NEW INEQUALITIES FOR DOUBLE AND PATH INTEGRALS ON GENERAL DOMAINS VIA GREEN'S IDENTITY

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Abstract. In this paper, by the use of the celebrated Green's identity for double and path integrals, we establish some inequalities for functions of two variables defined on closed and bounded subsets of the plane $\mathbb{R}^2$. Some examples for rectangles and disks are also provided.

1. Introduction

In paper [1], the authors obtained among others the following results concerning the difference between the double integral on the disk and the values in the center or the path integral on the circle:

Theorem 1. *If $f : D(C,R) \to \mathbb{R}$ has continuous partial derivatives on $D(C,R)$, the disk centered in the point $C = (a,b)$ with the radius $R > 0$, and*

$$\left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} := \sup_{(x,y)\in D(C,R)} \left| \frac{\partial f(x,y)}{\partial x} \right| < \infty,$$
$$\left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} := \sup_{(x,y)\in D(C,R)} \left| \frac{\partial f(x,y)}{\partial y} \right| < \infty;$$

then

$$\left| f(C) - \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\,dx\,dy \right| \leq \frac{4}{3\pi} R \left[\left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} \right]. \tag{1.1}$$

The constant $\frac{4}{3\pi}$ is sharp.

We also have

$$\left| \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\,dx\,dy - \frac{1}{2\pi R} \int_{\sigma(C,R)} f(\gamma)\,dl(\gamma) \right| \leq \frac{2R}{3\pi} \left[\left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} \right], \tag{1.2}$$

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where $\sigma(C,R)$ is the circle centered in $C=(a,b)$ with the radius $R>0$ and

$$\left|f(C)-\frac{1}{2\pi R}\int_{\sigma(C,R)}f(\gamma)\,dl(\gamma)\right| \leq \frac{2R}{\pi}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(C,R),\infty}+\left\|\frac{\partial f}{\partial y}\right\|_{D(C,R),\infty}\right]. \tag{1.3}$$

In the same paper [1] the authors also established the following Ostrowski type inequality:

Theorem 2. *If f has bounded partial derivatives on $D(0,1)$, then*

$$\begin{aligned}\left|f(u,v)-\frac{1}{\pi}\iint_{D(0,1)}f(x,y)\,dx\,dy\right| &\leq \frac{2}{\pi}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(0,1),\infty}\left(u\arcsin u+\frac{1}{3}\sqrt{1-u^2}\left(2+u^2\right)\right)\right.\\ &\quad\left.+\left\|\frac{\partial f}{\partial y}\right\|_{D(0,1),\infty}\left(v\arcsin v+\frac{1}{3}\sqrt{1-v^2}\left(2+v^2\right)\right)\right]\end{aligned} \tag{1.4}$$

for any $(u,v)\in D(0,1)$.

For other integral inequalities for double integrals see [2]-[13].

Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a region D. Let L and M be scalar functions defined at least on an open set containing D. Assume L and M have continuous first partial derivatives. Then the following equality is well known as the Green theorem (see for instance https://en.wikipedia.org/wiki/Green%27s_theorem)

$$\int\int_D\left(\frac{\partial M(x,y)}{\partial x}-\frac{\partial L(x,y)}{\partial y}\right)dxdy=\oint_{\partial D}\left(L(x,y)\,dx+M(x,y)\,dy\right). \tag{G}$$

By applying this equality for real and imaginary parts, we can also state it for complex valued functions P and Q.

Moreover, if the curve ∂D is described by the function $r(t)=(x(t),y(t))$, $t\in[a,b]$, with x, y differentiable on (a,b) then we can calculate the path integral as

$$\oint_{\partial D}\left(L(x,y)\,dx+M(x,y)\,dy\right)=\int_a^b\left[L(x(t),y(t))\,x'(t)+M(x(t),y(t))\,y'(t)\right]dt.$$

In the following, consider D a closed and bounded convex subset of $\mathbb{R}^2$. Define

$$A_D:=\int\int_D dxdy$$

the *area* of D and $(\overline{x_D},\overline{y_D})$ the *centre of mass* for D, where

$$\overline{x_D}:=\frac{1}{A_D}\int\int_D xdxdy,\ \overline{y_D}:=\frac{1}{A_D}\int\int_D ydxdy.$$

Consider the function of two variables $f=f(x,y)$ and denote by $\frac{\partial f}{\partial x}$ the partial derivative with respect to the variable x and $\frac{\partial f}{\partial y}$ the partial derivative with respect to the variable y.

Motivated by the above results, by the use of Green's identity (G), in this paper we establish some bounds for the absolute value of the difference

$$\frac{1}{A_D}\int\int_D f(x,y)\,dxdy-\frac{1}{2A_D}\oint_{\partial D}[(\beta-y)f(x,y)\,dx+(x-\alpha)f(x,y)\,dy]$$
$$-\frac{1}{2}\gamma(\alpha-\overline{x_D})-\frac{1}{2}\delta(\beta-\overline{y_D})$$

for α, β, γ, δ complex numbers, and, in particular,

$$\frac{1}{A_D}\int\int_D f(x,y)\,dxdy-\frac{1}{2A_D}\oint_{\partial D}[(\overline{y_D}-y)f(x,y)\,dx+(x-\overline{x_D})f(x,y)\,dy]$$

in the general case of closed and bounded subset of $\mathbb{R}^2$ and f is defined on an open set containing D and having continuous partial derivatives on D. Some examples for rectangles and disks are also provided.

2. Some Identities of Interest

We have the following identity of interest:

Lemma 1. *Let ∂D be a simple, closed counterclockwise curve bounding a region D and f defined on an open set containing D and having continuous partial derivatives on D. Then for any α, β, γ, $\delta\in\mathbb{C}$,*

$$\begin{aligned}\frac{1}{A_D}\int\int_D f(x,y)\,dxdy&\\ -\frac{1}{2A_D}\oint_{\partial D}[(\beta-y)f(x,y)\,dx+(x-\alpha)f(x,y)\,dy]&\\ -\frac{1}{2}\gamma(\alpha-\overline{x_D})-\frac{1}{2}\delta(\beta-\overline{y_D})&\\ =\frac{1}{2A_D}\int\int_D\left[(\alpha-x)\left(\frac{\partial f(x,y)}{\partial x}-\gamma\right)+(\beta-y)\left(\frac{\partial f(x,y)}{\partial y}-\delta\right)\right]dxdy.&\end{aligned}\tag{2.1}$$

In particular,

$$\begin{aligned}\frac{1}{A_D}\int\int_D f(x,y)\,dxdy&\\ -\frac{1}{2A_D}\oint_{\partial D}[(\overline{y_D}-y)f(x,y)\,dx+(x-\overline{x_D})f(x,y)\,dy]&\\ =\frac{1}{2A_D}\int\int_D\left[(\overline{x_D}-x)\left(\frac{\partial f(x,y)}{\partial x}-\gamma\right)+(\overline{y_D}-y)\left(\frac{\partial f(x,y)}{\partial y}-\delta\right)\right]dxdy.&\end{aligned}\tag{2.2}$$

Proof. Observe that

$$\frac{\partial}{\partial x}((x-\alpha)f(x,y))=f(x,y)+(x-\alpha)\frac{\partial f(x,y)}{\partial x}$$

and

$$\frac{\partial}{\partial y}\left((y-\beta) f(x,y)\right) = f(x,y) + (y-\beta)\frac{\partial f(x,y)}{\partial y}$$

for all $(x,y) \in D$ and is we add these equalities we get

$$\begin{aligned} \frac{\partial}{\partial x}\left((x-\alpha) f(x,y)\right) + \frac{\partial}{\partial y}\left((y-\beta) f(x,y)\right) \\ = 2f(x,y) + (x-\alpha)\frac{\partial f(x,y)}{\partial x} + (y-\beta)\frac{\partial f(x,y)}{\partial y}. \end{aligned} \tag{2.3}$$

Further, if we integrate on D the identity (2.3), then we obtain

$$\begin{aligned} \int\int_D \left[\frac{\partial}{\partial x}\left((x-\alpha) f(x,y)\right) + \frac{\partial}{\partial y}\left((y-\beta) f(x,y)\right)\right] dxdy \\ = 2\int\int_D f(x,y)\, dxdy \\ + \int\int_D \left[(x-\alpha)\frac{\partial f(x,y)}{\partial x} + (y-\beta)\frac{\partial f(x,y)}{\partial y}\right] dxdy. \end{aligned} \tag{2.4}$$

Now, if we apply Green's identity (G) for the functions $M(x,y) = (x-\alpha) f(x,y)$ and $L(x,y) = (\beta - y) f(x,y)$ then we get

$$\begin{aligned} &\int\int_D \left[\frac{\partial}{\partial x}\left((x-\alpha) f(x,y)\right) + \frac{\partial}{\partial y}\left((y-\beta) f(x,y)\right)\right] dxdy \\ &= \oint_{\partial D} \left[(\beta - y) f(x,y)\, dx + (x-\alpha) f(x,y)\, dy\right] \end{aligned}$$

and by (2.4) we obtain

$$\begin{aligned} 2\int\int_D f(x,y)\, dxdy + \int\int_D \left[(x-\alpha)\frac{\partial f(x,y)}{\partial x} + (y-\beta)\frac{\partial f(x,y)}{\partial y}\right] dxdy \\ = \oint_{\partial D} \left[(\beta - y) f(x,y)\, dx + (x-\alpha) f(x,y)\, dy\right] \end{aligned}$$

namely,

$$\begin{aligned} 2\int\int_D f(x,y)\, dxdy - \oint_{\partial D} \left[(\beta - y) f(x,y)\, dx + (x-\alpha) f(x,y)\, dy\right] \\ = \int\int_D \left[(\alpha - x)\frac{\partial f(x,y)}{\partial x} + (\beta - y)\frac{\partial f(x,y)}{\partial y}\right] dxdy. \end{aligned} \tag{2.5}$$

Moreover, we observe that

$$\begin{aligned}
&\int\int_D \left[(\alpha - x)\left(\frac{\partial f(x,y)}{\partial x} - \gamma\right) + (\beta - y)\left(\frac{\partial f(x,y)}{\partial y} - \delta\right)\right] dxdy \\
&= \int\int_D \left[(\alpha - x)\frac{\partial f(x,y)}{\partial x} + (\beta - y)\frac{\partial f(x,y)}{\partial y}\right] dxdy \\
&- \gamma \int\int_D (\alpha - x)\, dxdy - \delta \int\int_D (\beta - y)\, dxdy \\
&= \int\int_D \left[(\alpha - x)\frac{\partial f(x,y)}{\partial x} + (\beta - y)\frac{\partial f(x,y)}{\partial y}\right] dxdy \\
&- \gamma\left(\alpha A_D - \overline{x_D} A_D\right) - \delta\left(\beta A_D - \overline{y_D} A_D\right) \\
&= \int\int_D \left[(\alpha - x)\frac{\partial f(x,y)}{\partial x} + (\beta - y)\frac{\partial f(x,y)}{\partial y}\right] dxdy \\
&- \gamma A_D\left(\alpha - \overline{x_D}\right) - \delta A_D\left(\beta - \overline{y_D}\right),
\end{aligned}$$

namely

$$\begin{aligned}
&\int\int_D \left[(\alpha - x)\frac{\partial f(x,y)}{\partial x} + (\beta - y)\frac{\partial f(x,y)}{\partial y}\right] dxdy \\
&= \int\int_D \left[(\alpha - x)\left(\frac{\partial f(x,y)}{\partial x} - \gamma\right) + (\beta - y)\left(\frac{\partial f(x,y)}{\partial y} - \delta\right)\right] dxdy \\
&+ \gamma A_D\left(\alpha - \overline{x_D}\right) + \delta A_D\left(\beta - \overline{y_D}\right).
\end{aligned} \tag{2.6}$$

By utilising the identity (2.4) we then get

$$\begin{aligned}
&2\int\int_D f(x,y)\, dxdy - \oint_{\partial D} \left[(\beta - y) f(x,y)\, dx + (x - \alpha) f(x,y)\, dy\right] \\
&= \int\int_D \left[(\alpha - x)\left(\frac{\partial f(x,y)}{\partial x} - \gamma\right) + (\beta - y)\left(\frac{\partial f(x,y)}{\partial y} - \delta\right)\right] dxdy \\
&+ \gamma A_D\left(\alpha - \overline{x_D}\right) + \delta A_D\left(\beta - \overline{y_D}\right),
\end{aligned}$$

that is equivalent to (2.1). □

Corollary 1. *With the assumptions of Lemma 1 and if the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t \in [a,b]$, with x, y differentiable on (a,b), then*

$$\begin{aligned}
&\frac{1}{A_D}\int\int_D f(x,y)\, dxdy \\
&\quad - \frac{1}{2A_D}\int_a^b \left[(\beta - y(t))\, x'(t) + (x(t) - \alpha)\, y'(t)\right] f(x(t), y(t))\, dt \\
&\qquad - \frac{1}{2}\gamma\left(\alpha - \overline{x_D}\right) - \frac{1}{2}\delta\left(\beta - \overline{y_D}\right) \\
&= \frac{1}{2A_D}\int\int_D \left[(\alpha - x)\left(\frac{\partial f(x,y)}{\partial x} - \gamma\right) + (\beta - y)\left(\frac{\partial f(x,y)}{\partial y} - \delta\right)\right] dxdy.
\end{aligned} \tag{2.7}$$

In particular,

$$\begin{aligned}(2.8)\quad & \frac{1}{A_D}\int\int_D f(x,y)\,dxdy \\ & -\frac{1}{2A_D}\int_a^b \left[\left(\overline{y_D}-y(t)\right)x'(t)+\left(x(t)-\overline{x_D}\right)y'(t)\right] f(x(t),y(t))\,dt \\ = & \frac{1}{2A_D}\int\int_D \left[\left(\overline{x_D}-x\right)\left(\frac{\partial f(x,y)}{\partial x}-\gamma\right)+\left(\overline{y_D}-y\right)\left(\frac{\partial f(x,y)}{\partial y}-\delta\right)\right] dxdy.\end{aligned}$$

We define the quantities

$$x_{f,\partial D}:=\frac{\oint_{\partial D} xf(x,y)\,dy}{\oint_{\partial D} f(x,y)\,dy}=\frac{\int_a^b x(t)\,f(x(t),y(t))\,y'(t)\,dt}{\int_a^b f(x(t),y(t))\,y'(t)\,dt}$$

and

$$y_{f,\partial D}:=\frac{\oint_{\partial D} yf(x,y)\,dx}{\oint_{\partial D} f(x,y)\,dx}=\frac{\int_a^b y(t)\,f(x(t),y(t))\,x'(t)\,dt}{\int_a^b f(x(t),y(t))\,x'(t)\,dt}$$

provided the denominators are not zero.

Corollary 2. *With the assumptions of Lemma 1 we have the equality*

$$\begin{aligned}(2.9)\quad & \frac{1}{A_D}\int\int_D f(x,y)\,dxdy-\frac{1}{2}\gamma\left(x_{f,\partial D}-\overline{x_D}\right)-\frac{1}{2}\delta\left(y_{f,\partial D}-\overline{y_D}\right) \\ = & \frac{1}{2A_D}\int\int_D \left[\left(x_{f,\partial D}-x\right)\left(\frac{\partial f(x,y)}{\partial x}-\gamma\right)+\left(y_{f,\partial D}-y\right)\left(\frac{\partial f(x,y)}{\partial y}-\delta\right)\right] dxdy.\end{aligned}$$

The equality (2.9) follows by (2.1) on observing that

$$\oint_{\partial D}\left[\left(y_{f,\partial D}-y\right)f(x,y)\,dx+\left(x-x_{f,\partial D}\right)f(x,y)\,dy\right]=0.$$

3. Some Inequalities for Bounded Partial Derivatives

Let D be a closed and bounded subset of $\mathbb{R}^2$. Now, for ϕ, $\Phi\in\mathbb{C}$, define the sets of complex-valued functions

$$\begin{aligned}&\bar{U}_D(\phi,\Phi)\\ &:=\left\{f:D\to\mathbb{C}\,|\,\operatorname{Re}\left[(\Phi-f(x,y))\left(\overline{f(x,y)}-\overline{\phi}\right)\right]\geq 0\ \text{ for each }\ (x,y)\in D\right\}\end{aligned}$$

and

$$\bar{\Delta}_D(\phi,\Phi):=\left\{f:D\to\mathbb{C}\,|\,\left|f(x,y)-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}\left|\Phi-\phi\right|\ \text{ for each }\ (x,y)\in D\right\}.$$

The following representation result may be stated.

Proposition 1. *For any $\phi,\ \Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that $\bar{U}_D(\phi,\Phi)$ and $\bar{\Delta}_D(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_D(\phi,\Phi)=\bar{\Delta}_D(\phi,\Phi). \tag{3.1}$$

Proof. We observe that for any $w\in\mathbb{C}$ we have the equivalence

$$\left|w-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}\left|\Phi-\phi\right|$$

if and only if

$$\operatorname{Re}\left[(\Phi-w)\left(\overline{w}-\overline{\phi}\right)\right]\geq 0.$$

This follows by the equality

$$\frac{1}{4}\left|\Phi-\phi\right|^2-\left|w-\frac{\phi+\Phi}{2}\right|^2=\operatorname{Re}\left[(\Phi-w)\left(\overline{w}-\overline{\phi}\right)\right]$$

that holds for any $w\in\mathbb{C}$.

The equality (3.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 3. *For any $\phi,\ \Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that*

$$\begin{aligned}\bar{U}_D(\phi,\Phi)=\{f:D\to\mathbb{C}\mid\ &(\operatorname{Re}\Phi-\operatorname{Re}f(x,y))(\operatorname{Re}f(x,y)-\operatorname{Re}\phi)\\&+(\operatorname{Im}\Phi-\operatorname{Im}f(x,y))(\operatorname{Im}f(x,y)-\operatorname{Im}\phi)\geq 0 \textit{ for each } (x,y)\in D\}.\end{aligned}\tag{3.2}$$

Now, if we assume that $\operatorname{Re}(\Phi)\geq\operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi)\geq\operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_D(\phi,\Phi):=\{f:D\to\mathbb{C}\mid\ &\operatorname{Re}(\Phi)\geq\operatorname{Re}f(x,y)\geq\operatorname{Re}(\phi)\\&\text{and }\operatorname{Im}(\Phi)\geq\operatorname{Im}f(x,y)\geq\operatorname{Im}(\phi)\ \text{ for each }\ (x,y)\in D\}.\end{aligned}\tag{3.3}$$

One can easily observe that $\bar{S}_D(\phi,\Phi)$ is closed, convex and

$$\emptyset\neq\bar{S}_D(\phi,\Phi)\subseteq\bar{U}_D(\phi,\Phi). \tag{3.4}$$

We have:

Theorem 3. *Let ∂D be a simple, closed counterclockwise curve bounding a region D and f defined on an open set containing D and having continuous partial derivatives on D. Assume that $\frac{\partial f}{\partial x}\in\bar{\Delta}_D(\phi_1,\Phi_1)$ and $\frac{\partial f}{\partial y}\in\bar{\Delta}_D(\phi_2,\Phi_2)$ for some complex numbers $\phi_1\neq\Phi_1$ and $\phi_2\neq\Phi_2$. Then for any $\alpha,\ \beta\in\mathbb{C}$ we have*

$$\begin{aligned}&\left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy\right.\\&\quad-\frac{1}{2A_D}\oint_{\partial D}\left[(\beta-y)f(x,y)\,dx+(x-\alpha)f(x,y)\,dy\right]\\&\quad\left.-\frac{\phi_1+\Phi_1}{4}\left(\alpha-\overline{x_D}\right)-\frac{\phi_2+\Phi_2}{4}\left(\beta-\overline{y_D}\right)\right|\\&\leq\frac{1}{4A_D}\left[\left|\Phi_1-\phi_1\right|\int\int_D\left|\alpha-x\right|dxdy+\left|\Phi_2-\phi_2\right|\int\int_D\left|\beta-y\right|dxdy\right].\end{aligned}\tag{3.5}$$

In particular,

$$\left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy - \frac{1}{2A_D}\oint_{\partial D}\left[(\overline{y_D}-y)f(x,y)\,dx+(x-\overline{x_D})f(x,y)\,dy\right]\right| \leq \frac{1}{4A_D}\left[|\Phi_1-\phi_1|\int\int_D|\overline{x_D}-x|\,dxdy+|\Phi_2-\phi_2|\int\int_D|\overline{y_D}-y|\,dxdy\right] \tag{3.6}$$

and

$$\left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy - \frac{\phi_1+\Phi_1}{4}(x_{f,\partial D}-\overline{x_D})-\frac{\phi_2+\Phi_2}{4}(y_{f,\partial D}-\overline{y_D})\right| \leq \frac{1}{4A_D}\left[|\Phi_1-\phi_1|\int\int_D|x_{f,\partial D}-x|\,dxdy+|\Phi_2-\phi_2|\int\int_D|y_{f,\partial D}-y|\,dxdy\right]. \tag{3.7}$$

Proof. By taking the modulus in the equality (2.1) written for $\gamma = \frac{\phi_1+\Phi_1}{2}$ and $\delta = \frac{\phi_2+\Phi_2}{2}$ we get

$$\begin{aligned}
&\left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy-\frac{1}{2A_D}\oint_{\partial D}\left[(\beta-y)f(x,y)\,dx+(x-\alpha)f(x,y)\,dy\right]\right.\\
&\qquad\left.-\frac{\phi_1+\Phi_1}{4}(\alpha-\overline{x_D})-\frac{\phi_2+\Phi_2}{4}(\beta-\overline{y_D})\right|\\
&=\frac{1}{2A_D}\left|\int\int_D\left[(\alpha-x)\left(\frac{\partial f(x,y)}{\partial x}-\frac{\phi_1+\Phi_1}{2}\right)+(\beta-y)\left(\frac{\partial f(x,y)}{\partial y}-\frac{\phi_2+\Phi_2}{2}\right)\right]dxdy\right|\\
&\leq\frac{1}{2A_D}\int\int_D\left|\left[(\alpha-x)\left(\frac{\partial f(x,y)}{\partial x}-\frac{\phi_1+\Phi_1}{2}\right)+(\beta-y)\left(\frac{\partial f(x,y)}{\partial y}-\frac{\phi_2+\Phi_2}{2}\right)\right]\right|dxdy\\
&\leq\frac{1}{2A_D}\int\int_D\left[|\alpha-x|\left|\frac{\partial f(x,y)}{\partial x}-\frac{\phi_1+\Phi_1}{2}\right|+|\beta-y|\left|\frac{\partial f(x,y)}{\partial y}-\frac{\phi_2+\Phi_2}{2}\right|\right]dxdy\\
&=: B.
\end{aligned} \tag{3.8}$$

Since $\frac{\partial f}{\partial x}\in\bar{\Delta}_D(\phi_1,\Phi_1)$ and $\frac{\partial f}{\partial y}\in\bar{\Delta}_D(\phi_2,\Phi_2)$, hence

$$B\leq\frac{1}{4A_D}\left[|\Phi_1-\phi_1|\int\int_D|\alpha-x|\,dxdy+|\Phi_2-\phi_2|\int\int_D|\beta-y|\,dxdy\right]$$

and by (3.8) we get the desired result (3.5). $\square$

4. Inequalities for Lipschitzian Partial Derivatives

We assume that the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ satisfy the Lipschitz type conditions in the point $(u,v) \in D$

$$\left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(u,v)\right| \leq L_1 |x-u| + K_1 |y-v| \tag{4.1}$$

and

$$\left|\frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}(u,v)\right| \leq L_2 |x-u| + K_2 |y-v| \tag{4.2}$$

for any $(x,y) \in D$, where L_1, K_1, L_2 and K_2 are given positive constants.

Theorem 4. *Let ∂D be a simple, closed counterclockwise curve bounding a region D and f defined on an open set containing D and having continuous partial derivatives on D. Assume that $(u,v) \in D$ and $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ satisfy the Lipschitz type conditions (4.1) and (4.2). Then for any α, $\beta \in \mathbb{C}$ we have*

$$\begin{aligned}(4.3)\quad &\Bigg|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy \\ &\qquad - \frac{1}{2A_D}\oint_{\partial D}\left[(\beta-y)f(x,y)\,dx + (x-\alpha)f(x,y)\,dy\right] \\ &\qquad - \frac{1}{2}\frac{\partial f}{\partial x}(u,v)(\alpha-\overline{x_D}) - \frac{1}{2}\frac{\partial f}{\partial y}(u,v)(\beta-\overline{y_D})\Bigg| \\ &\leq \frac{L_1}{2A_D}\int\int_D |\alpha-x|\,|x-u|\,dxdy + \frac{K_1}{2A_D}\int\int_D |\alpha-x|\,|y-v|\,dxdy \\ &\quad + \frac{L_2}{2A_D}\int\int_D |\beta-y|\,|x-u|\,dxdy + \frac{K_2}{2A_D}\int\int_D |\beta-y|\,|y-v|\,dxdy.\end{aligned}$$

In particular,

$$\begin{aligned}(4.4)\quad &\Bigg|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy \\ &\qquad - \frac{1}{2A_D}\oint_{\partial D}\left[(\overline{y_D}-y)f(x,y)\,dx + (x-\overline{x_D})f(x,y)\,dy\right]\Bigg| \\ &\leq \frac{L_1}{2A_D}\int\int_D |\overline{x_D}-x|\,|x-u|\,dxdy + \frac{K_1}{2A_D}\int\int_D |\overline{x_D}-x|\,|y-v|\,dxdy \\ &\quad + \frac{L_2}{2A_D}\int\int_D |\overline{y_D}-y|\,|x-u|\,dxdy + \frac{K_2}{2A_D}\int\int_D |\overline{y_D}-y|\,|y-v|\,dxdy\end{aligned}$$

and

$$
\begin{aligned}
(4.5)\quad & \left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy \right.\\
& \left. -\frac{1}{2}\frac{\partial f}{\partial x}(u,v)\left(x_{f,\partial D}-\overline{x_D}\right)-\frac{1}{2}\frac{\partial f}{\partial y}(u,v)\left(y_{f,\partial D}-\overline{y_D}\right)\right| \\
& \leq \frac{L_1}{2A_D}\int\int_D \left|x_{f,\partial D}-x\right|\left|x-u\right|dxdy+\frac{K_1}{2A_D}\int\int_D \left|x_{f,\partial D}-x\right|\left|y-v\right|dxdy \\
& +\frac{L_2}{2A_D}\int\int_D \left|y_{f,\partial D}-y\right|\left|x-u\right|dxdy+\frac{K_2}{2A_D}\int\int_D \left|y_{f,\partial D}-y\right|\left|y-v\right|dxdy.
\end{aligned}
$$

Proof. Using the equality (2.1) for $\gamma=\frac{\partial f}{\partial x}(u,v)$ and $\delta=\frac{\partial f}{\partial y}(u,v)$ we get

$$
\begin{aligned}
(4.6)\quad & \frac{1}{A_D}\int\int_D f(x,y)\,dxdy \\
& -\frac{1}{2A_D}\oint_{\partial D}\left[(\beta-y)f(x,y)\,dx+(x-\alpha)f(x,y)\,dy\right] \\
& -\frac{1}{2}\frac{\partial f}{\partial x}(u,v)\left(\alpha-\overline{x_D}\right)-\frac{1}{2}\frac{\partial f}{\partial y}(u,v)\left(\beta-\overline{y_D}\right) \\
& =\frac{1}{2A_D}\int\int_D\left[(\alpha-x)\left(\frac{\partial f(x,y)}{\partial x}-\frac{\partial f}{\partial x}(u,v)\right)\right. \\
& \left. +(\beta-y)\left(\frac{\partial f(x,y)}{\partial y}-\frac{\partial f}{\partial y}(u,v)\right)\right]dxdy
\end{aligned}
$$

for any $\alpha,\ \beta\in\mathbb{C}$.

Taking the modulus in (4.6) we get

$$
\begin{aligned}
& \left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy\right. \\
& -\frac{1}{2A_D}\oint_{\partial D}\left[(\beta-y)f(x,y)\,dx+(x-\alpha)f(x,y)\,dy\right] \\
& \left.-\frac{1}{2}\frac{\partial f}{\partial x}(u,v)\left(\alpha-\overline{x_D}\right)-\frac{1}{2}\frac{\partial f}{\partial y}(u,v)\left(\beta-\overline{y_D}\right)\right| \\
& \leq\frac{1}{2A_D}\int\int_D\left|(\alpha-x)\left(\frac{\partial f(x,y)}{\partial x}-\frac{\partial f}{\partial x}(u,v)\right)\right. \\
& \left.+(\beta-y)\left(\frac{\partial f(x,y)}{\partial y}-\frac{\partial f}{\partial y}(u,v)\right)\right|dxdy \\
& \leq\frac{1}{2A_D}\int\int_D\left[\left|\alpha-x\right|\left|\frac{\partial f(x,y)}{\partial x}-\frac{\partial f}{\partial x}(u,v)\right|\right. \\
& \left.+\left|\beta-y\right|\left|\frac{\partial f(x,y)}{\partial y}-\frac{\partial f}{\partial y}(u,v)\right|\right]dxdy
\end{aligned}
$$

$$\leq \frac{1}{2A_D}\int\int_D |\alpha - x|\,[L_1\,|x-u| + K_1\,|y-v|]\,dxdy$$
$$+\frac{1}{2A_D}\int\int_D |\beta - y|\,[L_2\,|x-u| + K_2\,|y-v|]\,dxdy$$
$$= \frac{1}{2A_D}\left[L_1\int\int_D |\alpha - x|\,|x-u|\,dxdy + K_1\int\int_D |\alpha - x|\,|y-v|\,dxdy\right]$$
$$+\frac{1}{2A_D}\left[L_2\int\int_D |\beta - y|\,|x-u|\,dxdy + K_2\int\int_D |\beta - y|\,|y-v|\,dxdy\right],$$

which proves (4.3). □

Corollary 4. *With the assumptions of Theorem 4 and if* $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ *satisfy the Lipschitz type conditions (4.1) and (4.2) for* $(u,v) = (\overline{x_D}, \overline{y_D})$, *then*

$$\left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy - \frac{1}{2A_D}\oint_{\partial D}\left[(\overline{y_D} - y)\,f(x,y)\,dx + (x - \overline{x_D})\,f(x,y)\,dy\right]\right| \tag{4.7}$$
$$\leq \frac{L_1}{2A_D}\int\int_D (x - \overline{x_D})^2\,dxdy + \frac{K_2}{2A_D}\int\int_D (y - \overline{y_D})^2\,dxdy + \frac{K_1 + L_2}{2A_D}\int\int_D |x - \overline{x_D}|\,|y - \overline{y_D}|\,dxdy$$

and

$$\left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy - \frac{1}{2}\frac{\partial f}{\partial x}(\overline{x_D}, \overline{y_D})\,(x_{f,\partial D} - \overline{x_D}) - \frac{1}{2}\frac{\partial f}{\partial y}(\overline{x_D}, \overline{y_D})\,(y_{f,\partial D} - \overline{y_D})\right| \tag{4.8}$$
$$\leq \frac{L_1}{2A_D}\int\int_D |x_{f,\partial D} - x|\,|x - \overline{x_D}|\,dxdy + \frac{K_1}{2A_D}\int\int_D |x_{f,\partial D} - x|\,|y - \overline{y_D}|\,dxdy$$
$$+ \frac{L_2}{2A_D}\int\int_D |y_{f,\partial D} - y|\,|x - \overline{x_D}|\,dxdy + \frac{K_2}{2A_D}\int\int_D |y_{f,\partial D} - y|\,|y - \overline{y_D}|\,dxdy.$$

Assume that $f : D \to \mathbb{C}$ is twice differentiable on D, convex set, and the second partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are bounded on D. Put

$$\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty} := \sup_{(x,y)\in D}\left|\frac{\partial^2 f}{\partial x^2}(x,y)\right|, \quad \left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty} := \sup_{(x,y)\in D}\left|\frac{\partial^2 f}{\partial y^2}(x,y)\right|$$

and

$$\left\|\frac{\partial^2 f}{\partial x \partial y}\right\|_{D,\infty} := \sup_{(x,y)\in D}\left|\frac{\partial^2 f}{\partial x \partial y}(x,y)\right|,$$

then

$$\left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(u,v)\right| \leq \left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty}|x-u| + \left\|\frac{\partial^2 f}{\partial x \partial y}\right\|_{D,\infty}|y-v|$$

and

$$\left|\frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}(u,v)\right| \leq \left\|\frac{\partial^2 f}{\partial x \partial y}\right\|_{D,\infty}|x-u| + \left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty}|y-v|$$

for all $(x,y)\,,\,(u,v)\in D$.

Therefore the conditions (4.1) and (4.2) are valid for any $(u,v)\in D$ with

$$L_1=\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty},\ K_1=L_2=\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D,\infty}$$

and

$$K_2=\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty}.$$

Corollary 5. *With the assumptions of Theorem 4 and if $f:D\to\mathbb{C}$ is twice differentiable on D, convex set, and the second partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x\partial y}$ are bounded on D, then for any $\alpha,\ \beta\in\mathbb{C}$ and $(u,v)\in D$ we have*

$$\begin{aligned}(4.9)\quad &\left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy\right.\\ &-\frac{1}{2A_D}\oint_{\partial D}\left[(\beta-y)\,f(x,y)\,dx+(x-\alpha)\,f(x,y)\,dy\right]\\ &\left.-\frac{1}{2}\frac{\partial f}{\partial x}(u,v)\,(\alpha-\overline{x_D})-\frac{1}{2}\frac{\partial f}{\partial y}(u,v)\,(\beta-\overline{y_D})\right|\\ &\leq\frac{1}{2A_D}\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty}\int\int_D|\alpha-x|\,|x-u|\,dxdy\\ &+\frac{1}{2A_D}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D,\infty}\int\int_D|\alpha-x|\,|y-v|\,dxdy\\ &+\frac{1}{2A_D}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D,\infty}\int\int_D|\beta-y|\,|x-u|\,dxdy\\ &+\frac{1}{2A_D}\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty}\int\int_D|\beta-y|\,|y-v|\,dxdy.\end{aligned}$$

In particular,

$$\begin{aligned}(4.10)\quad &\left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy\right.\\ &\left.-\frac{1}{2A_D}\oint_{\partial D}\left[(\overline{y_D}-y)\,f(x,y)\,dx+(x-\overline{x_D})\,f(x,y)\,dy\right]\right|\\ &\leq\frac{1}{2A_D}\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty}\int\int_D|\overline{x_D}-x|\,|x-u|\,dxdy\\ &+\frac{1}{2A_D}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D,\infty}\int\int_D|\overline{x_D}-x|\,|y-v|\,dxdy\\ &+\frac{1}{2A_D}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D,\infty}\int\int_D|\overline{y_D}-y|\,|x-u|\,dxdy\\ &+\frac{1}{2A_D}\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty}\int\int_D|\overline{y_D}-y|\,|y-v|\,dxdy\end{aligned}$$

and

$$
\begin{aligned}
(4.11)\quad & \left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy\right. \\
& \left.-\frac{1}{2}\frac{\partial f}{\partial x}(u,v)\left(x_{f,\partial D}-\overline{x_D}\right)-\frac{1}{2}\frac{\partial f}{\partial y}(u,v)\left(y_{f,\partial D}-\overline{y_D}\right)\right| \\
& \leq \frac{1}{2A_D}\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty}\int\int_D \left|x_{f,\partial D}-x\right|\left|x-u\right|dxdy \\
& +\frac{1}{2A_D}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D,\infty}\int\int_D \left|x_{f,\partial D}-x\right|\left|y-v\right|dxdy \\
& +\frac{1}{2A_D}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D,\infty}\int\int_D \left|y_{f,\partial D}-y\right|\left|x-u\right|dxdy \\
& +\frac{1}{2A_D}\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty}\int\int_D \left|y_{f,\partial D}-y\right|\left|y-v\right|dxdy.
\end{aligned}
$$

Remark 1. *If we take in (4.10) and (4.11)* $(u,v)=\left(\overline{x_D},\overline{y_D}\right)$, *then we get*

$$
\begin{aligned}
(4.12)\quad & \left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy\right. \\
& \left.-\frac{1}{2A_D}\oint_{\partial D}\left[\left(\overline{y_D}-y\right)f(x,y)\,dx+\left(x-\overline{x_D}\right)f(x,y)\,dy\right]\right| \\
& \leq \frac{1}{2A_D}\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty}\int\int_D \left(x-\overline{x_D}\right)^2 dxdy \\
& +\frac{1}{2A_D}\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty}\int\int_D \left(y-\overline{y_D}\right)^2 dxdy \\
& +\frac{1}{A_D}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D,\infty}\int\int_D \left|x-\overline{x_D}\right|\left|y-\overline{y_D}\right|dxdy
\end{aligned}
$$

and

$$
\begin{aligned}
(4.13)\quad & \left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy\right. \\
& \left.-\frac{1}{2}\frac{\partial f}{\partial x}\left(\overline{x_D},\overline{y_D}\right)\left(x_{f,\partial D}-\overline{x_D}\right)-\frac{1}{2}\frac{\partial f}{\partial y}\left(\overline{x_D},\overline{y_D}\right)\left(y_{f,\partial D}-\overline{y_D}\right)\right| \\
& \leq \frac{1}{2A_D}\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty}\int\int_D \left|x_{f,\partial D}-x\right|\left|x-\overline{x_D}\right|dxdy \\
& +\frac{1}{2A_D}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D,\infty}\int\int_D \left|x_{f,\partial D}-x\right|\left|y-\overline{y_D}\right|dxdy \\
& +\frac{1}{2A_D}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D,\infty}\int\int_D \left|y_{f,\partial D}-y\right|\left|x-\overline{x_D}\right|dxdy \\
& +\frac{1}{2A_D}\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty}\int\int_D \left|y_{f,\partial D}-y\right|\left|y-\overline{y_D}\right|dxdy.
\end{aligned}
$$

5. Examples for Rectangles

Let $a<b$ and $c<d$. Put $A=(a,c)\,,\ B=(b,c)\,,\ C=(b,d)\,,\ D=(a,d)\in\mathbb{R}^2$ the vertices of the rectangle $ABCD=[a,b]\times[c,d]\,.$ Consider the counterclockwise segments

$$AB:\left\{\begin{array}{l} x=(1-t)\,a+tb \\ \\ y=c\end{array}\right.,\ \ t\in[0,1]$$

$$BC:\left\{\begin{array}{l} x=b \\ \\ y=(1-t)\,c+td\end{array}\right.,\ \ t\in[0,1]$$

$$CD:\left\{\begin{array}{l} x=(1-t)\,b+ta \\ \\ y=d\end{array}\right.,\ \ t\in[0,1]$$

and

$$DA:\left\{\begin{array}{l} x=a \\ \\ y=(1-t)\,d+tc\end{array}\right.,\ \ t\in[0,1]\,.$$

Therefore $\partial\,(ABCD)=AB\cup BC\cup CD\cup DA$.

If $\alpha,\ \beta\in\mathbb{R}$, then

$$\begin{aligned}&\oint_{AB}[(\beta-y)\,f\,(x,y)\,dx+(x-\alpha)\,f\,(x,y)\,dy]\\&=(b-a)\,(\beta-c)\int_0^1 f\,((1-t)\,a+tb,c)\,dt=(\beta-c)\int_a^b f\,(x,c)\,dx,\end{aligned}$$

$$\begin{aligned}&\oint_{BC}[(\beta-y)\,f\,(x,y)\,dx+(x-\alpha)\,f\,(x,y)\,dy]\\&=(d-c)\,(b-\alpha)\int_0^1 f\,(b,(1-t)\,c+td)\,dt=(b-\alpha)\int_c^d f\,(b,y)\,dy\end{aligned}$$

$$\begin{aligned}&\oint_{CD}[(\beta-y)\,f\,(x,y)\,dx+(x-\alpha)\,f\,(x,y)\,dy]\\&=(a-b)\,(\beta-d)\int_0^1 f\,((1-t)\,b+ta,d)\,dt=(\beta-d)\int_b^a f\,(x,d)\,dx\\&=(d-\beta)\int_a^b f\,(x,d)\,dx\end{aligned}$$

and

$$\oint_{DA} [(\beta - y) f(x,y)\,dx + (x-\alpha) f(x,y)\,dy]$$
$$= \int_0^1 (a-\alpha) f(a,(1-t)d + tc)(c-d)\,dt = (a-\alpha)\int_d^c f(a,y)\,dy$$
$$= (\alpha - a)\int_c^d f(a,y)\,dy.$$

Therefore

$$\oint_{ABCD} [(\beta - y) f(x,y)\,dx + (x-\alpha) f(x,y)\,dy]$$
$$= (\beta - c)\int_a^b f(x,c)\,dx + (d-\beta)\int_a^b f(x,d)\,dx$$
$$+ (b-\alpha)\int_c^d f(b,y)\,dy + (\alpha - a)\int_c^d f(a,y)\,dy$$

for all $\alpha,\ \beta \in \mathbb{R}$.

We also have $\overline{x_D} = \frac{a+b}{2}$ and $\overline{y_D} = \frac{c+d}{2}$, which imply that

$$\oint_{\partial(ABCD)} [(\overline{y_D} - y) f(x,y)\,dx + (x - \overline{x_D}) f(x,y)\,dy]$$
$$= (d-c)\int_a^b \left(\frac{f(x,c)+f(x,d)}{2}\right)dx + (b-a)\int_c^d \left(\frac{f(b,y)+f(a,y)}{2}\right)dy.$$

From the equality (2.1) we have

$$\begin{aligned} &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy \\ &- \frac{1}{2(b-a)(d-c)}\left[(\beta - c)\int_a^b f(x,c)\,dx + (d-\beta)\int_a^b f(x,d)\,dx\right] \\ &- \frac{1}{2(b-a)(d-c)}\left[(b-\alpha)\int_c^d f(b,y)\,dy + (\alpha - a)\int_c^d f(a,y)\,dy\right] \\ &- \frac{1}{2}\gamma\left(\alpha - \frac{a+b}{2}\right) - \frac{1}{2}\delta\left(\beta - \frac{c+d}{2}\right) \\ &= \frac{1}{2(b-a)(d-c)} \\ &\times \int_a^b\int_c^d \left[(\alpha - x)\left(\frac{\partial f(x,y)}{\partial x} - \gamma\right) + (\beta - y)\left(\frac{\partial f(x,y)}{\partial y} - \delta\right)\right] dxdy \end{aligned} \tag{5.1}$$

for all α, β, γ, $\delta \in \mathbb{R}$, while from (2.2) we get

$$\begin{aligned}(5.2)\quad &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy\\ &-\frac{1}{b-a}\int_a^b\left(\frac{f(x,c)+f(x,d)}{2}\right)dx-\frac{1}{d-c}\int_c^d\left(\frac{f(b,y)+f(a,y)}{2}\right)dy\\ &=\frac{1}{2(b-a)(d-c)}\int_a^b\int_c^d\left[\left(\frac{a+b}{2}-x\right)\left(\frac{\partial f(x,y)}{\partial x}-\gamma\right)\right.\\ &\left.+\left(\frac{c+d}{2}-y\right)\left(\frac{\partial f(x,y)}{\partial y}-\delta\right)\right]dxdy\end{aligned}$$

for all γ, $\delta \in \mathbb{C}$.

Assume that $\frac{\partial f}{\partial x} \in \bar{\Delta}_{[a,b]\times[c,d]}(\phi_1, \Phi_1)$ and $\frac{\partial f}{\partial y} \in \bar{\Delta}_{[a,b]\times[c,d]}(\phi_2, \Phi_2)$ for some complex numbers $\phi_1 \neq \Phi_1$ and $\phi_2 \neq \Phi_2$. Then by (5.2) we get

$$\begin{aligned}(5.3)\quad &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy\right.\\ &\left.-\frac{1}{b-a}\int_a^b\left(\frac{f(x,c)+f(x,d)}{2}\right)dx-\frac{1}{d-c}\int_c^d\left(\frac{f(b,y)+f(a,y)}{2}\right)dy\right|\\ &\leq\frac{1}{16}\left[|\Phi_1-\phi_1|(b-a)+|\Phi_2-\phi_2|(d-c)\right].\end{aligned}$$

If $f : [a,b] \times [c,d] \to \mathbb{C}$ is twice differentiable on $[a,b] \times [c,d]$ and the second partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are bounded on $[a,b] \times [c,d]$, then by utilising the inequality (4.12) we also have

$$\begin{aligned}(5.4)\quad &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy\right.\\ &\left.-\frac{1}{b-a}\int_a^b\left(\frac{f(x,c)+f(x,d)}{2}\right)dx-\frac{1}{d-c}\int_c^d\left(\frac{f(b,y)+f(a,y)}{2}\right)dy\right|\\ &\leq\frac{1}{24}(b-a)^2\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty}+\frac{1}{24}\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty}(d-c)^2\\ &+\frac{1}{16}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D,\infty}(b-a)(d-c).\end{aligned}$$

6. Examples for Disks

We consider the closed disk $D(C,R)$ centered in $C(a,b)$ and of radius $R > 0$. This is parametrized by

$$\begin{cases} x = r\cos\theta + a \\ y = r\sin\theta + b \end{cases},\ r \in [0,R],\ \theta \in [0,2\pi]$$

and the circle $\mathcal{C}(C,R)$ is parametrized by

$$\begin{cases} x = R\cos\theta + a \\ \\ y = R\sin\theta + b \end{cases}, \ \theta \in [0,2\pi].$$

Here $\overline{x_{D(C,R)}} = a$, $\overline{y_{D(C,R)}} = b$ and $A_{D(C,R)} = \pi R^2$.

Then

$$\begin{aligned}
&\frac{1}{A_D}\oint_{\partial D}\left[\left(\overline{y_D} - y\right) f(x,y)\,dx + \left(x - \overline{x_D}\right) f(x,y)\,dy\right] \\
&= \frac{1}{A_D}\int_a^b \left[\left(\overline{y_D} - y(t)\right) x'(t) + \left(x(t) - \overline{x_D}\right) y'(t)\right] f(x(t), y(t))\,dt \\
&= \frac{1}{\pi R^2}\int_0^{2\pi}\left[\sin^2\theta + \cos^2\theta\right] R^2 f(R\cos\theta + a, R\sin\theta + b)\,d\theta \\
&= \frac{1}{\pi}\int_0^{2\pi} f(R\cos\theta + a, R\sin\theta + b)\,d\theta
\end{aligned}$$

and by (3.6) we get

$$(6.1)\quad \left|\frac{1}{\pi R^2}\int\int_{D(C,R)} f(x,y)\,dxdy - \frac{1}{2\pi}\int_0^{2\pi} f(R\cos\theta + a, R\sin\theta + b)\,d\theta\right| \\ \leq \frac{R}{3\pi}\left[\left|\Phi_1 - \phi_1\right| + \left|\Phi_2 - \phi_2\right|\right],$$

provided that $\frac{\partial f}{\partial x} \in \bar{\Delta}_{D(C,R)}(\phi_1, \Phi_1)$ and $\frac{\partial f}{\partial y} \in \bar{\Delta}_{D(C,R)}(\phi_2, \Phi_2)$ for some complex numbers $\phi_1 \neq \Phi_1$ and $\phi_2 \neq \Phi_2$.

We also have for $D(C,R)$ that

$$\begin{aligned}
\frac{1}{A_{D(C,R)}}\int\int_{D(C,R)}\left(x - \overline{x_{D(C,R)}}\right)^2 dxdy &= \frac{1}{\pi R^2}\int_0^R\int_0^{2\pi} r^3\cos^2\theta\,drd\theta \\
&= \frac{R^2}{4\pi}\int_0^{2\pi}\cos^2\theta\,d\theta = \frac{R^2}{4},
\end{aligned}$$

$$\begin{aligned}
\frac{1}{A_{D(C,R)}}\int\int_{D(C,R)}\left(y - \overline{y_{D(C,R)}}\right)^2 dxdy &= \frac{1}{\pi R^2}\int_0^R\int_0^{2\pi} r^3\sin^2\theta\,drd\theta \\
&= \frac{R^2}{4\pi}\int_0^{2\pi}\sin^2\theta\,d\theta = \frac{R^2}{4}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{A_{D(C,R)}}\int\int_{D(C,R)}\left|x - \overline{x_{D(C,R)}}\right|\left|y - \overline{y_{D(C,R)}}\right| dxdy \\
&= \frac{1}{\pi R^2}\int_0^R\int_0^{2\pi} r^3\left|\cos\theta\sin\theta\right| drd\theta \\
&= \frac{R^2}{4\pi}\int_0^{2\pi}\left|\cos\theta\sin\theta\right| d\theta = \frac{R^2}{8\pi}\int_0^{2\pi}\left|\sin 2\theta\right| d\theta \\
&= \frac{4R^2}{8\pi}\int_0^{\frac{\pi}{2}}\sin 2\theta\,d\theta = \frac{R^2}{2\pi}.
\end{aligned}$$

From the inequality (4.12) we get

$$\left|\frac{1}{\pi R^2}\int\int_{D(C,R)} f(x,y)\,dxdy-\frac{1}{2\pi}\int_0^{2\pi} f(R\cos\theta+a,R\sin\theta+b)\,d\theta\right| \leq \frac{1}{8}R^2\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D(C,R),\infty}+\frac{1}{8}R^2\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D(C,R),\infty}+\frac{1}{2\pi}R^2\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D(C,R),\infty}, \tag{6.2}$$

provided that $f: D(C,R)\to\mathbb{C}$ is twice differentiable on $D(C,R)$ and the second partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x\partial y}$ are bounded on $D(C,R)$

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