-a)\int_c^d f(a,y)\,dy.\end{aligned}$$

Therefore

$$\begin{aligned}&\oint_{ABCD}\left[(\beta-y)\,f(x,y)\,dx+(x-\alpha)\,f(x,y)\,dy\right]\\&=(\beta-c)\int_a^b f(x,c)\,dx+(d-\beta)\int_a^b f(x,d)\,dx\\&+(b-\alpha)\int_c^d f(b,y)\,dy+(\alpha-a)\int_c^d f(a,y)\,dy\end{aligned}$$

for all $\alpha,\,\beta\in\mathbb{R}$.

We also have $\overline{x_D}=\frac{a+b}{2}$ and $\overline{y_D}=\frac{c+d}{2}$, which imply that

$$\begin{aligned}&\oint_{\partial(ABCD)}\left[(\overline{y_D}-y)\,f(x,y)\,dx+(x-\overline{x_D})\,f(x,y)\,dy\right]\\&=(d-c)\int_a^b\left(\frac{f(x,c)+f(x,d)}{2}\right)dx+(b-a)\int_c^d\left(\frac{f(b,y)+f(a,y)}{2}\right)dy.\end{aligned}$$

From the equality (2.1) we have

$$\int_a^b \int_c^d f(x,y)\,dxdy \tag{6.1}$$
$$= \frac{1}{2}\int_a^b \int_c^d \left[(\alpha - x)\frac{\partial f(x,y)}{\partial x} + (\beta - y)\frac{\partial f(x,y)}{\partial y}\right] dxdy$$
$$+ \frac{1}{2}\left[(\beta - c)\int_a^b f(x,c)\,dx + (d-\beta)\int_a^b f(x,d)\,dx\right]$$
$$+ \frac{1}{2}\left[(b-\alpha)\int_c^d f(b,y)\,dy + (\alpha - a)\int_c^d f(a,y)\,dy\right]$$

for all α, $\beta \in \mathbb{R}$, while from (2.2) we get

$$\int_a^b \int_c^d f(x,y)\,dxdy \tag{6.2}$$
$$= \frac{1}{2}\int_a^b \int_c^d \left[\left(\frac{a+b}{2} - x\right)\frac{\partial f(x,y)}{\partial x} + \left(\frac{c+d}{2} - y\right)\frac{\partial f(x,y)}{\partial y}\right] dxdy$$
$$+ \frac{1}{2}\left[(d-c)\int_a^b \left(\frac{f(x,c)+f(x,d)}{2}\right) dx + (b-a)\int_c^d \left(\frac{f(b,y)+f(a,y)}{2}\right) dy\right].$$

These imply the following perturbed identities

$$\frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x,y)\,dxdy - \lambda \tag{6.3}$$
$$- \frac{1}{2(b-a)(d-c)}\int_a^b \int_c^d \left[(\alpha - x)\frac{\partial f(x,y)}{\partial x} + (\beta - y)\frac{\partial f(x,y)}{\partial y}\right] dxdy$$
$$= \frac{1}{2(b-a)(d-c)}\left[(\beta - c)\int_a^b (f(x,c) - \lambda)\,dx + (d-\beta)\int_a^b (f(x,d) - \lambda)\,dx\right]$$
$$+ \frac{1}{2(b-a)(d-c)}\left[(b-\alpha)\int_c^d (f(b,y) - \lambda)\,dy + (\alpha - a)\int_c^d (f(a,y) - \lambda)\,dy\right]$$

and

$$\frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x,y)\,dxdy - \lambda \tag{6.4}$$
$$- \frac{1}{2(b-a)(d-c)}\int_a^b \int_c^d \left[\left(\frac{a+b}{2} - x\right)\frac{\partial f(x,y)}{\partial x} + \left(\frac{c+d}{2} - y\right)\frac{\partial f(x,y)}{\partial y}\right] dxdy$$
$$= \frac{1}{2}\left[\frac{1}{b-a}\int_a^b \left(\frac{f(x,c)+f(x,d)}{2} - \lambda\right) dx + \frac{1}{d-c}\int_c^d \left(\frac{f(b,y)+f(a,y)}{2} - \lambda\right) dy\right].$$

From (6.4) we can get, for instance, the simpler inequality

$$
\begin{aligned}
(6.5)\qquad &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy-\lambda\right.\\
&\left.-\frac{1}{2(b-a)(d-c)}\int_a^b\int_c^d\left[\left(\frac{a+b}{2}-x\right)\frac{\partial f(x,y)}{\partial x}+\left(\frac{c+d}{2}-y\right)\frac{\partial f(x,y)}{\partial y}\right]dxdy\right|\\
&\qquad\le\frac{1}{2}\left[\sup_{x\in[a,b]}\left|\frac{f(x,c)+f(x,d)}{2}-\lambda\right|dx+\sup_{y\in[a,b]}\left|\frac{f(b,y)+f(a,y)}{2}-\lambda\right|\right]
\end{aligned}
$$

for all $\lambda\in\mathbb{C}$.

If we take $\lambda=f(u,v)$ in (6.5), then we get

$$
\begin{aligned}
(6.6)\qquad &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy-f(u,v)\right.\\
&\left.-\frac{1}{2(b-a)(d-c)}\int_a^b\int_c^d\left[\left(\frac{a+b}{2}-x\right)\frac{\partial f(x,y)}{\partial x}+\left(\frac{c+d}{2}-y\right)\frac{\partial f(x,y)}{\partial y}\right]dxdy\right|\\
&\le\frac{1}{2}\left[\sup_{x\in[a,b]}\left|\frac{f(x,c)+f(x,d)}{2}-f(u,v)\right|dx+\sup_{y\in[a,b]}\left|\frac{f(b,y)+f(a,y)}{2}-f(u,v)\right|\right]
\end{aligned}
$$

and, in particular,

$$
\begin{aligned}
(6.7)\qquad &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\
&\left.-\frac{1}{2(b-a)(d-c)}\int_a^b\int_c^d\left[\left(\frac{a+b}{2}-x\right)\frac{\partial f(x,y)}{\partial x}+\left(\frac{c+d}{2}-y\right)\frac{\partial f(x,y)}{\partial y}\right]dxdy\right|\\
&\qquad\le\frac{1}{2}\left[\sup_{x\in[a,b]}\left|\frac{f(x,c)+f(x,d)}{2}-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|dx\right.\\
&\qquad\qquad\left.+\sup_{y\in[a,b]}\left|\frac{f(b,y)+f(a,y)}{2}-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right].
\end{aligned}
$$

Other inequalities may be also stated, however the details are not presented here.

7. Examples for Disks

We consider the closed disk $D(C,R)$ centered in $C(a,b)$ and of radius $R>0$. This is parametrized by

$$
\begin{cases} x=r\cos\theta+a \\ y=r\sin\theta+b \end{cases},\ r\in[0,R],\ \theta\in[0,2\pi]
$$

and the circle $\mathcal{C}(C,R)$ is parametrized by

$$
\begin{cases} x=R\cos\theta+a \\ y=R\sin\theta+b \end{cases},\ \theta\in[0,2\pi].
$$

Here $\overline{x_{D(C,R)}}=a$, $\overline{y_{D(C,R)}}=b$ and $A_{D(C,R)}=\pi R^2$.

Then

$$\begin{aligned}
&\frac{1}{A_D}\oint_{\partial D}\left[\left(\overline{y_D}-y\right)f\left(x,y\right)dx+\left(x-\overline{x_D}\right)f\left(x,y\right)dy\right] \\
&=\frac{1}{A_D}\int_a^b\left[\left(\overline{y_D}-y\left(t\right)\right)x'\left(t\right)+\left(x\left(t\right)-\overline{x_D}\right)y'\left(t\right)\right]f\left(x\left(t\right),y\left(t\right)\right)dt \\
&=\frac{1}{\pi R^2}\int_0^{2\pi}\left[\sin^2\theta+\cos^2\theta\right]R^2 f\left(R\cos\theta+a,R\sin\theta+b\right)d\theta \\
&=\frac{1}{\pi}\int_0^{2\pi}f\left(R\cos\theta+a,R\sin\theta+b\right)d\theta
\end{aligned}$$

and

$$\begin{aligned}
&\int\int_D\left[\left(\overline{x_D}-x\right)\frac{\partial f\left(x,y\right)}{\partial x}+\left(\overline{y_D}-y\right)\frac{\partial f\left(x,y\right)}{\partial y}\right]dxdy \\
&\qquad=-\int_0^R\int_0^{2\pi}\left[\cos\theta\frac{\partial f\left(r\cos\theta+a,r\sin\theta+b\right)}{\partial x}\right. \\
&\qquad\qquad\left.+\sin\theta\frac{\partial f\left(r\cos\theta+a,r\sin\theta+b\right)}{\partial y}\right]r^2 d\theta dr
\end{aligned}$$

and by the equality (2.2) we have

$$\begin{aligned}
(7.1)\quad &\frac{1}{\pi R^2}\int\int_{D(C,R)}f\left(x,y\right)dxdy \\
&=-\frac{1}{2}\int_0^R\int_0^{2\pi}\left[\cos\theta\frac{\partial f\left(r\cos\theta+a,r\sin\theta+b\right)}{\partial x}\right. \\
&\quad\left.+\sin\theta\frac{\partial f\left(r\cos\theta+a,r\sin\theta+b\right)}{\partial y}\right]r^2 d\theta dr \\
&\qquad+\frac{1}{2\pi}\int_0^{2\pi}f\left(R\cos\theta+a,R\sin\theta+b\right)d\theta.
\end{aligned}$$

We also have the perturbed identity

$$\begin{aligned}
(7.2)\quad &\frac{1}{\pi R^2}\int\int_{D(C,R)}f\left(x,y\right)dxdy-\lambda \\
&=-\frac{1}{2}\int_0^R\int_0^{2\pi}\left[\cos\theta\frac{\partial f\left(r\cos\theta+a,r\sin\theta+b\right)}{\partial x}\right. \\
&\quad\left.+\sin\theta\frac{\partial f\left(r\cos\theta+a,r\sin\theta+b\right)}{\partial y}\right]r^2 d\theta dr \\
&\qquad+\frac{1}{2\pi}\int_0^{2\pi}\left[f\left(R\cos\theta+a,R\sin\theta+b\right)-\lambda\right]d\theta
\end{aligned}$$

for all $\lambda\in\mathbb{C}$.

In particular, we have

$$\begin{aligned}
(7.3)\quad & \frac{1}{\pi R^2}\int\int_{D(C,R)} f(x,y)\,dxdy - f(a,b) \\
&= -\frac{1}{2}\int_0^R\int_0^{2\pi}\left[\cos\theta\frac{\partial f(r\cos\theta+a,r\sin\theta+b)}{\partial x}\right. \\
&\quad \left.+\sin\theta\frac{\partial f(r\cos\theta+a,r\sin\theta+b)}{\partial y}\right]r^2 d\theta dr \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi}\left[f(R\cos\theta+a,R\sin\theta+b)-f(a,b)\right]d\theta.
\end{aligned}$$

If we have the following boundedness condition on the boundary

$$|f(x,y)-f(a,b)| \le M \text{ for all } (x,y)\in\mathcal{C}(C,R)$$

then by (7.3) we get

$$\begin{aligned}
(7.4)\quad & \left|\frac{1}{\pi R^2}\int\int_{D(C,R)} f(x,y)\,dxdy - f(a,b)\right. \\
&\quad + \frac{1}{2}\int_0^R\int_0^{2\pi}\left[\cos\theta\frac{\partial f(r\cos\theta+a,r\sin\theta+b)}{\partial x}\right. \\
&\quad \left.\left.+\sin\theta\frac{\partial f(r\cos\theta+a,r\sin\theta+b)}{\partial y}\right]r^2 d\theta dr\right| \le M.
\end{aligned}$$

Also, if there exists the constants L, $K>0$ such that the following Lipschitz type condition holds

$$|f(R\cos\theta+a,R\sin\theta+b)-f(a,b)| \le R\left[|\cos\theta|L+|\sin\theta|K\right]$$

for all $\theta\in[0,2\pi]$, then by (7.3) we get

$$\begin{aligned}
(7.5)\quad & \left|\frac{1}{\pi R^2}\int\int_{D(C,R)} f(x,y)\,dxdy - f(a,b)\right. \\
&\quad + \frac{1}{2}\int_0^R\int_0^{2\pi}\left[\cos\theta\frac{\partial f(r\cos\theta+a,r\sin\theta+b)}{\partial x}\right. \\
&\quad \left.\left.+\sin\theta\frac{\partial f(r\cos\theta+a,r\sin\theta+b)}{\partial y}\right]r^2 d\theta dr\right| \\
&\le \frac{1}{2\pi}\int_0^{2\pi}|f(R\cos\theta+a,R\sin\theta+b)-f(a,b)|\,d\theta \\
&\le \frac{1}{2\pi}\int_0^{2\pi} R\left[|\cos\theta|L+|\sin\theta|K\right]d\theta = \frac{2R}{\pi}(L+K).
\end{aligned}$$

Other inequalities may be also stated, however the details are not presented here.

References

[1] Barnett, N. S.; Cîrstea, F. C. and Dragomir, S. S. Some inequalities for the integral mean of Hölder continuous functions defined on disks in a plane, in *Inequality Theory and Applications*, Vol. **2** (Chinju/Masan, 2001), 7–18, Nova Sci. Publ., Hauppauge, NY, 2003. Preprint *RGMIA Res. Rep. Coll.* **5** (2002), Nr. 1, Art. 7, 10 pp. [Online https://rgmia.org/papers/v5n1/BCD.pdf].

[2] Barnett, N. S.; Dragomir, S. S. An Ostrowski type inequality for double integrals and applications for cubature formulae. *Soochow J. Math.* **27** (2001), no. 1, 1–10.

[3] Barnett, N. S.; Dragomir, S. S.; Pearce, C. E. M. A quasi-trapezoid inequality for double integrals. *ANZIAM J.* **44** (2003), no. 3, 355–364.

[4] Budak, Hüseyin; Sarıkaya, Mehmet Zeki An inequality of Ostrowski-Grüss type for double integrals. *Stud. Univ. Babeş-Bolyai Math.* **62** (2017), no. 2, 163–173.

[5] Dragomir, S. S.; Cerone, P.; Barnett, N. S.; Roumeliotis, J. An inequality of the Ostrowski type for double integrals and applications for cubature formulae. *Tamsui Oxf. J. Math. Sci.* **16** (2000), no. 1, 1–16.

[6] Erden, Samet; Sarikaya, Mehmet Zeki On exponential Pompeiu's type inequalities for double integrals with applications to Ostrowski's inequality. *New Trends Math. Sci.* **4** (2016), no. 1, 256–267.

[7] Hanna, George Some results for double integrals based on an Ostrowski type inequality. *Ostrowski Type Inequalities and Applications in Numerical Integration,* 331–371, Kluwer Acad. Publ., Dordrecht, 2002.

[8] Hanna, G.; Dragomir, S. S.; Cerone, P. A general Ostrowski type inequality for double integrals. *Tamkang J. Math.* **33** (2002), no. 4, 319–333.

[9] Liu, Zheng A sharp general Ostrowski type inequality for double integrals. *Tamsui Oxf. J. Inf. Math. Sci.* **28** (2012), no. 2, 217–226.

[10] Özdemir, M. Emin; Akdemir, Ahmet Ocak; Set, Erhan A new Ostrowski-type inequality for double integrals. *J. Inequal. Spec. Funct.* **2** (2011), no. 1, 27–34.

[11] Pachpatte, B. G. A new Ostrowski type inequality for double integrals. *Soochow J. Math.* **32** (2006), no. 2, 317–322.

[12] Sarikaya, Mehmet Zeki On the Ostrowski type integral inequality for double integrals. *Demonstratio Math.* **45** (2012), no. 3, 533–540.

[13] Sarikaya, Mehmet Zeki; Ogunmez, Hasan On the weighted Ostrowski-type integral inequality for double integrals. *Arab. J. Sci. Eng.* **36** (2011), no. 6, 1153–1160.

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