PERTURBED OSTROWSKI TYPE INEQUALITIES FOR DOUBLE INTEGRAL ON GENERAL DOMAINS

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Abstract. In this paper, by making use of Green's identity for double integral, we establish some perturbed Ostrowski type inequalities for functions of two independent variables defined on closed and bounded convex subsets of the plane $\mathbb{R}^2$. Some examples for rectangles and disks are also provided.

1. Introduction

In the following, consider D a closed and *bounded convex subset* of $\mathbb{R}^2$. Define

$$A_D := \int\int_D dxdy,$$

the *area* of D and $(\overline{x_D}, \overline{y_D})$ the *centre of mass* for D, where

$$\overline{x_D} := \frac{1}{A_D}\int\int_D xdxdy,\ \overline{y_D} := \frac{1}{A_D}\int\int_D ydxdy.$$

Consider the function of two variables $f = f(x,y)$ and denote by $\frac{\partial f}{\partial x}$ the partial derivative with respect to the variable x and $\frac{\partial f}{\partial y}$ the partial derivative with respect to the variable y.

In the recent paper [6] we obtained, among others, the following Ostrowski type integral inequality for double integral:

Theorem 1. *Assume that $f : D \to \mathbb{C}$ is differentiable on D and $(u,v) \in D$. Then*

$$\begin{aligned}(1.1)\quad & \left|f(u,v) - \frac{1}{A_D}\iint_D f(x,y)\,dxdy\right| \\ & \leq \frac{1}{A_D}\iint_D |x-u| \left(\int_0^1 \left|\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)]\right| dt\right) dxdy \\ & + \frac{1}{A_D}\iint_D |y-v| \left(\int_0^1 \left|\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)]\right| dt\right) dxdy \\ & \leq \left\|\frac{\partial f}{\partial x}\right\|_{D,\infty} \frac{1}{A_D}\iint_D |x-u|\,dxdy + \left\|\frac{\partial f}{\partial y}\right\|_{D,\infty} \frac{1}{A_D}\iint_D |y-v|\,dxdy,\end{aligned}$$

provided

$$\left\|\frac{\partial f}{\partial x}\right\|_{D,\infty} := \sup_{(z,w)\in D}\left|\frac{\partial f}{\partial x}(z,w)\right| < \infty \text{ and } \left\|\frac{\partial f}{\partial y}\right\|_{D,\infty} := \sup_{(z,w)\in D}\left|\frac{\partial f}{\partial y}(z,w)\right| < \infty.$$

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In particular,

$$\left|f\left(\overline{x_D},\overline{y_D}\right)-\frac{1}{A_D}\iint_D f\left(x,y\right)dxdy\right| \tag{1.2}$$
$$\leq\frac{1}{A_D}\iint_D\left|x-\overline{x_D}\right|\left(\int_0^1\left|\frac{\partial f}{\partial x}\left[t\left(x,y\right)+\left(1-t\right)\left(\overline{x_D},\overline{y_D}\right)\right]\right|dt\right)dxdy$$
$$+\frac{1}{A_D}\iint_D\left|y-\overline{y_D}\right|\left(\int_0^1\left|\frac{\partial f}{\partial y}\left[t\left(x,y\right)+\left(1-t\right)\left(\overline{x_D},\overline{y_D}\right)\right]\right|dt\right)dxdy$$
$$\leq\left\|\frac{\partial f}{\partial x}\right\|_{D,\infty}\frac{1}{A_D}\iint_D\left|x-\overline{x_D}\right|dxdy+\left\|\frac{\partial f}{\partial y}\right\|_{D,\infty}\frac{1}{A_D}\iint_D\left|y-\overline{y_D}\right|dxdy.$$

For other Ostrowski type integral inequalities for double integrals see [2]-[14].

Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a region D. Let L and M be scalar functions defined at least on an open set containing D. Assume L and M have continuous first partial derivatives. Then the following equality is well known as the *Green theorem* (see for instance https://en.wikipedia.org/wiki/Green%27s_theorem)

$$\int\int_D\left(\frac{\partial M\left(x,y\right)}{\partial x}-\frac{\partial L\left(x,y\right)}{\partial y}\right)dxdy=\oint_{\partial D}\left(L\left(x,y\right)dx+M\left(x,y\right)dy\right). \tag{G}$$

By applying this equality for real and imaginary parts, we can also state it for complex valued functions P and Q.

Moreover, if the curve ∂D is described by the function $r\left(t\right)=\left(x\left(t\right),y\left(t\right)\right)$, $t\in\left[a,b\right]$, with x, y differentiable on $\left(a,b\right)$ then we can calculate the *path integral* as

$$\oint_{\partial D}\left(L\left(x,y\right)dx+M\left(x,y\right)dy\right)=\int_a^b\left[L\left(x\left(t\right),y\left(t\right)\right)x'\left(t\right)+M\left(x\left(t\right),y\left(t\right)\right)y'\left(t\right)\right]dt.$$

In this paper we establish some perturbed Ostrowski type inequalities for functions of two independent variables defined on closed and bounded convex subsets of the plane $\mathbb{R}^2$. Some examples for rectangles and disks are also provided.

2. Preliminary Facts

We have:

Lemma 1. *If $f:D\to\mathbb{C}$ is differentiable on the convex domain D, then for all $\left(x,y\right)$, $\left(u,v\right)\in D$ we have the equality*

$$f\left(u,v\right)=f\left(x,y\right)+\left(u-x\right)\frac{\partial f}{\partial x}\left(x,y\right)+\left(v-y\right)\frac{\partial f}{\partial y}\left(x,y\right) \tag{2.1}$$
$$+\left(u-x\right)\int_0^1\left(\frac{\partial f}{\partial x}\left[t\left(u,v\right)+\left(1-t\right)\left(x,y\right)\right]-\frac{\partial f}{\partial x}\left(x,y\right)\right)dt$$
$$+\left(v-y\right)\int_0^1\left(\frac{\partial f}{\partial y}\left[t\left(u,v\right)+\left(1-t\right)\left(x,y\right)\right]-\frac{\partial f}{\partial y}\left(x,y\right)\right)dt.$$

Proof. By Taylor's multivariate theorem with integral remainder, we have

$$f(u,v)=f(x,y)+(u-x)\int_0^1\frac{\partial f}{\partial x}[t(u,v)+(1-t)(x,y)]\,dt+(v-y)\int_0^1\frac{\partial f}{\partial y}[t(u,v)+(1-t)(x,y)]\,dt \tag{2.2}$$

for all (x,y), $(u,v)\in D$.

Since

$$\begin{aligned}&(u-x)\int_0^1\left(\frac{\partial f}{\partial x}[t(u,v)+(1-t)(x,y)]-\frac{\partial f}{\partial x}(x,y)\right)dt\\&=(u-x)\int_0^1\frac{\partial f}{\partial x}[t(u,v)+(1-t)(x,y)]\,dt-(u-x)\frac{\partial f}{\partial x}(x,y)\end{aligned}$$

and

$$\begin{aligned}&(v-y)\int_0^1\left(\frac{\partial f}{\partial y}[t(u,v)+(1-t)(x,y)]-\frac{\partial f}{\partial y}(x,y)\right)dt\\&=(v-y)\int_0^1\frac{\partial f}{\partial y}[t(u,v)+(1-t)(x,y)]\,dt-(v-y)\frac{\partial f}{\partial y}(x,y),\end{aligned}$$

hence by (2.2) we get the desired result (2.1). □

Corollary 1. *With the assumptions of Lemma 1 we have*

$$\begin{aligned}f(u,v)&=\frac{1}{A_D}\int\int_D f(x,y)\,dxdy\\&+\frac{1}{A_D}\int\int_D\left[(u-x)\frac{\partial f(x,y)}{\partial x}+(v-y)\frac{\partial f(x,y)}{\partial y}\right]dxdy\\&+\frac{1}{A_D}\int\int_D\left[(u-x)\int_0^1\left(\frac{\partial f}{\partial x}[t(u,v)+(1-t)(x,y)]-\frac{\partial f}{\partial x}(x,y)\right)dt\right]dxdy\\&+\frac{1}{A_D}\int\int_D\left[(v-y)\int_0^1\left(\frac{\partial f}{\partial y}[t(u,v)+(1-t)(x,y)]-\frac{\partial f}{\partial y}(x,y)\right)dt\right]dxdy\end{aligned} \tag{2.3}$$

for all $(u,v)\in D$.

The equality (2.3) follows by (2.1) on taking the integral mean over $(x,y)\in D$.

We also have the following identity of interest:

Lemma 2. *Let ∂D be a simple, closed counterclockwise curve bounding a region D and f defined on an open set containing D and having continuous partial derivatives on D. Then for any u, $v\in\mathbb{C}$,*

$$\begin{aligned}&\frac{1}{A_D}\int\int_D f(x,y)\,dxdy\\&=\frac{1}{2A_D}\int\int_D\left[(u-x)\frac{\partial f(x,y)}{\partial x}+(v-y)\frac{\partial f(x,y)}{\partial y}\right]dxdy\\&\quad+\frac{1}{2A_D}\oint_{\partial D}[(v-y)f(x,y)\,dx+(x-u)f(x,y)\,dy].\end{aligned} \tag{2.4}$$

Proof. Observe that

$$\frac{\partial}{\partial x}\left(\left(x-u\right)f\left(x,y\right)\right)=f\left(x,y\right)+\left(x-u\right)\frac{\partial f\left(x,y\right)}{\partial x}$$

and

$$\frac{\partial}{\partial y}\left(\left(y-v\right)f\left(x,y\right)\right)=f\left(x,y\right)+\left(y-v\right)\frac{\partial f\left(x,y\right)}{\partial y}$$

for all $(x,y)\in D$ and if we add these equalities, we get

$$\begin{aligned}\frac{\partial}{\partial x}\left(\left(x-u\right)f\left(x,y\right)\right)+\frac{\partial}{\partial y}\left(\left(y-v\right)f\left(x,y\right)\right)\\=2f\left(x,y\right)+\left(x-u\right)\frac{\partial f\left(x,y\right)}{\partial x}+\left(y-v\right)\frac{\partial f\left(x,y\right)}{\partial y}.\end{aligned}\tag{2.5}$$

Further, if we integrate on D the identity (2.5), then we obtain

$$\begin{aligned}\int\int_{D}\left[\frac{\partial}{\partial x}\left(\left(x-u\right)f\left(x,y\right)\right)+\frac{\partial}{\partial y}\left(\left(y-v\right)f\left(x,y\right)\right)\right]dxdy\\=2\int\int_{D}f\left(x,y\right)dxdy\\+\int\int_{D}\left[\left(x-u\right)\frac{\partial f\left(x,y\right)}{\partial x}+\left(y-v\right)\frac{\partial f\left(x,y\right)}{\partial y}\right]dxdy.\end{aligned}\tag{2.6}$$

Now, if we apply Green's identity (G) for the functions $M\left(x,y\right)=\left(x-u\right)f\left(x,y\right)$ and $L\left(x,y\right)=\left(v-y\right)f\left(x,y\right)$ then we get

$$\begin{aligned}&\int\int_{D}\left[\frac{\partial}{\partial x}\left(\left(x-u\right)f\left(x,y\right)\right)+\frac{\partial}{\partial y}\left(\left(y-v\right)f\left(x,y\right)\right)\right]dxdy\\&=\oint_{\partial D}\left[\left(v-y\right)f\left(x,y\right)dx+\left(x-u\right)f\left(x,y\right)dy\right]\end{aligned}$$

and by (2.6) we conclude that

$$\begin{aligned}2\int\int_{D}f\left(x,y\right)dxdy+\int\int_{D}\left[\left(x-u\right)\frac{\partial f\left(x,y\right)}{\partial x}+\left(y-v\right)\frac{\partial f\left(x,y\right)}{\partial y}\right]dxdy\\=\oint_{\partial D}\left[\left(v-y\right)f\left(x,y\right)dx+\left(x-u\right)f\left(x,y\right)dy\right],\end{aligned}$$

which is equivalent to the desired equality (2.4). $\square$

Lemma 3. *With the assumptions of Lemma 2 and if D is a convex domain, then we have*

$$\begin{aligned}(2.7)\quad &\frac{1}{A_D}\int\int_D f(x,y)\,dxdy\\ &= \frac{1}{3}f(u,v)+\frac{2}{3}\frac{1}{A_D}\oint_{\partial D}\left[\frac{(v-y)f(x,y)\,dx+(x-u)f(x,y)\,dy}{2}\right]\\ &+\frac{1}{3A_D}\int\int_D\left[(x-u)\int_0^1\left(\frac{\partial f}{\partial x}\left[t(u,v)+(1-t)(x,y)\right]-\frac{\partial f}{\partial x}(x,y)\right)dt\right]dxdy\\ &+\frac{1}{3A_D}\int\int_D\left[(y-v)\int_0^1\left(\frac{\partial f}{\partial y}\left[t(u,v)+(1-t)(x,y)\right]-\frac{\partial f}{\partial y}(x,y)\right)dt\right]dxdy\end{aligned}$$

and

$$\begin{aligned}(2.8)\quad &\frac{1}{A_D}\int\int_D\left[(u-x)\frac{\partial f}{\partial x}(x,y)+(v-y)\frac{\partial f}{\partial y}(x,y)\right]dxdy\\ &= \frac{2}{3}f(u,v)+\frac{1}{3}\frac{1}{A_D}\oint_{\partial D}\left[(y-v)f(x,y)\,dx+(u-x)f(x,y)\,dy\right]\\ &+\frac{2}{3A_D}\int\int_D\left[(x-u)\int_0^1\left(\frac{\partial f}{\partial x}\left[t(u,v)+(1-t)(x,y)\right]-\frac{\partial f}{\partial x}(x,y)\right)dt\right]dxdy\\ &+\frac{2}{3A_D}\int\int_D\left[(y-v)\int_0^1\left(\frac{\partial f}{\partial y}\left[t(u,v)+(1-t)(x,y)\right]-\frac{\partial f}{\partial y}(x,y)\right)dt\right]dxdy.\end{aligned}$$

Proof. The proof of the identity (2.7) follows by eliminating the quantity

$$\frac{1}{A_D}\int\int_D\left[(u-x)\frac{\partial f}{\partial x}(x,y)+(v-y)\frac{\partial f}{\partial y}(x,y)\right]dxdy$$

in the equations (2.3) and (2.4).

The proof of the identity (2.8) follows eliminating the quantity

$$\frac{1}{A_D}\int\int_D f(x,y)\,dxdy$$

in the equations (2.3) and (2.4). □

We have the centre of mass identity:

Corollary 2. *With the assumptions of Lemma 3 we have*

$$\begin{aligned}(2.9)\quad &\frac{1}{A_D}\int\int_D f(x,y)\,dxdy\\ &= \frac{1}{3}f\left(\overline{x_D},\overline{y_D}\right)+\frac{2}{3}\frac{1}{A_D}\oint_{\partial D}\left[\frac{(\overline{y_D}-y)f(x,y)\,dx+(x-\overline{x_D})f(x,y)\,dy}{2}\right]\\ &+\frac{1}{3A_D}\int\int_D\left[(x-\overline{x_D})\int_0^1\left(\frac{\partial f}{\partial x}\left[t\left(\overline{x_D},\overline{y_D}\right)+(1-t)(x,y)\right]-\frac{\partial f}{\partial x}(x,y)\right)dt\right]dxdy\\ &+\frac{1}{3A_D}\int\int_D\left[(y-\overline{y_D})\int_0^1\left(\frac{\partial f}{\partial y}\left[t\left(\overline{x_D},\overline{y_D}\right)+(1-t)(x,y)\right]-\frac{\partial f}{\partial y}(x,y)\right)dt\right]dxdy\end{aligned}$$

and

$$\frac{1}{A_D}\int\int_D\left[\left(\overline{x_D}-x\right)\frac{\partial f}{\partial x}(x,y)+\left(\overline{y_D}-y\right)\frac{\partial f}{\partial y}(x,y)\right]dxdy \tag{2.10}$$
$$=\frac{2}{3}f\left(\overline{x_D},\overline{y_D}\right)+\frac{1}{3}\frac{1}{A_D}\oint_{\partial D}\left[\left(y-\overline{y_D}\right)f(x,y)\,dx+\left(\overline{x_D}-x\right)f(x,y)\,dy\right]$$
$$+\frac{2}{3A_D}\int\int_D\left[\left(x-\overline{x_D}\right)\int_0^1\left(\frac{\partial f}{\partial x}\left[t\left(\overline{x_D},\overline{y_D}\right)+(1-t)(x,y)\right]-\frac{\partial f}{\partial x}(x,y)\right)dt\right]dxdy$$
$$+\frac{2}{3A_D}\int\int_D\left[\left(y-\overline{y_D}\right)\int_0^1\left(\frac{\partial f}{\partial y}\left[t\left(\overline{x_D},\overline{y_D}\right)+(1-t)(x,y)\right]-\frac{\partial f}{\partial y}(x,y)\right)dt\right]dxdy.$$

We define the quantities

$$x_{f,\partial D}:=\frac{\oint_{\partial D}xf(x,y)\,dy}{\oint_{\partial D}f(x,y)\,dy}\text{ and }y_{f,\partial D}:=\frac{\oint_{\partial D}yf(x,y)\,dx}{\oint_{\partial D}f(x,y)\,dx}$$

provided the denominators are not zero.

Corollary 3. *With the assumptions of Lemma 3 we have*

$$\frac{1}{A_D}\int\int_D f(x,y)\,dxdy=\frac{1}{3}f\left(x_{f,\partial D},y_{f,\partial D}\right) \tag{2.11}$$
$$+\frac{1}{3A_D}\int\int_D\left[\left(x-x_{f,\partial D}\right)\int_0^1\left(\frac{\partial f}{\partial x}\left[t\left(x_{f,\partial D},y_{f,\partial D}\right)+(1-t)(x,y)\right]-\frac{\partial f}{\partial x}(x,y)\right)dt\right]dxdy$$
$$+\frac{1}{3A_D}\int\int_D\left[\left(y-y_{f,\partial D}\right)\int_0^1\left(\frac{\partial f}{\partial y}\left[t\left(x_{f,\partial D},y_{f,\partial D}\right)+(1-t)(x,y)\right]-\frac{\partial f}{\partial y}(x,y)\right)dt\right]dxdy.$$

The proof follows by (2.7) on observing that

$$\oint_{\partial D}\left[\frac{\left(y_{f,\partial D}-y\right)f(x,y)\,dx+\left(x-x_{f,\partial D}\right)f(x,y)\,dy}{2}\right]=0.$$

We define

$$x_{\partial f,D}:=\frac{\iint_D x\frac{\partial f}{\partial x}(x,y)\,dxdy}{\iint_D \frac{\partial f}{\partial x}(x,y)\,dxdy}\text{ and }y_{\partial f,D}:=\frac{\iint_D y\frac{\partial f}{\partial y}(x,y)\,dxdy}{\iint_D \frac{\partial f}{\partial y}(x,y)\,dxdy},$$

provided the denominators are not zero.

Corollary 4. *With the assumptions of Lemma 3 we have*

$$
\begin{aligned}
(2.12)\quad & f\left(x_{\partial f,D}, y_{\partial f,D}\right) \\
&= \frac{1}{2}\frac{1}{A_D}\oint_{\partial D}\left[\left(y_{\partial f,D}-y\right)f(x,y)\,dx+\left(x-x_{\partial f,D}\right)f(x,y)\,dy\right] \\
&+\frac{1}{A_D}\int\int_D\left[\left(x_{\partial f,D}-x\right)\int_0^1\left(\frac{\partial f}{\partial x}\left[t\left(x_{\partial f,D}, y_{\partial f,D}\right)+(1-t)(x,y)\right]-\frac{\partial f}{\partial x}(x,y)\right)dt\right]dxdy \\
&+\frac{1}{A_D}\int\int_D\left[\left(y_{\partial f,D}-y\right)\int_0^1\left(\frac{\partial f}{\partial y}\left[t\left(x_{\partial f,D}, y_{\partial f,D}\right)+(1-t)(x,y)\right]-\frac{\partial f}{\partial y}(x,y)\right)dt\right]dxdy.
\end{aligned}
$$

The proof follows by (2.7) on observing that

$$\int\int_D\left[\left(x_{\partial f,D}-x\right)\frac{\partial f}{\partial x}(x,y)+\left(y_{\partial f,D}-y\right)\frac{\partial f}{\partial y}(x,y)\right]dxdy=0.$$

3. Some Inequalities

We assume that the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ satisfy the *Lipschitz type conditions*

$$\left|\frac{\partial f}{\partial x}(x,y)-\frac{\partial f}{\partial x}(u,v)\right|\le L_1|x-u|+K_1|y-v| \tag{3.1}$$

and

$$\left|\frac{\partial f}{\partial y}(x,y)-\frac{\partial f}{\partial y}(u,v)\right|\le L_2|x-u|+K_2|y-v| \tag{3.2}$$

for any (x,y), $(u,v)\in D$, where L_1, K_1, L_2 and K_2 are given positive constants.

Theorem 2. *Let ∂D be a simple, closed counterclockwise curve bounding a convex region D and f defined on an open set containing D and having continuous partial derivatives on D. If the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ satisfy the Lipschitz type conditions (3.1) and (3.2), then*

$$
\begin{aligned}
(3.3)\quad & \left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy\right. \\
&\left.-\frac{1}{3}f(u,v)-\frac{2}{3}\frac{1}{A_D}\oint_{\partial D}\left[\frac{(v-y)f(x,y)\,dx+(x-u)f(x,y)\,dy}{2}\right]\right| \\
&\le\frac{1}{6}\left[L_1\frac{1}{A_D}\iint_D(x-u)^2\,dxdy+K_2\frac{1}{A_D}\iint_D(y-v)^2\,dxdy\right] \\
&\qquad+\frac{K_1+L_2}{6}\frac{1}{A_D}\iint_D|x-u|\,|y-v|\,dxdy
\end{aligned}
$$

and

$$\left|\frac{1}{A_D}\int\int_D\left[(u-x)\frac{\partial f}{\partial x}(x,y)+(v-y)\frac{\partial f}{\partial y}(x,y)\right]dxdy\right.\tag{3.4}$$
$$\left.-\frac{2}{3}f(u,v)-\frac{1}{3}\frac{1}{A_D}\oint_{\partial D}[(y-v)f(x,y)dx+(u-x)f(x,y)dy]\right|$$
$$\leq\frac{1}{3}\left[L_1\frac{1}{A_D}\iint_D(x-u)^2dxdy+K_2\frac{1}{A_D}\iint_D(y-v)^2dxdy\right]$$
$$+\frac{K_1+L_2}{3}\frac{1}{A_D}\iint_D|x-u||y-v|dxdy$$

for all $(u,v)\in D$.

Proof. From the identity (2.7) we have

$$\left|\frac{1}{A_D}\int\int_D f(x,y)dxdy\right.$$
$$\left.-\frac{1}{3}f(u,v)-\frac{2}{3}\frac{1}{A_D}\oint_{\partial D}\left[\frac{(v-y)f(x,y)dx+(x-u)f(x,y)dy}{2}\right]\right|$$
$$=\left|\frac{1}{3A_D}\int\int_D\left[(x-u)\int_0^1\left(\frac{\partial f}{\partial x}[t(u,v)+(1-t)(x,y)]-\frac{\partial f}{\partial x}(x,y)\right)dt\right]dxdy\right.$$
$$\left.+\frac{1}{3A_D}\int\int_D\left[(y-v)\int_0^1\left(\frac{\partial f}{\partial y}[t(u,v)+(1-t)(x,y)]-\frac{\partial f}{\partial y}(x,y)\right)dt\right]dxdy\right|$$

$$\leq\frac{1}{3A_D}\iint_D\left|(x-u)\left(\int_0^1\left(\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)]-\frac{\partial f}{\partial x}(u,v)\right)dt\right)\right|dxdy$$
$$+\frac{1}{3A_D}\iint_D\left|(y-v)\left(\int_0^1\left(\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)]-\frac{\partial f}{\partial y}(u,v)\right)dt\right)\right|dxdy$$
$$\leq\frac{1}{3A_D}\iint_D|x-u|\int_0^1\left|\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)]-\frac{\partial f}{\partial x}(u,v)\right|dtdxdy$$
$$+\frac{1}{3A_D}\iint_D|y-v|\left(\int_0^1\left|\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)]-\frac{\partial f}{\partial y}(u,v)\right|dt\right)dxdy$$
$$\leq\frac{1}{6}\frac{1}{A_D}\iint_D|x-u|[L_1|x-u|+K_1|y-v|]dxdy$$
$$+\frac{1}{6}\frac{1}{A_D}\iint_D|y-v|[L_2|x-u|+K_2|y-v|]dxdy$$
$$=\frac{1}{6}\left[L_1\frac{1}{A_D}\iint_D(x-u)^2dxdy+K_2\frac{1}{A_D}\iint_D(y-v)^2dxdy\right]$$
$$+\frac{K_1+L_2}{6}\frac{1}{A_D}\iint_D|x-u||y-v|dxdy,$$

which proves the desired result (3.3).

The inequality (3.4) follows in a similar way from the identity (2.8). □

Assume that $f : D \to \mathbb{C}$ is twice differentiable on D and the second partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are bounded on D. Put

$$\left\| \frac{\partial^2 f}{\partial x^2} \right\|_{D,\infty} := \sup_{(x,y)\in D} \left| \frac{\partial^2 f}{\partial x^2} (x,y) \right|, \quad \left\| \frac{\partial^2 f}{\partial y^2} \right\|_{D,\infty} := \sup_{(x,y)\in D} \left| \frac{\partial^2 f}{\partial y^2} (x,y) \right|$$

and

$$\left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_{D,\infty} := \sup_{(x,y)\in D} \left| \frac{\partial^2 f}{\partial x \partial y} (x,y) \right|,$$

then

$$\left| \frac{\partial f}{\partial x} (x,y) - \frac{\partial f}{\partial x} (u,v) \right| \leq \left\| \frac{\partial^2 f}{\partial x^2} \right\|_{D,\infty} |x-u| + \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_{D,\infty} |y-v|$$

and

$$\left| \frac{\partial f}{\partial y} (x,y) - \frac{\partial f}{\partial y} (u,v) \right| \leq \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_{D,\infty} |x-u| + \left\| \frac{\partial^2 f}{\partial y^2} \right\|_{D,\infty} |y-v|$$

for all (x,y), $(u,v) \in D$.

Therefore the conditions (3.1) and (3.2) are valid for

$$L_1 = \left\| \frac{\partial^2 f}{\partial x^2} \right\|_{D,\infty}, \; K_1 = L_2 = \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_{D,\infty}$$

and

$$K_2 = \left\| \frac{\partial^2 f}{\partial y^2} \right\|_{D,\infty}.$$

Corollary 5. *Let ∂D be a simple, closed counterclockwise curve bounding a convex region D and f defined on an open set containing D and twice differentiable on D. If the second partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are bounded, then*

$$\begin{aligned} &\left| \frac{1}{A_D} \int\int_D f(x,y)\,dxdy \right. \\ &\quad \left. - \frac{1}{3} f(u,v) - \frac{2}{3} \frac{1}{A_D} \oint_{\partial D} \left[\frac{(v-y) f(x,y)\,dx + (x-u) f(x,y)\,dy}{2} \right] \right| \\ &\leq \frac{1}{6} \left[\left\| \frac{\partial^2 f}{\partial x^2} \right\|_{D,\infty} \frac{1}{A_D} \iint_D (x-u)^2\,dxdy + \left\| \frac{\partial^2 f}{\partial y^2} \right\|_{D,\infty} \frac{1}{A_D} \iint_D (y-v)^2\,dxdy \right] \\ &\qquad + \frac{1}{3} \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_{D,\infty} \frac{1}{A_D} \iint_D |x-u|\,|y-v|\,dxdy \end{aligned} \tag{3.5}$$

and

$$\left|\frac{1}{A_D}\int\int_D\left[(u-x)\frac{\partial f}{\partial x}(x,y)+(v-y)\frac{\partial f}{\partial y}(x,y)\right]dxdy\right.$$
$$\left.-\frac{2}{3}f(u,v)-\frac{1}{3}\frac{1}{A_D}\oint_{\partial D}[(y-v)f(x,y)\,dx+(u-x)f(x,y)\,dy]\right|$$
$$\leq\frac{1}{3}\left[\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty}\frac{1}{A_D}\iint_D(x-u)^2\,dxdy+\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty}\frac{1}{A_D}\iint_D(y-v)^2\,dxdy\right]$$
$$+\frac{2}{3}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D,\infty}\frac{1}{A_D}\iint_D|x-u|\,|y-v|\,dxdy \tag{3.6}$$

for all $(u,v)\in D$.

Remark 1. *With the assumptions of Corollary 5 we have the following particular inequalities of interest*

$$\left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy\right.$$
$$\left.-\frac{1}{3}f(\overline{x_D},\overline{y_D})-\frac{2}{3}\frac{1}{A_D}\oint_{\partial D}\left[\frac{(\overline{y_D}-y)f(x,y)\,dx+(x-\overline{x_D})f(x,y)\,dy}{2}\right]\right|$$
$$\leq\frac{1}{6}\left[\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty}\frac{1}{A_D}\iint_D(x-\overline{x_D})^2\,dxdy+\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty}\frac{1}{A_D}\iint_D(\overline{y_D}-v)^2\,dxdy\right]$$
$$+\frac{1}{3}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D,\infty}\frac{1}{A_D}\iint_D|x-\overline{x_D}|\,|y-\overline{y_D}|\,dxdy, \tag{3.7}$$

$$\left|\frac{1}{A_D}\int\int_D\left[(\overline{x_D}-x)\frac{\partial f}{\partial x}(x,y)+(\overline{y_D}-y)\frac{\partial f}{\partial y}(x,y)\right]dxdy\right.$$
$$\left.-\frac{2}{3}f(\overline{x_D},\overline{y_D})-\frac{1}{3}\frac{1}{A_D}\oint_{\partial D}[(y-\overline{y_D})f(x,y)\,dx+(\overline{x_D}-x)f(x,y)\,dy]\right|$$
$$\leq\frac{1}{3}\left[\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty}\frac{1}{A_D}\iint_D(x-\overline{x_D})^2\,dxdy+\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty}\frac{1}{A_D}\iint_D(\overline{y_D}-y)^2\,dxdy\right]$$
$$+\frac{2}{3}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D,\infty}\frac{1}{A_D}\iint_D|x-\overline{x_D}|\,|y-\overline{y_D}|\,dxdy, \tag{3.8}$$

$$\left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy-\frac{1}{3}f(x_{f,\partial D},y_{f,\partial D})\right|$$
$$\leq\frac{1}{6}\left[\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty}\frac{1}{A_D}\iint_D(x-x_{f,\partial D})^2\,dxdy+\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty}\frac{1}{A_D}\iint_D(y-y_{f,\partial D})^2\,dxdy\right]$$
$$+\frac{1}{3}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D,\infty}\frac{1}{A_D}\iint_D|x-x_{f,\partial D}|\,|y-y_{f,\partial D}|\,dxdy \tag{3.9}$$

$$(3.10)\quad \left|\frac{1}{A_D}\int\int_D\left[(x_{f,\partial D}-x)\frac{\partial f}{\partial x}(x,y)+(y_{f,\partial D}-y)\frac{\partial f}{\partial y}(x,y)\right]dxdy -\frac{2}{3}f(x_{f,\partial D},y_{f,\partial D})\right|$$
$$\leq\frac{1}{3}\left[\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty}\frac{1}{A_D}\iint_D(x-x_{f,\partial D})^2dxdy+\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty}\frac{1}{A_D}\iint_D(y-y_{f,\partial D})^2dxdy\right]$$
$$+\frac{2}{3}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D,\infty}\frac{1}{A_D}\iint_D|x-x_{f,\partial D}||y-y_{f,\partial D}|dxdy$$

and

$$(3.11)\quad \left|f(x_{\partial f,D},y_{\partial f,D}) -\frac{1}{2}\frac{1}{A_D}\oint_{\partial D}[(y_{\partial f,D}-y)f(x,y)dx+(x-x_{\partial f,D})f(x,y)dy]\right|$$
$$\leq\frac{1}{2}\left[\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty}\frac{1}{A_D}\iint_D(x-x_{\partial f,D})^2dxdy+\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty}\frac{1}{A_D}\iint_D(y-y_{\partial f,D})^2dxdy\right]$$
$$+\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D,\infty}\frac{1}{A_D}\iint_D|x-x_{\partial f,D}||y-y_{\partial f,D}|dxdy.$$

4. Examples for Rectangles

Let $a<b$ and $c<d$. Put $A=(a,c)$, $B=(b,c)$, $C=(b,d)$, $D=(a,d)\in\mathbb{R}^2$ the vertices of the rectangle $ABCD=[a,b]\times[c,d]$. Consider the counterclockwise segments

$$AB:\begin{cases}x=(1-t)a+tb\\ y=c\end{cases},\ t\in[0,1]$$
$$BC:\begin{cases}x=b\\ y=(1-t)c+td\end{cases},\ t\in[0,1]$$
$$CD:\begin{cases}x=(1-t)b+ta\\ y=d\end{cases},\ t\in[0,1]$$

and

$$DA:\begin{cases}x=a\\ y=(1-t)d+tc\end{cases},\ t\in[0,1].$$

Therefore $\partial(ABCD)=AB\cup BC\cup CD\cup DA$.

If α, $\beta\in\mathbb{R}$, then

$$\oint_{AB}[(\beta-y)f(x,y)dx+(x-\alpha)f(x,y)dy]$$
$$=(b-a)(\beta-c)\int_0^1 f((1-t)a+tb,c)dt=(\beta-c)\int_a^b f(x,c)dx,$$

$$\begin{aligned}
&\oint_{BC} [(\beta - y) f(x,y)\,dx + (x-\alpha) f(x,y)\,dy] \\
&= (d-c)(b-\alpha)\int_0^1 f(b,(1-t)c+td)\,dt = (b-\alpha)\int_c^d f(b,y)\,dy,
\end{aligned}$$

$$\begin{aligned}
&\oint_{CD} [(\beta - y) f(x,y)\,dx + (x-\alpha) f(x,y)\,dy] \\
&= (a-b)(\beta-d)\int_0^1 f((1-t)b+ta,d)\,dt = (\beta-d)\int_b^a f(x,d)\,dx \\
&= (d-\beta)\int_a^b f(x,d)\,dx
\end{aligned}$$

and

$$\begin{aligned}
&\oint_{DA} [(\beta - y) f(x,y)\,dx + (x-\alpha) f(x,y)\,dy] \\
&= \int_0^1 (a-\alpha) f(a,(1-t)d+tc)(c-d)\,dt = (a-\alpha)\int_d^c f(a,y)\,dy \\
&= (\alpha-a)\int_c^d f(a,y)\,dy.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\oint_{ABCD} [(\beta - y) f(x,y)\,dx + (x-\alpha) f(x,y)\,dy] \\
&= (\beta-c)\int_a^b f(x,c)\,dx + (d-\beta)\int_a^b f(x,d)\,dx \\
&+ (b-\alpha)\int_c^d f(b,y)\,dy + (\alpha-a)\int_c^d f(a,y)\,dy
\end{aligned}$$

for all α, $\beta \in \mathbb{R}$.

We also have $\overline{x_D} = \frac{a+b}{2}$ and $\overline{y_D} = \frac{c+d}{2}$, which imply that

$$\begin{aligned}
&\oint_{\partial(ABCD)} [(\overline{y_D} - y) f(x,y)\,dx + (x-\overline{x_D}) f(x,y)\,dy] \\
&= (d-c)\int_a^b \left(\frac{f(x,c)+f(x,d)}{2}\right) dx + (b-a)\int_c^d \left(\frac{f(b,y)+f(a,y)}{2}\right) dy.
\end{aligned}$$

Let f defined on an open set containing $D = [a,b]\times[c,d]$ and twice differentiable on D. If the second partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are bounded on D, then by Theorem 2 we get

$$
\begin{aligned}
(4.1)\quad & \left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy-\frac{1}{3}f(u,v)\right. \\
& -\frac{2}{3}\frac{1}{(b-a)(d-c)}\left[(v-c)\int_a^b f(x,c)\,dx+(d-v)\int_a^b f(x,d)\,dx\right. \\
& \left.\left.+(b-u)\int_c^d f(b,y)\,dy+(u-a)\int_c^d f(a,y)\,dy\right]\right| \\
& \leq \frac{1}{6}\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty}\left[\frac{1}{12}+\left(\frac{u-\frac{b+a}{2}}{b-a}\right)^2\right](b-a)^2 \\
& +\frac{1}{6}\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty}\left[\frac{1}{12}+\left(\frac{v-\frac{d+c}{2}}{b-a}\right)^2\right](d-c)^2 \\
& +\frac{1}{3}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D,\infty}\left[\frac{1}{4}+\left(\frac{u-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{4}+\left(\frac{v-\frac{c+d}{2}}{d-c}\right)^2\right](b-a)(d-c)
\end{aligned}
$$

and

$$
\begin{aligned}
(4.2)\quad & \left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\left[(u-x)\frac{\partial f}{\partial x}(x,y)+(v-y)\frac{\partial f}{\partial y}(x,y)\right]dxdy\right. \\
& -\frac{2}{3}f(u,v)-\frac{1}{3}\frac{1}{(b-a)(d-c)}\left[(v-c)\int_a^b f(x,c)\,dx+(d-v)\int_a^b f(x,d)\,dx\right. \\
& \left.\left.+(b-u)\int_c^d f(b,y)\,dy+(u-a)\int_c^d f(a,y)\,dy\right]\right| \\
& \leq \frac{1}{3}\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty}\left[\frac{1}{12}+\left(\frac{u-\frac{b+a}{2}}{b-a}\right)^2\right](b-a)^2 \\
& +\frac{1}{3}\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty}\left[\frac{1}{12}+\left(\frac{v-\frac{d+c}{2}}{b-a}\right)^2\right](d-c)^2 \\
& +\frac{2}{3}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D,\infty}\left[\frac{1}{4}+\left(\frac{u-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{4}+\left(\frac{v-\frac{c+d}{2}}{d-c}\right)^2\right](b-a)(d-c)
\end{aligned}
$$

for all $(u,v)\in D$.

In particular, we have

$$\begin{aligned}
(4.3)\quad & \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dxdy - \frac{1}{3} f\left(\frac{a+b}{2}, \frac{d+c}{2}\right) \right. \\
& - \frac{2}{3} \left[\frac{1}{b-a} \int_a^b \left(\frac{f(x,c)+f(x,d)}{2} \right) dx \right. \\
& \left. \left. + \frac{1}{d-c} \int_c^d \left(\frac{f(b,y)+f(a,y)}{2} \right) dy \right] \right| \\
& \leq \frac{1}{72} \left\| \frac{\partial^2 f}{\partial x^2} \right\|_{D,\infty} (b-a)^2 + \frac{1}{72} \left\| \frac{\partial^2 f}{\partial y^2} \right\|_{D,\infty} (d-c)^2 \\
& \qquad + \frac{1}{48} \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_{D,\infty} (b-a)(d-c)
\end{aligned}$$

and

$$\begin{aligned}
(4.4)\quad & \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left(\frac{a+b}{2} - x \right) \frac{\partial f}{\partial x}(x,y)\,dxdy \right. \\
& + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left(\frac{d+c}{2} - y \right) \frac{\partial f}{\partial y}(x,y)\,dxdy \\
& - \frac{2}{3} f\left(\frac{a+b}{2}, \frac{d+c}{2}\right) - \frac{1}{3} \left[\frac{1}{b-a} \int_a^b \left(\frac{f(x,c)+f(x,d)}{2} \right) dx \right. \\
& \left. \left. + \frac{1}{d-c} \int_c^d \left(\frac{f(b,y)+f(a,y)}{2} \right) dy \right] \right| \\
& \leq \frac{1}{36} \left\| \frac{\partial^2 f}{\partial x^2} \right\|_{D,\infty} (b-a)^2 + \frac{1}{36} \left\| \frac{\partial^2 f}{\partial y^2} \right\|_{D,\infty} (d-c)^2 \\
& \qquad + \frac{1}{24} \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_{D,\infty} (b-a)(d-c).
\end{aligned}$$

5. Examples for Disks

We consider the closed disk $D(C,R)$ centered in $C(a,b)$ and of radius $R>0$. This is parametrized by

$$\begin{cases} x = r\cos\theta + a \\ y = r\sin\theta + b \end{cases}, \quad r \in [0,R],\ \theta \in [0,2\pi]$$

and the circle $\mathcal{C}(C,R)$ is parametrized by

$$\begin{cases} x = R\cos\theta + a \\ y = R\sin\theta + b \end{cases}, \quad \theta \in [0,2\pi].$$

Here $\overline{x_{D(C,R)}} = a$, $\overline{y_{D(C,R)}} = b$ and $A_{D(C,R)} = \pi R^2$.

Then

$$\begin{aligned}
&\frac{1}{A_D}\int_a^b \left[\left(\overline{y_D}-y(t)\right)x'(t)+\left(x(t)-\overline{x_D}\right)y'(t)\right]f(x(t),y(t))\,dt \\
&=\frac{1}{\pi R^2}\int_0^{2\pi}\left[\sin^2\theta+\cos^2\theta\right]R^2 f(R\cos\theta+a,R\sin\theta+b)\,d\theta \\
&=\frac{1}{\pi}\int_0^{2\pi} f(R\cos\theta+a,R\sin\theta+b)\,d\theta
\end{aligned}$$

and

$$\begin{aligned}
&\int\int_D\left[\left(\overline{x_D}-x\right)\frac{\partial f(x,y)}{\partial x}+\left(\overline{y_D}-y\right)\frac{\partial f(x,y)}{\partial y}\right]dxdy \\
&\qquad=-\int_0^R\int_0^{2\pi}\left[\frac{\partial f(r\cos\theta+a,r\sin\theta+b)}{\partial x}\cos\theta\right. \\
&\qquad\qquad\left.+\frac{\partial f(r\cos\theta+a,r\sin\theta+b)}{\partial y}\sin\theta\right]r^2 d\theta dr.
\end{aligned}$$

Moreover,

$$\frac{1}{A_D}\iint_D\left(x-\overline{x_D}\right)^2dxdy=\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}r^3\cos^2\theta drd\theta=\frac{1}{4}R^2,$$

$$\frac{1}{A_D}\iint_D\left(\overline{y_D}-y\right)^2dxdy=\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}r^3\sin^2\theta drd\theta=\frac{1}{4}R^2$$

and

$$\frac{1}{A_D}\iint_D\left|x-\overline{x_D}\right|\left|y-\overline{y_D}\right|dxdy=\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}r^3\left|\sin\theta\cos\theta\right|drd\theta=\frac{1}{2\pi}R^2.$$

From Corollary 5 we then have

$$\begin{aligned}
&\left|\frac{1}{\pi R^2}\int\int_{D(C,R)}f(x,y)\,dxdy-\frac{1}{3}f(a,b)\right. \\
&\qquad\left.-\frac{2}{3}\frac{1}{\pi}\int_0^{2\pi}f(R\cos\theta+a,R\sin\theta+b)\,d\theta\right| \\
&\qquad\le\frac{1}{24}\left[\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty}+\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty}\right]R^2+\frac{1}{6\pi}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D,\infty}R^2
\end{aligned}\tag{5.1}$$

and

$$\begin{aligned}
&\left|\int_0^R\int_0^{2\pi}\left[\frac{\partial f(r\cos\theta+a,r\sin\theta+b)}{\partial x}\cos\theta\right.\right. \\
&\qquad\left.+\frac{\partial f(r\cos\theta+a,r\sin\theta+b)}{\partial y}\sin\theta\right]r^2d\theta dr \\
&\qquad\left.+\frac{2}{3}f(a,b)-\frac{1}{3\pi}\int_0^{2\pi}f(R\cos\theta+a,R\sin\theta+b)\,d\theta\right| \\
&\qquad\le\frac{1}{12}\left[\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty}+\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty}\right]R^2+\frac{1}{3\pi}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{D,\infty}R^2.
\end{aligned}\tag{5.2}$$

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