SOME TRIPLE INTEGRAL INEQUALITIES FOR FUNCTIONS DEFINED ON 3-DIMENSIONAL BODIES VIA GAUSS-OSTROGRADSKY IDENTITY

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Abstract. In this paper, by the use of Gauss-Ostrogradsky identity, we establish some inequalities for functions of three variables defined on closed and bounded bodies of the Euclidean space $\mathbb{R}^3$. Some examples for 3-dimensional balls are also provided.

1. Introduction

Recall the following inequalities of Hermite-Hadamard's type for convex functions defined on a ball $B(C,R)$, where $C=(a,b,c)\in\mathbb{R}^3$, $R>0$ and

$$B(C,R):=\left\{(x,y,z)\in\mathbb{R}^3\,\middle|\,(x-a)^2+(y-b)^2+(z-c)^2\le R^2\right\}.$$

The following theorem holds [10].

Theorem 1. *Let $f:B(C,R)\to\mathbb{R}$ be a convex function on the ball $B(C,R)$. Then we have the inequality:*

$$f(a,b,c)\le\frac{1}{V(B(C,R))}\iiint_{B(C,R)}f(x,y,z)\,dxdydz\le\frac{1}{\sigma(B(C,R))}\iint_{S(C,R)}f(x,y,z)\,dS, \tag{1.1}$$

where

$$S(C,R):=\left\{(x,y,z)\in\mathbb{R}^3\,\middle|\,(x-a)^2+(y-b)^2+(z-c)^2=R^2\right\}$$

and

$$V(B(C,R))=\frac{4\pi R^3}{3},\ \sigma(B(C,R))=4\pi R^2.$$

If the assumption of convexity is dropped, then one can prove the following Ostrowski type inequality for the centre of the ball as well, see [11].

Theorem 2. *Assume that $f:B(C,R)\to\mathbb{C}$ is differentiable on $B(C,R)$. Then*

$$\left|f(a,b,c)-\frac{1}{V(B(C,R))}\iiint_{B(C,R)}f(x,y,z)\,dxdydz\right|\le\frac{3}{8}R\left[\left\|\frac{\partial f}{\partial x}\right\|_{B(C,R),\infty}+\left\|\frac{\partial f}{\partial y}\right\|_{B(C,R),\infty}+\left\|\frac{\partial f}{\partial z}\right\|_{B(C,R),\infty}\right], \tag{1.2}$$

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provided

$$\left\|\frac{\partial f}{\partial x}\right\|_{B(C,R),\infty}:=\sup_{(x,y,z)\in B(C,R)}\left|\frac{\partial f(x,y,z)}{\partial x}\right|<\infty\ ,$$

$$\left\|\frac{\partial f}{\partial y}\right\|_{B(C,R),\infty}:=\sup_{(x,y,z)\in B(C,R)}\left|\frac{\partial f(x,y,z)}{\partial y}\right|<\infty$$

and

$$\left\|\frac{\partial f}{\partial z}\right\|_{B(C,R),\infty}:=\sup_{(x,y,z)\in B(C,R)}\left|\frac{\partial f(x,y,z)}{\partial y}\right|<\infty.$$

This fact can be furthermore generalized to the following Ostrowski type inequality for any point in a convex body $B\subset\mathbb{R}^3$, see [11].

Theorem 3. *Assume that $f:B\to\mathbb{C}$ is differentiable on the convex body B and $(u,v,w)\in B$. If $V(B)$ is the volume of B, then*

$$\begin{aligned}
&\left|f(u,v,w)-\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz\right| \qquad (1.3)\\
&\quad\le\frac{1}{V(B)}\iiint_B|x-u|\left(\int_0^1\left|\frac{\partial f}{\partial x}[t(x,y,z)+(1-t)(u,v,w)]\right|dt\right)dxdydz\\
&\quad+\frac{1}{V(B)}\iiint_B|y-v|\left(\int_0^1\left|\frac{\partial f}{\partial y}[t(x,y,z)+(1-t)(u,v,w)]\right|dt\right)dxdydz\\
&\quad+\frac{1}{V(B)}\iiint_B|z-w|\left(\int_0^1\left|\frac{\partial f}{\partial y}[t(x,y,z)+(1-t)(u,v,w)]\right|dt\right)dxdydz\\
&\le\left\|\frac{\partial f}{\partial x}\right\|_{B,\infty}\frac{1}{V(B)}\iiint_B|x-u|\,dxdydz+\left\|\frac{\partial f}{\partial y}\right\|_{B,\infty}\frac{1}{V(B)}\iiint_B|y-v|\,dxdydz\\
&\qquad\qquad+\left\|\frac{\partial f}{\partial z}\right\|_{B,\infty}\frac{1}{V(B)}\iiint_B|z-w|\,dxdydz
\end{aligned}$$

provided

$$\left\|\frac{\partial f}{\partial x}\right\|_{B,\infty},\left\|\frac{\partial f}{\partial y}\right\|_{B,\infty},\left\|\frac{\partial f}{\partial z}\right\|_{B,\infty}<\infty.$$

In particular,

$$\begin{aligned}
&\left|f(\overline{x_B},\overline{y_B},\overline{z_B})-\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz\right| \qquad (1.4)\\
&\le\frac{1}{V(B)}\iiint_B|x-\overline{x_B}|\left(\int_0^1\left|\frac{\partial f}{\partial x}[t(x,y,z)+(1-t)(\overline{x_B},\overline{y_B},\overline{z_B})]\right|dt\right)dxdydz\\
&+\frac{1}{V(B)}\iiint_B|y-\overline{y_B}|\left(\int_0^1\left|\frac{\partial f}{\partial y}[t(x,y,z)+(1-t)(\overline{x_B},\overline{y_B},\overline{z_B})]\right|dt\right)dxdydz\\
&+\frac{1}{V(B)}\iiint_B|z-\overline{z_B}|\left(\int_0^1\left|\frac{\partial f}{\partial y}[t(x,y,z)+(1-t)(\overline{x_B},\overline{y_B},\overline{z_B})]\right|dt\right)dxdydz\\
&\le\left\|\frac{\partial f}{\partial x}\right\|_{B,\infty}\frac{1}{V(B)}\iiint_B|x-\overline{x_B}|\,dxdydz+\left\|\frac{\partial f}{\partial y}\right\|_{B,\infty}\frac{1}{V(B)}\iiint_B|y-\overline{y_B}|\,dxdydz\\
&\qquad\qquad+\left\|\frac{\partial f}{\partial z}\right\|_{B,\infty}\frac{1}{V(B)}\iiint_B|z-\overline{z_B}|\,dxdydz,
\end{aligned}$$

where

$$\overline{x_B} := \frac{1}{V(B)} \iiint_B x dx dy dz, \ \overline{y_B} = \frac{1}{V(B)} \iiint_B y dx dy dz, \ \overline{z_B} = \frac{1}{V(B)} \iiint_B z dx dy dz$$

are the centre of gravity coordinates for the convex body B.

For some Hermite-Hadamard type inequalities for multiple integrals see [2], [6], [8], [9], [10], [17], [18], [19], [20], [25], [26] and [27]. For some Ostrowski type inequalities see [3], [4], [5], [7], [11], [12], [13], [14], [15], [16], [21], [22], [23] and [24].

In this paper we establish some error bounds in approximating the triple integral

$$\frac{1}{V(B)} \iiint_B f(x,y,z) \, dx dy dz$$

by either the surface integrals

$$\frac{1}{3} \left[\int \int_S (x-\alpha) f(x,y,z) \, dy \wedge dz + \int \int_S (y-\beta) f(x,y,z) \, dz \wedge dx \right. \\ \left. + \int \int_S (z-\gamma) f(x,y,z) \, dx \wedge dy \right] \tag{1.5}$$

or by, the possibly simpler, triple integrals

$$\frac{1}{3} \iiint_B \left[(\alpha - x) \frac{\partial f(x,y,z)}{\partial x} + (\beta - y) \frac{\partial f(x,y,z)}{\partial y} \right. \\ \left. + (\gamma - z) \frac{\partial f(x,y,z)}{\partial z} \right] dx dy dz \tag{1.6}$$

for some α, β and γ complex numbers.

Examples for functions defined on a ball $B(C,R)$ centered in $C = (a,b,c) \in \mathbb{R}^3$ and with the radius $R > 0$ are also provided.

2. Some Preliminary Facts

Following Apostol [1], consider a surface described by the vector equation

$$r(u,v) = x(u,v) \overrightarrow{i} + y(u,v) \overrightarrow{j} + z(u,v) \overrightarrow{k} \tag{2.1}$$

where $(u,v) \in [a,b] \times [c,d]$.

If x, y, z are differentiable on $[a,b] \times [c,d]$ we consider the two vectors

$$\frac{\partial r}{\partial u} = \frac{\partial x}{\partial u} \overrightarrow{i} + \frac{\partial y}{\partial u} \overrightarrow{j} + \frac{\partial z}{\partial u} \overrightarrow{k}$$

and

$$\frac{\partial r}{\partial v} = \frac{\partial x}{\partial v} \overrightarrow{i} + \frac{\partial y}{\partial v} \overrightarrow{j} + \frac{\partial z}{\partial v} \overrightarrow{k}.$$

The *cross product* of these two vectors $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ will be referred to as the fundamental vector product of the representation r. Its components can be expressed as *Jacobian* *determinants*. In fact, we have [1, p. 420]

$$\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}=\begin{vmatrix}\frac{\partial y}{\partial u} & \frac{\partial z}{\partial u}\\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v}\end{vmatrix}\overrightarrow{i}+\begin{vmatrix}\frac{\partial z}{\partial u} & \frac{\partial x}{\partial u}\\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v}\end{vmatrix}\overrightarrow{j}+\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix}\overrightarrow{k} \tag{2.2}$$

$$=\frac{\partial(y,z)}{\partial(u,v)}\overrightarrow{i}+\frac{\partial(z,x)}{\partial(u,v)}\overrightarrow{j}+\frac{\partial(x,y)}{\partial(u,v)}\overrightarrow{k}.$$

Let $S=r(T)$ be a parametric surface described by a vector-valued function r defined on the box $T=[a,b]\times[c,d]$. The area of S denoted A_S is defined by the double integral [1, p. 424-425]

$$A_S=\int_a^b\int_c^d\left\|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right\|dudv \tag{2.3}$$

$$=\int_a^b\int_c^d\sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2+\left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2+\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2}dudv.$$

We define surface integrals in terms of a parametric representation for the surface. One can prove that under certain general conditions the value of the integral is independent of the representation.

Let $S=r(T)$ be a parametric surface described by a vector-valued differentiable function r defined on the box $T=[a,b]\times[c,d]$ and let $f:S\to\mathbb{C}$ defined and bounded on S. The surface integral of f over S is defined by [1, p. 430]

$$\int\int_S fdS=\int_a^b\int_c^d f(x,y,z)\left\|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right\|dudv \tag{2.4}$$

$$=\int_a^b\int_c^d f(x(u,v),y(u,v),z(u,v))$$

$$\times\sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2+\left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2+\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2}dudv.$$

If $S=r(T)$ is a parametric surface, the fundamental vector product $N=\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}$ is normal to S at each regular point of the surface. At each such point there are two unit normals, a unit normal n_1, which has the same direction as N, and a unit normal n_2 which has the opposite direction. Thus

$$n_1=\frac{N}{\|N\|}\text{ and }n_2=-n_1.$$

Let n be one of the two normals n_1 or n_2. Let also F be a vector field defined on S and assume that the surface integral,

$$\int\int_S(F\cdot n)\,dS,$$

called *the flux surface integral,* exists. Here $F\cdot n$ is the dot or inner product.

We can write [1, p. 434]

$$\int\int_S(F\cdot n)\,dS=\pm\int_a^b\int_c^d F(r(u,v))\cdot\left(\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right)dudv$$

where the sign " $+$ " is used if $n=n_1$ and the " $-$ " sign is used if $n=n_2$.

If

$$F(x,y,z) = P(x,y,z)\overrightarrow{i} + Q(x,y,z)\overrightarrow{j} + R(x,y,z)\overrightarrow{k}$$

and

$$r(u,v) = x(u,v)\overrightarrow{i} + y(u,v)\overrightarrow{j} + z(u,v)\overrightarrow{k} \text{ where } (u,v) \in [a,b]\times[c,d]$$

then the flux surface integral for $n = n_1$ can be explicitly calculated as [1, p. 435]

$$\begin{aligned} \int\int_S (F\cdot n)\,dS = &\int_a^b\int_c^d P(x(u,v),y(u,v),z(u,v))\frac{\partial(y,z)}{\partial(u,v)}dudv \\ &+\int_a^b\int_c^d Q(x(u,v),y(u,v),z(u,v))\frac{\partial(z,x)}{\partial(u,v)}dudv \\ &+\int_a^b\int_c^d R(x(u,v),y(u,v),z(u,v))\frac{\partial(x,y)}{\partial(u,v)}dudv. \end{aligned} \tag{2.5}$$

The sum of the double integrals on the right is often written more briefly as [1, p. 435]

$$\int\int_S P(x,y,z)\,dy\wedge dz + \int\int_S Q(x,y,z)\,dz\wedge dx + \int\int_S R(x,y,z)\,dx\wedge dy.$$

Let $B\subset\mathbb{R}^3$ be a solid in 3-space bounded by an orientable closed surface S, and let n be the unit outer normal to S. If F is a continuously differentiable vector field defined on B, we have the *Gauss-Ostrogradsky identity*

$$\iiint_B (\operatorname{div} F)\,dV = \int\int_S (F\cdot n)\,dS. \tag{GO}$$

If we express

$$F(x,y,z) = P(x,y,z)\overrightarrow{i} + Q(x,y,z)\overrightarrow{j} + R(x,y,z)\overrightarrow{k},$$

then (GO) can be written as

$$\begin{aligned} &\iiint_B \left(\frac{\partial P(x,y,z)}{\partial x} + \frac{\partial Q(x,y,z)}{\partial y} + \frac{\partial R(x,y,z)}{\partial z}\right)dxdydz \\ &\quad= \int\int_S P(x,y,z)\,dy\wedge dz + \int\int_S Q(x,y,z)\,dz\wedge dx \\ &\qquad\qquad + \int\int_S R(x,y,z)\,dx\wedge dy. \end{aligned} \tag{2.6}$$

By taking the real and imaginary part, we can extend the above inequality for complex valued functions P, Q, R defined on B.

3. Identities of Interest

We have:

Lemma 1. *Let B be a solid in the three dimensional space $\mathbb{R}^3$ bounded by an orientable closed surface S. If $f : B \to \mathbb{C}$ is a continuously differentiable function* *defined on a open set containing B, then we have the equality*

$$\begin{aligned}(3.1)\quad & \iiint_B f(x,y,z)\,dxdydz \\ &= \frac{1}{3}\iiint_B \left[(\alpha - x)\frac{\partial f(x,y,z)}{\partial x} + (\beta - y)\frac{\partial f(x,y,z)}{\partial y} \right. \\ &\qquad \left. + (\gamma - z)\frac{\partial f(x,y,z)}{\partial z}\right] dxdydz \\ &+ \frac{1}{3}\left[\int\int_S (x-\alpha) f(x,y,z)\,dy\wedge dz + \int\int_S (y-\beta) f(x,y,z)\,dz\wedge dx \right. \\ &\qquad \left. + \int\int_S (z-\gamma) f(x,y,z)\,dx\wedge dy\right]\end{aligned}$$

for all α, β and γ complex numbers.

In particular, we have

$$\begin{aligned}(3.2)\quad & \iiint_B f(x,y,z)\,dxdydz \\ &= \frac{1}{3}\iiint_B \left[(\overline{x_B} - x)\frac{\partial f(x,y,z)}{\partial x} + (\overline{y_B} - y)\frac{\partial f(x,y,z)}{\partial y} \right. \\ &\qquad \left. + (\overline{z_B} - z)\frac{\partial f(x,y,z)}{\partial z}\right] dxdydz \\ &+ \frac{1}{3}\left[\int\int_S (x-\overline{x_B}) f(x,y,z)\,dy\wedge dz + \int\int_S (y-\overline{y_B}) f(x,y,z)\,dz\wedge dx \right. \\ &\qquad \left. + \int\int_S (z-\overline{z_B}) f(x,y,z)\,dx\wedge dy\right].\end{aligned}$$

Proof. We have

$$\frac{\partial[(x-\alpha) f(x,y,z)]}{\partial x} = f(x,y,z) + (x-\alpha)\frac{\partial f(x,y,z)}{\partial x},$$
$$\frac{\partial[(y-\beta) f(x,y,z)]}{\partial y} = f(x,y,z) + (y-\beta)\frac{\partial f(x,y,z)}{\partial y}$$

and

$$\frac{\partial[(z-\gamma) f(x,y,z)]}{\partial z} = f(x,y,z) + (z-\gamma)\frac{\partial f(x,y,z)}{\partial z}.$$

By adding these three equalities we get

$$\begin{aligned}(3.3)\quad & \frac{\partial[(x-\alpha) f(x,y,z)]}{\partial x} + \frac{\partial[(y-\beta) f(x,y,z)]}{\partial y} + \frac{\partial[(z-\gamma) f(x,y,z)]}{\partial z} \\ &= 3f(x,y,z) \\ &+ (x-\alpha)\frac{\partial f(x,y,z)}{\partial x} + (y-\beta)\frac{\partial f(x,y,z)}{\partial y} + (z-\gamma)\frac{\partial f(x,y,z)}{\partial z}\end{aligned}$$

for all $(x,y,z) \in B$.

Integrating this equality on B we get

$$
\begin{aligned}
(3.4)\quad & \iiint_B \left(\frac{\partial\left[(x-\alpha) f(x,y,z)\right]}{\partial x} + \frac{\partial\left[(y-\beta) f(x,y,z)\right]}{\partial y} \right.\\
&\qquad \left. + \frac{\partial\left[(z-\gamma) f(x,y,z)\right]}{\partial z} \right) dxdydz \\
&\qquad = 3 \iiint_B f(x,y,z)\, dxdydz \\
&\qquad + \iiint_B \left[(x-\alpha) \frac{\partial f(x,y,z)}{\partial x} + (y-\beta) \frac{\partial f(x,y,z)}{\partial y} \right.\\
&\qquad\qquad \left. + (z-\gamma) \frac{\partial f(x,y,z)}{\partial z} \right] dxdydz.
\end{aligned}
$$

Applying the *Gauss-Ostrogradsky identity (2.6)* for the functions

$$P(x,y,z) = (x-\alpha) f(x,y,z),\ Q(x,y,z) = (y-\beta) f(x,y,z)$$

and

$$R(x,y,z) = (z-\gamma) f(x,y,z)$$

we obtain

$$
\begin{aligned}
(3.5)\quad & \iiint_B \left(\frac{\partial\left[(x-\alpha) f(x,y,z)\right]}{\partial x} + \frac{\partial\left[(y-\beta) f(x,y,z)\right]}{\partial y} \right.\\
&\qquad \left. + \frac{\partial\left[(z-\gamma) f(x,y,z)\right]}{\partial z} \right) dxdydz \\
&= \int\int_S (x-\alpha) f(x,y,z)\, dy\wedge dz + \int\int_S (y-\beta) f(x,y,z)\, dz\wedge dx \\
&\qquad + \int\int_S (z-\gamma) f(x,y,z)\, dx\wedge dy.
\end{aligned}
$$

By (3.4) and (3.5) we get

$$
\begin{aligned}
& 3 \iiint_B f(x,y,z)\, dxdydz \\
& + \iiint_B \left[(x-\alpha) \frac{\partial f(x,y,z)}{\partial x} + (y-\beta) \frac{\partial f(x,y,z)}{\partial y} + (z-\gamma) \frac{\partial f(x,y,z)}{\partial z} \right] dxdydz \\
&= \int\int_S (x-\alpha) f(x,y,z)\, dy\wedge dz + \int\int_S (y-\beta) f(x,y,z)\, dz\wedge dx \\
&\qquad + \int\int_S (z-\gamma) f(x,y,z)\, dx\wedge dy,
\end{aligned}
$$

which is equivalent to the desired result (3.1). □

Remark 1. *For a function f as in Lemma 1 above, we define the points*

$$x_{B,\partial f} := \frac{\iiint_B x \frac{\partial f(x,y,z)}{\partial x} dxdydz}{\iiint_B \frac{\partial f(x,y,z)}{\partial x} dxdydz},\ y_{B,\partial f} := \frac{\iiint_B y \frac{\partial f(x,y,z)}{\partial y} dxdydz}{\iiint_B \frac{\partial f(x,y,z)}{\partial y} dxdydz},$$

and

$$z_{B,\partial f} := \frac{\iiint_B z \frac{\partial f(x,y,z)}{\partial z} dxdydz}{\iiint_B \frac{\partial f(x,y,z)}{\partial z} dxdydz}$$

provided the denominators are not zero.

If we take $\alpha = x_{B,\partial f}$, $\beta = y_{B,\partial f}$ *and* $\gamma = z_{B,\partial f}$ *in (3.1), then we get*

$$\iiint_B f(x,y,z)\,dxdydz \tag{3.6}$$
$$= \frac{1}{3}\left[\int\int_S (x - x_{B,\partial f}) f(x,y,z)\,dy \wedge dz + \int\int_S (y - \beta y_{B,\partial f}) f(x,y,z)\,dz \wedge dx \right.$$
$$\left. + \int\int_S (z - z_{B,\partial f}) f(x,y,z)\,dx \wedge dy\right],$$

since, obviously,

$$\iiint_B \left[(x_{B,\partial f} - x)\frac{\partial f(x,y,z)}{\partial x} + (y_{B,\partial f} - y)\frac{\partial f(x,y,z)}{\partial y}\right.$$
$$\left. + (z_{B,\partial f} - z)\frac{\partial f(x,y,z)}{\partial z}\right] dxdydz = 0.$$

We also have the following dual approach:

Remark 2. *For a function* f *as in Lemma 1 above, we define the points*

$$x_{S,f} := \frac{\int\int_S xf(x,y,z)\,dy \wedge dz}{\int\int_S f(x,y,z)\,dy \wedge dz},\quad y_{S,f} := \frac{\int\int_S yf(x,y,z)\,dz \wedge dx}{\int\int_S f(x,y,z)\,dz \wedge dx}$$

and

$$z_{S,f} := \frac{\int\int_S zf(x,y,z)\,dx \wedge dy}{\int\int_S f(x,y,z)\,dx \wedge dy}$$

provided the denominators are not zero.

If we take $\alpha = x_{S,f}$, $\beta = y_{S,f}$ *and* $\gamma = z_{S,f}$ *in (3.1), then we get*

$$\iiint_B f(x,y,z)\,dxdydz \tag{3.7}$$
$$= \frac{1}{3}\iiint_B \left[(x_{S,f} - x)\frac{\partial f(x,y,z)}{\partial x} + (y_{S,f} - y)\frac{\partial f(x,y,z)}{\partial y}\right.$$
$$\left. + (z_{S,f} - z)\frac{\partial f(x,y,z)}{\partial z}\right] dxdydz$$

since, obviously,

$$\int\int_S (x - x_{S,f}) f(x,y,z)\,dy \wedge dz + \int\int_S (y - y_{S,f}) f(x,y,z)\,dz \wedge dx$$
$$+ \int\int_S (z - z_{S,f}) f(x,y,z)\,dx \wedge dy = 0.$$

4. Integral Inequalities

For a measurable function $g : B \to \mathbb{C}$ we define the *Lebesgue norms*

$$\|g\|_{B,p} := \left(\iiint_B |g(x,y,z)|^p\,dxdydz\right)^{1/p} < \infty$$

for $p \geq 1$ and

$$\|g\|_{B,\infty} := \sup_{(x,y,z)\in B} |g(x,y,z)| < \infty$$

for $p=\infty$.

We have:

Theorem 4. *Let B be a solid in the three dimensional space $\mathbb{R}^3$ bounded by an orientable closed surface S. If $f:B\to\mathbb{C}$ is a continuously differentiable function defined on a open set containing B, then for all α, β, γ complex numbers we have the inequality*

$$\begin{aligned}(4.1)\quad & \left|\iiint_B f(x,y,z)\,dxdydz-\frac{1}{3}\left[\int\int_S (x-\alpha)f(x,y,z)\,dy\wedge dz\right.\right.\\ &\left.\left.+\int\int_S (y-\beta)f(x,y,z)\,dz\wedge dx+\int\int_S (z-\gamma)f(x,y,z)\,dx\wedge dy\right]\right|\\ &\le\frac{1}{3}\iiint_B\left[|\alpha-x|\left|\frac{\partial f(x,y,z)}{\partial x}\right|+|\beta-y|\left|\frac{\partial f(x,y,z)}{\partial y}\right|\right.\\ &\left.+|\gamma-z|\left|\frac{\partial f(x,y,z)}{\partial z}\right|\right]dxdydz=:M(\alpha,\beta,\gamma;f).\end{aligned}$$

Moreover, we have the bounds

$$(4.2)\quad M(\alpha,\beta,\gamma;f)$$

$$\le\frac{1}{3}\begin{cases}\left\|\frac{\partial f}{\partial x}\right\|_{B,\infty}\iiint_B|\alpha-x|\,dxdydz+\left\|\frac{\partial f}{\partial y}\right\|_{B,\infty}\iiint_B|\beta-y|\,dxdydz\\ +\left\|\frac{\partial f}{\partial z}\right\|_{B,\infty}\iiint_B|\gamma-z|\,dxdydz;\\ \\ \left\|\frac{\partial f}{\partial x}\right\|_{B,p}\left(\iiint_B|\alpha-x|^q\,dxdydz\right)^{1/q}+\left\|\frac{\partial f}{\partial y}\right\|_{B,p}\left(\iiint_B|\beta-y|^q\,dxdydz\right)^{1/q}\\ +\left\|\frac{\partial f}{\partial z}\right\|_{B,p}\left(\iiint_B|\gamma-z|\,dxdydz\right)^{1/q},\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \sup_{(x,y,z)\in}|\alpha-x|\left\|\frac{\partial f}{\partial x}\right\|_{B,1}+\sup_{(x,y,z)\in}|\beta-y|\left\|\frac{\partial f}{\partial y}\right\|_{B,1}\\ +\sup_{(x,y,z)\in}|\gamma-z|\left\|\frac{\partial f}{\partial z}\right\|_{B,1}.\end{cases}$$

Proof. From the identity (3.1) we have

$$\begin{aligned}&\left|\iiint_B f(x,y,z)\,dxdydz-\frac{1}{3}\left[\int\int_S (x-\alpha)f(x,y,z)\,dy\wedge dz\right.\right.\\ &\left.\left.+\int\int_S (y-\beta)f(x,y,z)\,dz\wedge dx+\int\int_S (z-\gamma)f(x,y,z)\,dx\wedge dy\right]\right|\\ &=\frac{1}{3}\left|\iiint_B\left[(\alpha-x)\frac{\partial f(x,y,z)}{\partial x}+(\beta-y)\frac{\partial f(x,y,z)}{\partial y}\right.\right.\\ &\left.\left.+(\gamma-z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz\right|\end{aligned}$$

$$\leq \frac{1}{3}\iiint_B \left|\left[(\alpha - x)\frac{\partial f(x,y,z)}{\partial x} + (\beta - y)\frac{\partial f(x,y,z)}{\partial y} + (\gamma - z)\frac{\partial f(x,y,z)}{\partial z}\right]\right| dxdydz$$

$$\leq \frac{1}{3}\iiint_B \left[\left|(\alpha - x)\frac{\partial f(x,y,z)}{\partial x}\right| + \left|(\beta - y)\frac{\partial f(x,y,z)}{\partial y}\right| + \left|(\gamma - z)\frac{\partial f(x,y,z)}{\partial z}\right|\right] dxdydz = M(\alpha,\beta,\gamma;f),$$

which proves the inequality (4.1).

By Hölder's multiple integral inequality we also have

$$\iiint_B \left|(\alpha - x)\frac{\partial f(x,y,z)}{\partial x}\right| dxdydz$$

$$\leq \begin{cases} \left\|\frac{\partial f}{\partial x}\right\|_{B,\infty} \iiint_B |\alpha - x|\, dxdydz; \\ \left\|\frac{\partial f}{\partial x}\right\|_{B,p} \left(\iiint_B |\alpha - x|^q\, dxdydz\right)^{1/q}, \ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sup_{(x,y,z)\in} |\alpha - x| \left\|\frac{\partial f}{\partial x}\right\|_{B,1} \end{cases}$$

and the other two similar inequalities for the partial derivatives $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$, which, by addition, provide the bound from (4.2). □

Corollary 1. *With the assumptions of Theorem 4 we have the inequalities*

$$\left|\iiint_B f(x,y,z)\,dxdydz - \frac{1}{3}\left[\int\int_S (x - \overline{x_B}) f(x,y,z)\,dy\wedge dz + \int\int_S (\overline{y_B} - \beta) f(x,y,z)\,dz\wedge dx + \int\int_S (z - \overline{z_B}) f(x,y,z)\,dx\wedge dy\right]\right|$$
$$\leq \frac{1}{3}\iiint_B \left[|\overline{x_B} - x|\left|\frac{\partial f(x,y,z)}{\partial x}\right| + |\overline{y_B} - y|\left|\frac{\partial f(x,y,z)}{\partial y}\right| + |\overline{z_B} - z|\left|\frac{\partial f(x,y,z)}{\partial z}\right|\right] dxdydz =: M(\overline{x_B},\overline{y_B},\overline{z_B};f) \tag{4.3}$$

with

$$M(\overline{x_B},\overline{y_B},\overline{z_B};f) \tag{4.4}$$

$$\leq \frac{1}{3}\begin{cases} \left\|\frac{\partial f}{\partial x}\right\|_{B,\infty} \iiint_B |\overline{x_B}-x|\,dxdydz + \left\|\frac{\partial f}{\partial y}\right\|_{B,\infty} \iiint_B |\overline{y_B}-y|\,dxdydz \\ +\left\|\frac{\partial f}{\partial z}\right\|_{B,\infty} \iiint_B |\overline{z_B}-z|\,dxdydz; \\ \\ \left\|\frac{\partial f}{\partial x}\right\|_{B,p} \left(\iiint_B |\overline{x_B}-x|^q\,dxdydz\right)^{1/q} + \left\|\frac{\partial f}{\partial y}\right\|_{B,p} \left(\iiint_B |\overline{y_B}-y|^q\,dxdydz\right)^{1/q} \\ +\left\|\frac{\partial f}{\partial z}\right\|_{B,p} \left(\iiint_B |\overline{z_B}-z|\,dxdydz\right)^{1/q},\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sup_{(x,y,z)\in} |\overline{x_B}-x| \left\|\frac{\partial f}{\partial x}\right\|_{B,1} + \sup_{(x,y,z)\in} |\overline{y_B}-y| \left\|\frac{\partial f}{\partial y}\right\|_{B,1} \\ +\sup_{(x,y,z)\in} |\overline{z_B}-z| \left\|\frac{\partial f}{\partial z}\right\|_{B,1}. \end{cases}$$

We also have

$$\left|\iiint_B f(x,y,z)\,dxdydz\right| \tag{4.5}$$
$$\leq \frac{1}{3}\iiint_B \left[|x_{S,f}-x|\left|\frac{\partial f(x,y,z)}{\partial x}\right| + |y_{S,f}-y|\left|\frac{\partial f(x,y,z)}{\partial y}\right| + |z_{S,f}-z|\left|\frac{\partial f(x,y,z)}{\partial z}\right|\right] dxdydz =: M(x_{S,f},y_{S,f},z_{S,f};f)$$

with

$$M(x_{S,f},y_{S,f},z_{S,f};f) \tag{4.6}$$
$$\leq \frac{1}{3}\begin{cases} \left\|\frac{\partial f}{\partial x}\right\|_{B,\infty} \iiint_B |x_{S,f}-x|\,dxdydz + \left\|\frac{\partial f}{\partial y}\right\|_{B,\infty} \iiint_B |y_{S,f}-y|\,dxdydz \\ +\left\|\frac{\partial f}{\partial z}\right\|_{B,\infty} \iiint_B |z_{S,f}-z|\,dxdydz; \\ \\ \left\|\frac{\partial f}{\partial x}\right\|_{B,p} \left(\iiint_B |x_{S,f}-x|^q\,dxdydz\right)^{1/q} + \left\|\frac{\partial f}{\partial y}\right\|_{B,p} \left(\iiint_B |y_{S,f}-y|^q\,dxdydz\right)^{1/q} \\ +\left\|\frac{\partial f}{\partial z}\right\|_{B,p} \left(\iiint_B |z_{S,f}-z|\,dxdydz\right)^{1/q},\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sup_{(x,y,z)\in} |x_{S,f}-x| \left\|\frac{\partial f}{\partial x}\right\|_{B,1} + \sup_{(x,y,z)\in} |y_{S,f}-y| \left\|\frac{\partial f}{\partial y}\right\|_{B,1} \\ +\sup_{(x,y,z)\in} |z_{S,f}-z| \left\|\frac{\partial f}{\partial z}\right\|_{B,1}. \end{cases}$$

Remark 3. *Using the discrete Hölder's inequality we have*

$$|\alpha-x|\left|\frac{\partial f(x,y,z)}{\partial x}\right| + |\beta-y|\left|\frac{\partial f(x,y,z)}{\partial y}\right| + |\gamma-z|\left|\frac{\partial f(x,y,z)}{\partial z}\right|$$
$$\leq \begin{cases} \max\{|\alpha-x|,|\beta-y|,|\gamma-z|\}\left[\left|\frac{\partial f(x,y,z)}{\partial x}\right| + \left|\frac{\partial f(x,y,z)}{\partial y}\right| + \left|\frac{\partial f(x,y,z)}{\partial z}\right|\right]; \\ \\ \left(|\alpha-x|^q+|\beta-y|^q+|\gamma-z|^q\right)^{1/q}\left[\left|\frac{\partial f(x,y,z)}{\partial x}\right|^p + \left|\frac{\partial f(x,y,z)}{\partial y}\right|^p + \left|\frac{\partial f(x,y,z)}{\partial z}\right|^p\right]^{1/p} \\ \text{for } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \max\left\{\left|\frac{\partial f(x,y,z)}{\partial x}\right|, \left|\frac{\partial f(x,y,z)}{\partial y}\right|, \left|\frac{\partial f(x,y,z)}{\partial z}\right|\right\}\left[|\alpha-x|+|\beta-y|+|\gamma-z|\right] \end{cases}$$

for all $(x,y,z)\in B$ and all α, β, γ complex numbers.

By taking the integral we get

$$M(\alpha,\beta,\gamma;f) \leq \frac{1}{3}\begin{cases} \iiint_B \max\{|\alpha-x|,|\beta-y|,|\gamma-z|\} \\ \times\left[\left|\frac{\partial f(x,y,z)}{\partial x}\right|+\left|\frac{\partial f(x,y,z)}{\partial y}\right|+\left|\frac{\partial f(x,y,z)}{\partial z}\right|\right]dxdydz; \\ \\ \iiint_B \left(|\alpha-x|^q+|\beta-y|^q+|\gamma-z|^q\right)^{1/q} \\ \times\left[\left|\frac{\partial f(x,y,z)}{\partial x}\right|^p+\left|\frac{\partial f(x,y,z)}{\partial y}\right|^p+\left|\frac{\partial f(x,y,z)}{\partial z}\right|^p\right]^{1/p}dxdydz \\ \text{for } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \iiint_B \max\left\{\left|\frac{\partial f(x,y,z)}{\partial x}\right|,\left|\frac{\partial f(x,y,z)}{\partial y}\right|,\left|\frac{\partial f(x,y,z)}{\partial z}\right|\right\} \\ \times\left[|\alpha-x|+|\beta-y|+|\gamma-z|\right]dxdydz \end{cases}$$

for all α, β, γ complex numbers.

One can separate the factors in the above inequality by using Hölder's integral inequality. For instance, we have

$$\begin{aligned} &\iiint_B \left(|\alpha-x|^q+|\beta-y|^q+|\gamma-z|^q\right)^{1/q} \\ &\qquad\times\left[\left|\frac{\partial f(x,y,z)}{\partial x}\right|^p+\left|\frac{\partial f(x,y,z)}{\partial y}\right|^p+\left|\frac{\partial f(x,y,z)}{\partial z}\right|^p\right]^{1/p}dxdydz \\ &\qquad\leq\left(\iiint_B\left[\left(|\alpha-x|^q+|\beta-y|^q+|\gamma-z|^q\right)^{1/q}\right]^q dxdydz\right)^{1/q} \\ &\times\left(\iiint_B\left(\left[\left|\frac{\partial f(x,y,z)}{\partial x}\right|^p+\left|\frac{\partial f(x,y,z)}{\partial y}\right|^p+\left|\frac{\partial f(x,y,z)}{\partial z}\right|^p\right]^{1/p}\right)^p dxdydz\right)^{1/p} \\ &\qquad=\left(\iiint_B\left(|\alpha-x|^q+|\beta-y|^q+|\gamma-z|^q\right)dxdydz\right)^{1/q} \\ &\qquad\times\left(\iiint_B\left[\left|\frac{\partial f(x,y,z)}{\partial x}\right|^p+\left|\frac{\partial f(x,y,z)}{\partial y}\right|^p+\left|\frac{\partial f(x,y,z)}{\partial z}\right|^p\right]dxdydz\right)^{1/p}, \end{aligned}$$

which gives

$$\begin{aligned} M(\alpha,\beta,\gamma;f) &\leq \frac{1}{3}\left(\iiint_B\left(|\alpha-x|^q+|\beta-y|^q+|\gamma-z|^q\right)dxdydz\right)^{1/q} \\ &\times\left(\iiint_B\left[\left|\frac{\partial f(x,y,z)}{\partial x}\right|^p+\left|\frac{\partial f(x,y,z)}{\partial y}\right|^p+\left|\frac{\partial f(x,y,z)}{\partial z}\right|^p\right]dxdydz\right)^{1/p}, \end{aligned} \tag{4.7}$$

for p, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$.

We also have:

Theorem 5. *Let B be a solid in the three dimensional space $\mathbb{R}^3$ bounded by an orientable closed surface S described by the vector equation*

$$r(u,v)=x(u,v)\overrightarrow{i}+y(u,v)\overrightarrow{j}+z(u,v)\overrightarrow{k},\quad (u,v)\in[a,b]\times[c,d]$$

where $x(u,v)$, $y(u,v)$, $z(u,v)$ are differentiable. If $f:B\to\mathbb{C}$ is a continuously differentiable function defined on a open set containing B, then for all α, β, γ complex numbers we have the inequality

$$\begin{aligned}(4.8)\quad &\left|\iiint_B f(x,y,z)\,dxdydz-\frac{1}{3}\iiint_B\left[(\alpha-x)\frac{\partial f(x,y,z)}{\partial x}\right.\right.\\ &\left.\left.+(\beta-y)\frac{\partial f(x,y,z)}{\partial y}+(\gamma-z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz\right|\\ &\leq\frac{1}{3}\left[\int_a^b\int_c^d|f(x(u,v),y(u,v),z(u,v))|\,|x(u,v)-\alpha|\left|\frac{\partial(y,z)}{\partial(u,v)}\right|dudv\right.\\ &+\int_a^b\int_c^d|f(x(u,v),y(u,v),z(u,v))|\,|y(u,v)-\beta|\left|\frac{\partial(z,x)}{\partial(u,v)}\right|dudv\\ &\left.+\int_a^b\int_c^d|f(x(u,v),y(u,v),z(u,v))|\,|z(u,v)-\gamma|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv\right]\\ &=:N(\alpha,\beta,\gamma;f).\end{aligned}$$

Moreover, if we put $\square:=[a,b]\times[c,d]$, then we have the bounds

$$\begin{aligned}(4.9)\quad N(\alpha,\beta,\gamma;f)&\leq\frac{1}{3}\|f\|_{S,\infty}\left[\int_a^b\int_c^d|x(u,v)-\alpha|\left|\frac{\partial(y,z)}{\partial(u,v)}\right|\right.\\ &\left.+|y(u,v)-\beta|\left|\frac{\partial(z,x)}{\partial(u,v)}\right|+|z(u,v)-\gamma|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv\right]\\ &\leq\frac{1}{3}\|f\|_{S,\infty}\\ &\times\begin{cases}\left\|\frac{\partial(y,z)}{\partial(\cdot,\cdot)}\right\|_{\square,\infty}\|x-\alpha\|_{\square,1}+\left\|\frac{\partial(z,x)}{\partial(\cdot,\cdot)}\right\|_{\square,\infty}\|y-\beta\|_{\square,1}\\ \quad+\left\|\frac{\partial(x,y)}{\partial(\cdot,\cdot)}\right\|_{\square,\infty}\|z-\gamma\|_{\square,1},\\ \\ \left\|\frac{\partial(y,z)}{\partial(\cdot,\cdot)}\right\|_{\square,p}\|x-\alpha\|_{\square,q}+\left\|\frac{\partial(z,x)}{\partial(\cdot,\cdot)}\right\|_{\square,p}\|y-\beta\|_{\square,q}\\ \quad+\left\|\frac{\partial(x,y)}{\partial(\cdot,\cdot)}\right\|_{\square,p}\|z-\gamma\|_{\square,q},\\ \\ \left\|\frac{\partial(y,z)}{\partial(\cdot,\cdot)}\right\|_{\square,1}\|x-\alpha\|_{\square,\infty}+\left\|\frac{\partial(z,x)}{\partial(\cdot,\cdot)}\right\|_{\square,p}\|y-\beta\|_{\square,\infty}\\ \quad+\left\|\frac{\partial(x,y)}{\partial(\cdot,\cdot)}\right\|_{\square,1}\|z-\gamma\|_{\square,\infty}.\end{cases}\end{aligned}$$

Proof. From the identity (3.1) we get

$$\begin{aligned}(4.10)\quad &\iiint_B f(x,y,z)\,dxdydz\\ &-\frac{1}{3}\iiint_B\left[(\alpha-x)\frac{\partial f(x,y,z)}{\partial x}+(\beta-y)\frac{\partial f(x,y,z)}{\partial y}\right.\\ &\left.+(\gamma-z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3}\left[\int_a^b\int_c^d (x(u,v)-\alpha) f(x(u,v),y(u,v),z(u,v)) \frac{\partial(y,z)}{\partial(u,v)} dudv\right.\\
&\qquad + \int_a^b\int_c^d (y(u,v)-\beta) f(x(u,v),y(u,v),z(u,v)) \frac{\partial(z,x)}{\partial(u,v)} dudv\\
&\qquad\qquad \left. + \int_a^b\int_c^d (z(u,v)-\gamma) f(x(u,v),y(u,v),z(u,v)) \frac{\partial(x,y)}{\partial(u,v)} dudv\right]
\end{aligned}
$$

for all α, β, γ complex numbers.

By taking the modulus in (4.10) we get

$$
\begin{aligned}
&\left|\iiint_B f(x,y,z)\,dxdydz - \frac{1}{3}\iiint_B \left[(\alpha-x)\frac{\partial f(x,y,z)}{\partial x}\right.\right.\\
&\qquad\left.\left. + (\beta-y)\frac{\partial f(x,y,z)}{\partial y} + (\gamma-z)\frac{\partial f(x,y,z)}{\partial z}\right] dxdydz\right|\\
&\le \frac{1}{3}\left[\int_a^b\int_c^d \left|(x(u,v)-\alpha) f(x(u,v),y(u,v),z(u,v)) \frac{\partial(y,z)}{\partial(u,v)}\right| dudv\right.\\
&\quad + \int_a^b\int_c^d \left|(y(u,v)-\beta) f(x(u,v),y(u,v),z(u,v)) \frac{\partial(z,x)}{\partial(u,v)}\right| dudv\\
&\quad \left. + \int_a^b\int_c^d \left|(z(u,v)-\gamma) f(x(u,v),y(u,v),z(u,v)) \frac{\partial(x,y)}{\partial(u,v)}\right| dudv\right]\\
&= \frac{1}{3}\left[\int_a^b\int_c^d |x(u,v)-\alpha| \left|f(x(u,v),y(u,v),z(u,v))\right| \left|\frac{\partial(y,z)}{\partial(u,v)}\right| dudv\right.\\
&\quad + \int_a^b\int_c^d |y(u,v)-\beta| \left|f(x(u,v),y(u,v),z(u,v))\right| \left|\frac{\partial(z,x)}{\partial(u,v)}\right| dudv\\
&\quad \left. + \int_a^b\int_c^d |z(u,v)-\gamma| \left|f(x(u,v),y(u,v),z(u,v))\right| \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv\right]\\
&\qquad\qquad = N(\alpha,\beta,\gamma;f),
\end{aligned}
$$

which proves the first inequality in (4.8).

We have

$$
\begin{aligned}
N(\alpha,\beta,\gamma;f) &\le \frac{1}{3}\|f\|_{S,\infty}\left[\int_a^b\int_c^d |x(u,v)-\alpha| \left|\frac{\partial(y,z)}{\partial(u,v)}\right| dudv\right.\\
&\qquad + \int_a^b\int_c^d |y(u,v)-\beta| \left|\frac{\partial(z,x)}{\partial(u,v)}\right| dudv\\
&\qquad\qquad \left. + \int_a^b\int_c^d |z(u,v)-\gamma| \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv\right]
\end{aligned}
$$

and by Hölder's inequality for each integral we get the last part of (4.9). □

Corollary 2. *With the assumptions of Theorem 5 we have*

$$N(\alpha,\beta,\gamma;f) \leq \frac{1}{3}\left(\int\int_S |f(x,y,z)|^2 dS\right)^{1/2} \times \left(\int\int_S \left(|x-\alpha|^2+|y-\beta|^2+|z-\gamma|^2\right) dS\right)^{1/2} \tag{4.11}$$

for all α, β, γ complex numbers.

Proof. We have, by Cauchy-Bunyakovsky-Schwarz (CBS) discrete inequality, that

$$\begin{aligned}
&|x(u,v)-\alpha|\left|\frac{\partial(y,z)}{\partial(u,v)}\right|+|y(u,v)-\beta|\left|\frac{\partial(z,x)}{\partial(u,v)}\right|+|z(u,v)-\gamma|\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \\
&\leq \left(|x(u,v)-\alpha|^2+|y(u,v)-\beta|^2+|z(u,v)-\gamma|^2\right)^{1/2} \\
&\times\left(\left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2+\left|\frac{\partial(z,x)}{\partial(u,v)}\right|^2+\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2\right)^{1/2}
\end{aligned}$$

for $(u,v)\in[a,b]\times[c,d]$.

Therefore we get

$$\begin{aligned}
N(\alpha,\beta,\gamma;f) \leq{}& \int_a^b\int_c^d |f(x(u,v),y(u,v),z(u,v))| \\
&\times\left(|x(u,v)-\alpha|^2+|y(u,v)-\beta|^2+|z(u,v)-\gamma|^2\right)^{1/2} \\
&\times\left(\left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2+\left|\frac{\partial(z,x)}{\partial(u,v)}\right|^2+\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2\right)^{1/2} dudv \\
=:{}& P(\alpha,\beta,\gamma;f).
\end{aligned}$$

By using CBS weighted integral inequality we get

$$\begin{aligned}
&P(\alpha,\beta,\gamma;f) \\
&\leq \left(\int_a^b\int_c^d |f(x(u,v),y(u,v),z(u,v))|^2\right. \\
&\left.\times\left(\left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2+\left|\frac{\partial(z,x)}{\partial(u,v)}\right|^2+\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2\right)^{1/2} dudv\right)^{1/2} \\
&\times\left(\int_a^b\int_c^d \left(|x(u,v)-\alpha|^2+|y(u,v)-\beta|^2+|z(u,v)-\gamma|^2\right)\right. \\
&\left.\times\left(\left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2+\left|\frac{\partial(z,x)}{\partial(u,v)}\right|^2+\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2\right)^{1/2} dudv\right)^{1/2} \\
&= \left(\int\int_S |f(x,y,z)|^2 dS\right)^{1/2}\left(\int\int_S\left(|x-\alpha|^2+|y-\beta|^2+|z-\gamma|^2\right)dS\right)^{1/2},
\end{aligned}$$

which proves the desired result (4.11). □

Remark 4. *From (4.8) we get*

$$\left|\iiint_B f(x,y,z)\,dxdydz - \frac{1}{3}\iiint_B \left[(\overline{x_B}-x)\frac{\partial f(x,y,z)}{\partial x} + (\overline{y_B}-y)\frac{\partial f(x,y,z)}{\partial y} + (\overline{z_B}-z)\frac{\partial f(x,y,z)}{\partial z}\right] dxdydz\right|$$
$$\le \frac{1}{3}\left[\int_a^b\int_c^d |f(x(u,v),y(u,v),z(u,v))|\,|x(u,v)-\overline{x_B}|\left|\frac{\partial(y,z)}{\partial(u,v)}\right| dudv + \int_a^b\int_c^d |f(x(u,v),y(u,v),z(u,v))|\,|y(u,v)-\overline{y_B}|\left|\frac{\partial(z,x)}{\partial(u,v)}\right| dudv + \int_a^b\int_c^d |f(x(u,v),y(u,v),z(u,v))|\,|z(u,v)-\overline{z_B}|\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv\right] =: N(\alpha,\beta,\gamma;f). \tag{4.12}$$

Moreover, if we put $\square := [a,b]\times[c,d]$, *then we have the bounds*

$$N(\alpha,\beta,\gamma;f) \le \frac{1}{3}\|f\|_{S,\infty}\left[\int_a^b\int_c^d |x(u,v)-\overline{x_B}|\left|\frac{\partial(y,z)}{\partial(u,v)}\right| + |y(u,v)-\overline{y_B}|\left|\frac{\partial(z,x)}{\partial(u,v)}\right| + |z(u,v)-\overline{z_B}|\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv\right]$$
$$\le \frac{1}{3}\|f\|_{S,\infty} \times \begin{cases} \left\|\frac{\partial(y,z)}{\partial(\cdot,\cdot)}\right\|_{\square,\infty}\|x-\overline{x_B}\|_{\square,1} + \left\|\frac{\partial(z,x)}{\partial(\cdot,\cdot)}\right\|_{\square,\infty}\|y-\overline{y_B}\|_{\square,1} + \left\|\frac{\partial(x,y)}{\partial(\cdot,\cdot)}\right\|_{\square,\infty}\|z-\overline{z_B}\|_{\square,1}, \\ \\ \left\|\frac{\partial(y,z)}{\partial(\cdot,\cdot)}\right\|_{\square,p}\|x-\overline{x_B}\|_{\square,q} + \left\|\frac{\partial(z,x)}{\partial(\cdot,\cdot)}\right\|_{\square,p}\|y-\overline{y_B}\|_{\square,q} + \left\|\frac{\partial(x,y)}{\partial(\cdot,\cdot)}\right\|_{\square,p}\|z-\overline{z_B}\|_{\square,q}, \\ \\ \left\|\frac{\partial(y,z)}{\partial(\cdot,\cdot)}\right\|_{\square,1}\|x-\overline{x_B}\|_{\square,\infty} + \left\|\frac{\partial(z,x)}{\partial(\cdot,\cdot)}\right\|_{\square,p}\|y-\overline{y_B}\|_{\square,\infty} + \left\|\frac{\partial(x,y)}{\partial(\cdot,\cdot)}\right\|_{\square,1}\|z-\overline{z_B}\|_{\square,\infty}. \end{cases} \tag{4.13}$$

We also observe that under the assumptions of Theorem 5 we have

$$\left|\iiint_B f(x,y,z)\,dxdydz\right| \le \frac{1}{3}\left[\int_a^b\int_c^d |f(x(u,v),y(u,v),z(u,v))|\,|x(u,v)-x_{B,\partial f}|\left|\frac{\partial(y,z)}{\partial(u,v)}\right| dudv \right. \tag{4.14}$$

$$+\int_a^b\int_c^d |f(x(u,v),y(u,v),z(u,v))|\,|y(u,v)-y_{B,\partial f}|\left|\frac{\partial(z,x)}{\partial(u,v)}\right|dudv$$
$$\left.+\int_a^b\int_c^d |f(x(u,v),y(u,v),z(u,v))|\,|z(u,v)-z_{B,\partial f}|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv\right]$$
$$=: N(x_{B,\partial f},y_{B,\partial f},z_{B,\partial f};f).$$

Moreover, we have the bounds

$$N(x_{B,\partial f},y_{B,\partial f},z_{B,\partial f};f)\le\frac{1}{3}\|f\|_{S,\infty}\left[\int_a^b\int_c^d |x(u,v)-x_{B,\partial f}|\left|\frac{\partial(y,z)}{\partial(u,v)}\right|\right.$$
$$\left.+|y(u,v)-y_{B,\partial f}|\left|\frac{\partial(z,x)}{\partial(u,v)}\right|+|z(u,v)-z_{B,\partial f}|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv\right]$$
$$\le\frac{1}{3}\|f\|_{S,\infty}$$
$$\times\begin{cases}\left\|\frac{\partial(y,z)}{\partial(\cdot,\cdot)}\right\|_{\Box,\infty}\|x-x_{B,\partial f}\|_{\Box,1}+\left\|\frac{\partial(z,x)}{\partial(\cdot,\cdot)}\right\|_{\Box,\infty}\|y-y_{B,\partial f}\|_{\Box,1}\\ \qquad+\left\|\frac{\partial(x,y)}{\partial(\cdot,\cdot)}\right\|_{\Box,\infty}\|z-z_{B,\partial f}\|_{\Box,1},\\[2ex] \left\|\frac{\partial(y,z)}{\partial(\cdot,\cdot)}\right\|_{\Box,p}\|x-x_{B,\partial f}\|_{\Box,q}+\left\|\frac{\partial(z,x)}{\partial(\cdot,\cdot)}\right\|_{\Box,p}\|y-y_{B,\partial f}\|_{\Box,q}\\ \qquad+\left\|\frac{\partial(x,y)}{\partial(\cdot,\cdot)}\right\|_{\Box,p}\|z-z_{B,\partial f}\|_{\Box,q},\\[2ex] \left\|\frac{\partial(y,z)}{\partial(\cdot,\cdot)}\right\|_{\Box,1}\|x-x_{B,\partial f}\|_{\Box,\infty}+\left\|\frac{\partial(z,x)}{\partial(\cdot,\cdot)}\right\|_{\Box,p}\|y-y_{B,\partial f}\|_{\Box,\infty}\\ \qquad+\left\|\frac{\partial(x,y)}{\partial(\cdot,\cdot)}\right\|_{\Box,1}\|z-z_{B,\partial f}\|_{\Box,\infty}.\end{cases}\tag{4.15}$$

5. Applications for Three Dimensional Balls

Now, let us compute the surface integral

$$K(S(C,R),f):=\iint_{S(C,R)} f(x,y,z)\,dS,$$

where

$$S(C,R):=\left\{(x,y,z)\in\mathbb{R}^3\middle|\,(x-a)^2+(y-b)^2+(z-c)^2=R^2\right\}.$$

If we consider the parametrization of $S(C,R)$ given by:

$$S(C,R):\begin{cases}x=R\cos\psi\cos\varphi+a\\ y=R\cos\psi\sin\varphi+b\\ z=R\sin\psi+c\end{cases};\quad(\psi,\varphi)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi]$$

and putting

$$A:=\begin{vmatrix}\frac{\partial y}{\partial\psi} & \frac{\partial z}{\partial\psi}\\ \frac{\partial y}{\partial\varphi} & \frac{\partial z}{\partial\varphi}\end{vmatrix}=-R^2\cos^2\psi\cos\varphi,$$

$$B:=\begin{vmatrix}\frac{\partial x}{\partial\psi} & \frac{\partial z}{\partial\psi}\\ \frac{\partial x}{\partial\varphi} & \frac{\partial z}{\partial\varphi}\end{vmatrix}=R^2\cos^2\psi\sin\varphi,$$

and

$$C := \begin{vmatrix} \frac{\partial x}{\partial \psi} & \frac{\partial y}{\partial \psi} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = -R^2 \sin\psi \cos\psi,$$

we have that

$$A^2 + B^2 + C^2 = R^4 \cos^2 \psi \text{ for all } (\psi, \varphi) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [0, 2\pi].$$

Thus,

$$\begin{aligned} (5.1) \quad K(S(C,R), f) &= \iint_{S(C,R)} f(x,y,z)\, dS \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} [f(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c) \\ &\qquad \times \sqrt{A^2+B^2+C^2}\Big] d\psi d\varphi \\ &= R^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \cos\psi f(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c)\, d\psi d\varphi. \end{aligned}$$

We also have

$$\begin{aligned} (5.2) \quad L(S(C,R), f) &:= \int\int_{S(C,R)} (x-a) f(x,y,z)\, dy \wedge dz \\ &+ \int\int_{S(C,R)} (y-b) f(x,y,z)\, dz \wedge dx + \int\int_{S(C,R)} (z-c) f(x,y,z)\, dx \wedge dy \\ &= -R^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \cos^3\psi \cos^2\varphi \\ &\times f(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c)\, d\psi d\varphi \\ &+ R^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \cos^3\psi \sin^2\varphi \\ &\times f(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c)\, d\psi d\varphi \\ &- R^3 \int\int_S \sin^2\psi \cos\psi f(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c)\, d\psi d\varphi. \end{aligned}$$

Let us consider the transformation $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ given by:

$$T_2(r, \psi, \varphi) := (r\cos\psi\cos\varphi + a, r\cos\psi\sin\varphi + b, r\sin\psi + c).$$

It is well known that the Jacobian of T_2 is

$$J(T_2) = r^2 \cos\psi$$

and T_2 is a one-to-one mapping defined on the interval of $\mathbb{R}^3$, $[0,R] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [0, 2\pi]$, with values in the ball $B(C,R)$ from $\mathbb{R}^3$. Thus we have the change of variable:

$$\begin{aligned} (5.3) \quad I(B(C,R), f) &:= \iiint_{B(C,R)} f(x,y,z)\, dx dy dz \\ &= \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} f(r\cos\psi\cos\varphi + a, r\cos\psi\sin\varphi + b, r\sin\psi + c)\, r^2 \cos\psi dr d\psi d\varphi. \end{aligned}$$

We also have

$$\iiint_{B(C,R)} |a-x|\, dxdydz = \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} r^3 \cos^2\psi \,|\cos\varphi|\, drd\psi d\varphi = \frac{\pi}{2} R^4$$

and, similarly

$$\iiint_{B(C,R)} |b-y|\, dxdydz = \iiint_{B(C,R)} |c-z|\, dxdydz = \frac{\pi}{2} R^4.$$

Therefore

$$\begin{aligned}
&\left\| \frac{\partial f}{\partial x} \right\|_{B(C,R),\infty} \iiint_{B(C,R)} |\overline{x_B} - x|\, dxdydz \\
&\qquad + \left\| \frac{\partial f}{\partial y} \right\|_{B(C,R),\infty} \iiint_{B(C,R)} |\overline{y_B} - y|\, dxdydz \\
&\qquad + \left\| \frac{\partial f}{\partial z} \right\|_{B(C,R),\infty} \iiint_{B(C,R)} |\overline{z_B} - z|\, dxdydz \\
&\qquad = \frac{\pi}{2} R^4 \left(\left\| \frac{\partial f}{\partial x} \right\|_{B(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{B(C,R),\infty} + \left\| \frac{\partial f}{\partial z} \right\|_{B(C,R),\infty} \right)
\end{aligned}$$

and by the inequalities (4.3) and (4.4) we get

$$\begin{aligned}
&\left| I(B(C,R), f) - \frac{1}{3} L(S(C,R), f) \right| \\
&\qquad \le \frac{\pi}{6} R^4 \left(\left\| \frac{\partial f}{\partial x} \right\|_{B(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{B(C,R),\infty} + \left\| \frac{\partial f}{\partial z} \right\|_{B(C,R),\infty} \right)
\end{aligned} \tag{5.4}$$

provided $f : B(C,R) \to \mathbb{C}$ is a continuously differentiable function defined on a open set containing $B(C,R)$.

We also consider

$$\begin{aligned}
T(B(C,R), f) :&= \iiint_B \left[(\overline{x_B} - x) \frac{\partial f(x,y,z)}{\partial x} \right. \\
&\quad \left. + (\overline{y_B} - y) \frac{\partial f(x,y,z)}{\partial y} + (\overline{z_B} - z) \frac{\partial f(x,y,z)}{\partial z} \right] dxdydz \\
&= -\int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \left(r^3 \cos^2\psi \cos\varphi \right) \\
&\quad \times \frac{\partial f(r\cos\psi\cos\varphi + a, r\cos\psi\sin\varphi + b, r\sin\psi + c)}{\partial x} drd\psi d\varphi \\
&\quad - \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \left(r^3 \cos^2\psi \sin\varphi \right) \\
&\quad \times \frac{\partial f(r\cos\psi\cos\varphi + a, r\cos\psi\sin\varphi + b, r\sin\psi + c)}{\partial y} drd\psi d\varphi \\
&\quad - \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} r^3 \sin\psi \cos\psi \\
&\quad \times \frac{\partial f(r\cos\psi\cos\varphi + a, r\cos\psi\sin\varphi + b, r\sin\psi + c)}{\partial z} drd\psi d\varphi.
\end{aligned} \tag{5.5}$$

By the inequality (4.11) we obtain

$$N(a,b,c;f) \leq \frac{1}{3}\left(\int\int_{S(C,R)} |f(x,y,z)|^2 \, dS\right)^{1/2} \times \left(\int\int_{S(C,R)} \left(|x-a|^2 + |y-b|^2 + |z-c|^2\right) dS\right)^{1/2} = \frac{2}{3}\sqrt{\pi}R^2 \left[K\left(S(C,R), |f|^2\right)\right]^{1/2}$$

and by utilising (4.12) we also get

$$\left|I(B(C,R),f) - \frac{1}{3}T(B(C,R),f)\right| \leq \frac{2}{3}\sqrt{\pi}R^2 \left[K\left(S(C,R), |f|^2\right)\right]^{1/2}. \tag{5.6}$$

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