SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR CONVEX FUNCTIONS DEFINED ON CONVEX BODIES VIA GAUSS-OSTROGRADSKY IDENTITY

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper, by the use of Gauss-Ostrogradsky identity, we establish some integral inequalities of Hermite-Hadamard type for functions of three variables defined on closed and bounded convex bodies of the Euclidean space $\mathbb{R}^3$. Some examples for 3-dimensional balls are also provided.

1. Introduction

Recall the following inequalities of Hermite-Hadamard's type for convex functions defined on a ball $B(C,R)$, where $C=(a,b,c)\in\mathbb{R}^3$, $R>0$ and

$$B(C,R):=\left\{(x,y,z)\in\mathbb{R}^3\middle|\,(x-a)^2+(y-b)^2+(z-c)^2\leq R^2\right\}.$$

The following theorem holds [6].

Theorem 1. *Let $f:B(C,R)\to\mathbb{R}$ be a convex mapping on the ball $B(C,R)$. Then we have the inequalities:*

$$f(a,b,c)\leq\frac{1}{V(B(C,R))}\iiint_{B(C,R)}f(x,y,z)\,dxdydz\leq\frac{1}{A(B(C,R))}\iint_{S(C,R)}f(x,y,z)\,dS, \tag{1.1}$$

where

$$S(C,R):=\left\{(x,y,z)\in\mathbb{R}^3\middle|\,(x-a)^2+(y-b)^2+(z-c)^2=R^2\right\}$$

and

$$V(B(C,R))=\frac{4\pi R^3}{3},\;A(B(C,R))=4\pi R^2.$$

Let D be a bounded convex domain from $\mathbb{R}^3$ with a piecewise smooth boundary S. We use the notations

$$A(S):=\iint_S dS,\;V(D)=\iiint_D dV,$$

$$\overline{x_D}:=\frac{1}{V(D)}\iiint_D xdV,\;\overline{y_D}:=\frac{1}{V(D)}\iiint_D ydV,\;\overline{z_D}:=\frac{1}{V(D)}\iiint_D zdV$$

1991 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Hermite-Hadamard inequality, Triple integral inequalities, Gauss-Ostrogradsky identity.

RGMIA Res. Rep. Coll. 22 (2019), Art. 61, 15 pp. Received 13/06/19

and

$$\overline{x_S} := \frac{1}{A(S)} \iint_S x dS, \ \overline{y_S} := \frac{1}{A(S)} \iint_S y dS \text{ and } \overline{z_S} := \frac{1}{A(S)} \iint_S z dS.$$

Let us assume that the surface S is oriented with the aid of the unit normal h directed to the exterior of D

$$h = (\cos\alpha, \cos\beta, \cos\gamma).$$

The following Hermite-Hadamard type inequalities for convex functions defined on general convex domains were obtained by B. Gavrea in 2000, [9]:

Theorem 2. *Let $f : D \to \mathbb{R}$ be a convex function on D, a bounded convex domain from $\mathbb{R}^3$ with a piecewise smooth boundary S. Then we have the inequalities*

$$(1.2) \quad f(\overline{x_D}, \overline{y_D}, \overline{z_D}) \leq \frac{1}{V(D)} \iiint_D f dV \leq \frac{1}{4A(S)} \iint_S f dS + \frac{1}{4V(D)} \iint_S [(x - \overline{x_S}) \cos\alpha + (y - \overline{y_S}) \cos\beta + (z - \overline{z_S}) \cos\gamma] f(x, y, z) dS.$$

For other multivariate Hermite-Hadamard type inequalities, see [2]-[4] and [10]-[16].

Motivated by the above results, we obtain in this paper other integral inequalities of Hermite-Hadamard type for convex functions defined on convex domains from the space. Some examples for three dimensional balls are also provided.

2. Some Preliminary Facts and Results

Following Apostol [1], consider a surface described by the vector equation

$$(2.1) \qquad r(u, v) = x(u, v) \overrightarrow{i} + y(u, v) \overrightarrow{j} + z(u, v) \overrightarrow{k}$$

where $(u, v) \in [a, b] \times [c, d]$.

If x, y, z are differentiable on $[a, b] \times [c, d]$ we consider the two vectors

$$\frac{\partial r}{\partial u} = \frac{\partial x}{\partial u} \overrightarrow{i} + \frac{\partial y}{\partial u} \overrightarrow{j} + \frac{\partial z}{\partial u} \overrightarrow{k}$$

and

$$\frac{\partial r}{\partial v} = \frac{\partial x}{\partial v} \overrightarrow{i} + \frac{\partial y}{\partial v} \overrightarrow{j} + \frac{\partial z}{\partial v} \overrightarrow{k}.$$

The *cross product* of these two vectors $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ will be referred to as the fundamental vector product of the representation r. Its components can be expressed as *Jacobian determinants*. In fact, we have [1, p. 420]

$$(2.2) \qquad \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \overrightarrow{i} + \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix} \overrightarrow{j} + \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \overrightarrow{k}$$

$$= \frac{\partial(y, z)}{\partial(u, v)} \overrightarrow{i} + \frac{\partial(z, x)}{\partial(u, v)} \overrightarrow{j} + \frac{\partial(x, y)}{\partial(u, v)} \overrightarrow{k}.$$

Let $S = r(T)$ be a parametric surface described by a vector-valued function r defined on the box $T = [a, b] \times [c, d]$. The area of S denoted A_S is defined by the double integral [1, p. 424-425]

$$A_S = \int_a^b \int_c^d \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| dudv \tag{2.3}$$
$$= \int_a^b \int_c^d \sqrt{\left(\frac{\partial (y, z)}{\partial (u, v)} \right)^2 + \left(\frac{\partial (z, x)}{\partial (u, v)} \right)^2 + \left(\frac{\partial (x, y)}{\partial (u, v)} \right)^2} dudv.$$

We define surface integrals in terms of a parametric representation for the surface. One can prove that under certain general conditions the value of the integral is independent of the representation.

Let $S = r(T)$ be a parametric surface described by a vector-valued differentiable function r defined on the box $T = [a, b] \times [c, d]$ and let $f : S \to \mathbb{C}$ defined and bounded on S. The surface integral of f over S is defined by [1, p. 430]

$$\int\int_S f dS = \int_a^b \int_c^d f(x, y, z) \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| dudv \tag{2.4}$$
$$= \int_a^b \int_c^d f(x(u, v), y(u, v), z(u, v))$$
$$\times \sqrt{\left(\frac{\partial (y, z)}{\partial (u, v)} \right)^2 + \left(\frac{\partial (z, x)}{\partial (u, v)} \right)^2 + \left(\frac{\partial (x, y)}{\partial (u, v)} \right)^2} dudv.$$

If $S = r(T)$ is a parametric surface, the fundamental vector product $N = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ is normal to S at each regular point of the surface. At each such point there are two unit normals, a unit normal n_1, which has the same direction as N, and a unit normal n_2 which has the opposite direction. Thus

$$n_1 = \frac{N}{\|N\|} \text{ and } n_2 = -n_1.$$

Let n be one of the two normals n_1 or n_2. Let also F be a vector field defined on S and assume that the surface integral,

$$\int\int_S (F \cdot n)\, dS,$$

called the flux surface integral, exists. Here $F \cdot n$ is the dot or inner product.

We can write [1, p. 434]

$$\int\int_S (F \cdot n)\, dS = \pm \int_a^b \int_c^d F(r(u, v)) \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right) dudv$$

where the sign " + " is used if $n = n_1$ and the " − " sign is used if $n = n_2$.

If

$$F(x, y, z) = P(x, y, z) \overrightarrow{i} + Q(x, y, z) \overrightarrow{j} + R(x, y, z) \overrightarrow{k}$$

and

$$r(u, v) = x(u, v) \overrightarrow{i} + y(u, v) \overrightarrow{j} + z(u, v) \overrightarrow{k} \text{ where } (u, v) \in [a, b] \times [c, d]$$

then the flux surface integral for $n=n_1$ can be explicitly calculated as [1, p. 435]

$$\begin{aligned}\int\int_S (F\cdot n)\,dS &= \int_a^b\int_c^d P(x(u,v),y(u,v),z(u,v))\frac{\partial(y,z)}{\partial(u,v)}dudv\\ &+\int_a^b\int_c^d Q(x(u,v),y(u,v),z(u,v))\frac{\partial(z,x)}{\partial(u,v)}dudv\\ &+\int_a^b\int_c^d R(x(u,v),y(u,v),z(u,v))\frac{\partial(x,y)}{\partial(u,v)}dudv.\end{aligned}\tag{2.5}$$

The sum of the double integrals on the right is often written more briefly as [1, p. 435]

$$\int\int_S P(x,y,z)\,dy\wedge dz+\int\int_S Q(x,y,z)\,dz\wedge dx+\int\int_S R(x,y,z)\,dx\wedge dy.$$

Let $B\subset\mathbb{R}^3$ be a solid in 3-space bounded by an orientable closed surface S, and let n be the unit outer normal to S. If F is a continuously differentiable vector field defined on B, we have the *Gauss-Ostrogradsky identity*

$$\iiint_B(\operatorname{div}F)\,dV=\int\int_S(F\cdot n)\,dS.\tag{GO}$$

If we express

$$F(x,y,z)=P(x,y,z)\overrightarrow{i}+Q(x,y,z)\overrightarrow{j}+R(x,y,z)\overrightarrow{k},$$

then (GO) can be written as

$$\begin{aligned}&\iiint_B\left(\frac{\partial P(x,y,z)}{\partial x}+\frac{\partial Q(x,y,z)}{\partial y}+\frac{\partial R(x,y,z)}{\partial z}\right)dxdydz\\ &=\int\int_S P(x,y,z)\,dy\wedge dz+\int\int_S Q(x,y,z)\,dz\wedge dx\\ &\qquad+\int\int_S R(x,y,z)\,dx\wedge dy.\end{aligned}\tag{2.6}$$

By taking the real and imaginary part, we can extend the above inequality for complex valued functions P, Q, R defined on B.

For the body B we consider the coordinates for the *center of gravity* $G(\overline{x_B},\overline{y_B},\overline{z_B})$ defined by

$$\overline{x_B}:=\frac{1}{V(B)}\iiint_B xdxdydz,\ \overline{y_B}:=\frac{1}{V(B)}\iiint_B ydxdydz$$

and

$$\overline{z_B}:=\frac{1}{V(B)}\iiint_B zdxdydz.$$

We have:

Lemma 1. *Let B be a solid in the three dimensional space $\mathbb{R}^3$ bounded by an orientable closed surface S. If $f:B\to\mathbb{C}$ is a continuously differentiable function* *defined on a open set containing B, then we have the equality*

$$\begin{aligned}(2.7)\quad &\iiint_B f(x,y,z)\,dxdydz \\ &= \frac{1}{3}\iiint_B \left[(\alpha - x)\frac{\partial f(x,y,z)}{\partial x} + (\beta - y)\frac{\partial f(x,y,z)}{\partial y} + (\gamma - z)\frac{\partial f(x,y,z)}{\partial z}\right] dxdydz \\ &+ \frac{1}{3}\left[\int\int_S (x-\alpha) f(x,y,z)\,dy\wedge dz + \int\int_S (y-\beta) f(x,y,z)\,dz\wedge dx + \int\int_S (z-\gamma) f(x,y,z)\,dx\wedge dy\right]\end{aligned}$$

for all α, β and γ complex numbers.

In particular, we have

$$\begin{aligned}(2.8)\quad &\iiint_B f(x,y,z)\,dxdydz \\ &= \frac{1}{3}\iiint_B \left[(\overline{x_B} - x)\frac{\partial f(x,y,z)}{\partial x} + (\overline{y_B} - y)\frac{\partial f(x,y,z)}{\partial y} + (\overline{z_B} - z)\frac{\partial f(x,y,z)}{\partial z}\right] dxdydz \\ &+ \frac{1}{3}\left[\int\int_S (x-\overline{x_B}) f(x,y,z)\,dy\wedge dz + \int\int_S (y-\overline{y_B}) f(x,y,z)\,dz\wedge dx + \int\int_S (z-\overline{z_B}) f(x,y,z)\,dx\wedge dy\right].\end{aligned}$$

Proof. We have

$$\frac{\partial[(x-\alpha) f(x,y,z)]}{\partial x} = f(x,y,z) + (x-\alpha)\frac{\partial f(x,y,z)}{\partial x},$$
$$\frac{\partial[(y-\beta) f(x,y,z)]}{\partial y} = f(x,y,z) + (y-\beta)\frac{\partial f(x,y,z)}{\partial y}$$

and

$$\frac{\partial[(z-\gamma) f(x,y,z)]}{\partial z} = f(x,y,z) + (z-\gamma)\frac{\partial f(x,y,z)}{\partial z}.$$

By adding these three equalities we get

$$\begin{aligned}(2.9)\quad &\frac{\partial[(x-\alpha) f(x,y,z)]}{\partial x} + \frac{\partial[(y-\beta) f(x,y,z)]}{\partial y} + \frac{\partial[(z-\gamma) f(x,y,z)]}{\partial z} \\ &= 3f(x,y,z) + (x-\alpha)\frac{\partial f(x,y,z)}{\partial x} + (y-\beta)\frac{\partial f(x,y,z)}{\partial y} + (z-\gamma)\frac{\partial f(x,y,z)}{\partial z}\end{aligned}$$

for all $(x,y,z) \in B$.

Integrating this equality on B we get

$$\iiint_B \left(\frac{\partial \left[(x-\alpha) f(x,y,z) \right]}{\partial x} + \frac{\partial \left[(y-\beta) f(x,y,z) \right]}{\partial y} + \frac{\partial \left[(z-\gamma) f(x,y,z) \right]}{\partial z} \right) dxdydz \tag{2.10}$$
$$= 3 \iiint_B f(x,y,z)\, dxdydz + \iiint_B \left[(x-\alpha) \frac{\partial f(x,y,z)}{\partial x} + (y-\beta) \frac{\partial f(x,y,z)}{\partial y} + (z-\gamma) \frac{\partial f(x,y,z)}{\partial z} \right] dxdydz.$$

Applying the *Gauss-Ostrogradsky identity (2.6)* for the functions

$$P(x,y,z) = (x-\alpha) f(x,y,z),\ Q(x,y,z) = (y-\beta) f(x,y,z)$$

and

$$R(x,y,z) = (z-\gamma) f(x,y,z)$$

we obtain

$$\iiint_B \left(\frac{\partial \left[(x-\alpha) f(x,y,z) \right]}{\partial x} + \frac{\partial \left[(y-\beta) f(x,y,z) \right]}{\partial y} + \frac{\partial \left[(z-\gamma) f(x,y,z) \right]}{\partial z} \right) dxdydz \tag{2.11}$$
$$= \int\int_S (x-\alpha) f(x,y,z)\, dy \wedge dz + \int\int_S (y-\beta) f(x,y,z)\, dz \wedge dx + \int\int_S (z-\gamma) f(x,y,z)\, dx \wedge dy.$$

By (2.10) and (2.11) we get

$$3 \iiint_B f(x,y,z)\, dxdydz + \iiint_B \left[(x-\alpha) \frac{\partial f(x,y,z)}{\partial x} + (y-\beta) \frac{\partial f(x,y,z)}{\partial y} + (z-\gamma) \frac{\partial f(x,y,z)}{\partial z} \right] dxdydz$$
$$= \int\int_S (x-\alpha) f(x,y,z)\, dy \wedge dz + \int\int_S (y-\beta) f(x,y,z)\, dz \wedge dx + \int\int_S (z-\gamma) f(x,y,z)\, dx \wedge dy,$$

which is equivalent to the desired result (2.7). □

Remark 1. *For a function f as in Lemma 1 above, we define the points*

$$x_{B,\partial f} := \frac{\iiint_B x \frac{\partial f(x,y,z)}{\partial x} dxdydz}{\iiint_B \frac{\partial f(x,y,z)}{\partial x} dxdydz},\ y_{B,\partial f} := \frac{\iiint_B y \frac{\partial f(x,y,z)}{\partial y} dxdydz}{\iiint_B \frac{\partial f(x,y,z)}{\partial y} dxdydz},$$

and

$$z_{B,\partial f} := \frac{\iiint_B z \frac{\partial f(x,y,z)}{\partial z} dxdydz}{\iiint_B \frac{\partial f(x,y,z)}{\partial z} dxdydz}$$

provided the denominators are not zero.

If we take $\alpha = x_{B,\partial f}$, $\beta = y_{B,\partial f}$ *and* $\gamma = z_{B,\partial f}$ *in (2.7), then we get*

$$\iiint_B f(x,y,z)\,dxdydz = \frac{1}{3}\left[\int\int_S (x - x_{B,\partial f}) f(x,y,z)\,dy\wedge dz + \int\int_S (y - \beta y_{B,\partial f}) f(x,y,z)\,dz\wedge dx + \int\int_S (z - z_{B,\partial f}) f(x,y,z)\,dx\wedge dy\right], \tag{2.12}$$

since, obviously,

$$\iiint_B \left[(x_{B,\partial f} - x)\frac{\partial f(x,y,z)}{\partial x} + (y_{B,\partial f} - y)\frac{\partial f(x,y,z)}{\partial y} + (z_{B,\partial f} - z)\frac{\partial f(x,y,z)}{\partial z}\right] dxdydz = 0.$$

We also have the following dual approach:

Remark 2. *For a function* f *as in Lemma 1 above, we define the points*

$$x_{S,f} := \frac{\int\int_S xf(x,y,z)\,dy\wedge dz}{\int\int_S f(x,y,z)\,dy\wedge dz},\quad y_{S,f} := \frac{\int\int_S yf(x,y,z)\,dz\wedge dx}{\int\int_S f(x,y,z)\,dz\wedge dx}$$

and

$$z_{S,f} := \frac{\int\int_S zf(x,y,z)\,dx\wedge dy}{\int\int_S f(x,y,z)\,dx\wedge dy}$$

provided the denominators are not zero.

If we take $\alpha = x_{S,f}$, $\beta = y_{S,f}$ *and* $\gamma = z_{S,f}$ *in (2.7), then we get*

$$\iiint_B f(x,y,z)\,dxdydz = \frac{1}{3}\iiint_B \left[(x_{S,f} - x)\frac{\partial f(x,y,z)}{\partial x} + (y_{S,f} - y)\frac{\partial f(x,y,z)}{\partial y} + (z_{S,f} - z)\frac{\partial f(x,y,z)}{\partial z}\right] dxdydz \tag{2.13}$$

since, obviously,

$$\int\int_S (x - x_{S,f}) f(x,y,z)\,dy\wedge dz + \int\int_S (y - y_{S,f}) f(x,y,z)\,dz\wedge dx + \int\int_S (z - z_{S,f}) f(x,y,z)\,dx\wedge dy = 0.$$

3. Inequalities for Convex Functions

We have the following result:

Theorem 3. *Let* B *be a convex body in the three dimensional space* $\mathbb{R}^3$ *bounded by an orientable closed surface* S *and* $f : B \to \mathbb{C}$ *a continuously differentiable function* *defined on a open set containing B. If f is convex on B, then for any $(u,v,w)\in B$ we have*

$$f(u,v,w)+(\overline{x_B}-u)\frac{\partial f(u,v,w)}{\partial x}+(\overline{y_B}-v)\frac{\partial f(u,v,w)}{\partial y}+(\overline{z_B}-w)\frac{\partial f(u,v,w)}{\partial z}\leq\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz\leq\frac{1}{4}f(u,v,w)+\frac{1}{4}\frac{1}{V(B)}\left[\int\int_S(x-u)f(x,y,z)\,dy\wedge dz+\int\int_S(y-v)f(x,y,z)\,dz\wedge dx+\int\int_S(z-w)f(x,y,z)\,dx\wedge dy\right].\tag{3.1}$$

In particular, we have

$$f(\overline{x_B},\overline{y_B},\overline{z_B})\leq\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz\leq\frac{1}{4}f(\overline{x_B},\overline{y_B},\overline{z_B})+\frac{1}{4}\frac{1}{V(B)}\left[\int\int_S(x-\overline{x_B})f(x,y,z)\,dy\wedge dz+\int\int_S(y-\overline{y_B})f(x,y,z)\,dz\wedge dx+\int\int_S(z-\overline{z_B})f(x,y,z)\,dx\wedge dy\right].\tag{3.2}$$

Proof. By the gradient inequality for the convex function f on the convex set B we have

$$(x-u)\frac{\partial f(u,v,w)}{\partial x}+(y-v)\frac{\partial f(u,v,w)}{\partial y}+(z-w)\frac{\partial f(u,v,w)}{\partial z}\leq f(x,y,z)-f(u,v,w)\leq(x-u)\frac{\partial f(x,y,z)}{\partial x}+(y-v)\frac{\partial f(x,y,z)}{\partial y}+(z-w)\frac{\partial f(x,y,z)}{\partial z}$$

for all $(u,v,w)\,,\,(x,y,z)\in B$.

If we take the integral mean over the variables $(x,y,z)\in B$, we get

$$\frac{1}{V(B)}\iiint_B(x-u)\frac{\partial f(u,v,w)}{\partial x}dxdydz+\frac{1}{V(B)}\iiint_B(y-v)\frac{\partial f(u,v,w)}{\partial y}dxdydz+\frac{1}{V(B)}\iiint_B(z-w)\frac{\partial f(u,v,w)}{\partial z}dxdydz\leq\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz-\frac{1}{V(B)}\iiint_B f(u,v,w)\,dxdydz\leq\frac{1}{V(B)}\iiint_B(x-u)\frac{\partial f(x,y,z)}{\partial x}dxdydz+\frac{1}{V(B)}\iiint_B(y-v)\frac{\partial f(x,y,z)}{\partial y}dxdydz+\frac{1}{V(B)}\iiint_B(z-w)\frac{\partial f(x,y,z)}{\partial z}dxdydz$$

namely

$$(3.3)\quad (\overline{x_B}-u)\frac{\partial f(u,v,w)}{\partial x}+(\overline{y_B}-v)\frac{\partial f(u,v,w)}{\partial y}+(\overline{z_B}-w)\frac{\partial f(u,v,w)}{\partial z}$$
$$\leq \frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz - f(u,v,w)$$
$$\leq \frac{1}{V(B)}\iiint_B (x-u)\frac{\partial f(x,y,z)}{\partial x}dxdydz + \frac{1}{V(B)}\iiint_B (y-v)\frac{\partial f(x,y,z)}{\partial y}dxdydz$$
$$+\frac{1}{V(B)}\iiint_B (z-w)\frac{\partial f(x,y,z)}{\partial z}dxdydz$$

for all $(u,v,w)\in B$, which is an inequality of interest in itself.

The first inequality in (3.3) gives now the first part of (3.1).

From the identity (2.7) we get for $(\alpha,\beta,\gamma)=(u,v,w)$ that

$$\iiint_B f(x,y,z)\,dxdydz$$
$$=\frac{1}{3}\iiint_B\left[(u-x)\frac{\partial f(x,y,z)}{\partial x}+(v-y)\frac{\partial f(x,y,z)}{\partial y}\right.$$
$$\left.+(w-z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz$$
$$+\frac{1}{3}\left[\int\int_S (x-u)f(x,y,z)\,dy\wedge dz+\int\int_S (y-v)f(x,y,z)\,dz\wedge dx\right.$$
$$\left.+\int\int_S (z-w)f(x,y,z)\,dx\wedge dy\right],$$

namely

$$\frac{1}{V(B)}\iiint_B\left[(x-u)\frac{\partial f(x,y,z)}{\partial x}+(y-v)\frac{\partial f(x,y,z)}{\partial y}\right.$$
$$\left.+(z-w)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz=$$
$$+\frac{1}{V(B)}\left[\int\int_S (x-u)f(x,y,z)\,dy\wedge dz+\int\int_S (y-v)f(x,y,z)\,dz\wedge dx\right.$$
$$\left.+\int\int_S (z-w)f(x,y,z)\,dx\wedge dy\right]-3\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz.$$

From the second part of (3.3) we get

$$\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz-f(u,v,w)$$
$$\leq\frac{1}{V(B)}\left[\int\int_S (x-u)f(x,y,z)\,dy\wedge dz+\int\int_S (y-v)f(x,y,z)\,dz\wedge dx\right.$$
$$\left.+\int\int_S (z-w)f(x,y,z)\,dx\wedge dy\right]-3\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz,$$

namely

$$\frac{4}{V(B)}\iiint_B f(x,y,z)\,dxdydz \le f(u,v,w) + \frac{1}{V(B)}\left[\int\int_S (x-u) f(x,y,z)\,dy\wedge dz + \int\int_S (y-v) f(x,y,z)\,dz\wedge dx + \int\int_S (z-w) f(x,y,z)\,dx\wedge dy\right],$$

which gives the second part of the inequality (3.1). □

Remark 3. *The first inequality in (3.2) is the same as the one from inequality (1.2).*

Corollary 1. *With the assumptions of Theorem 3 and if* $(x_{S,f}, y_{S,f}, z_{S,f}) \in B$, *then*

$$f(x_{S,f}, y_{S,f}, z_{S,f}) + (\overline{x_B} - x_{S,f})\frac{\partial f(x_{S,f}, y_{S,f}, z_{S,f})}{\partial x} + (\overline{y_B} - y_{S,f})\frac{\partial f(x_{S,f}, y_{S,f}, z_{S,f})}{\partial y} + (\overline{z_B} - z_{S,f})\frac{\partial f(x_{S,f}, y_{S,f}, z_{S,f})}{\partial z} \le \frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz \le \frac{1}{4} f(x_{S,f}, y_{S,f}, z_{S,f}). \tag{3.4}$$

The proof follows by (3.1) observing that

$$\int\int_S (x - x_{S,f}) f(x,y,z)\,dy\wedge dz + \int\int_S (y - y_{S,f}) f(x,y,z)\,dz\wedge dx + \int\int_S (z - z_{S,f}) f(x,y,z)\,dx\wedge dy = 0.$$

Corollary 2. *With the assumptions of Theorem 3, we have*

$$\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz \le \frac{1}{3}\frac{1}{V(B)}\left[\int\int_S (x-\overline{x_B}) f(x,y,z)\,dy\wedge dz + \int\int_S (y-\overline{y_B}) f(x,y,z)\,dz\wedge dx + \int\int_S (z-\overline{z_B}) f(x,y,z)\,dx\wedge dy\right]. \tag{3.5}$$

Proof. From (3.2) we get

$$\begin{aligned}\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz &\le \frac{1}{4} f(\overline{x_B}, \overline{y_B}, \overline{z_B}) + \frac{1}{4}\frac{1}{V(B)}\left[\int\int_S (x-\overline{x_B}) f(x,y,z)\,dy\wedge dz + \int\int_S (y-\overline{y_B}) f(x,y,z)\,dz\wedge dx + \int\int_S (z-\overline{z_B}) f(x,y,z)\,dx\wedge dy\right]\\ &\le \frac{1}{4}\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz + \frac{1}{4}\frac{1}{V(B)}\left[\int\int_S (x-\overline{x_B}) f(x,y,z)\,dy\wedge dz + \int\int_S (y-\overline{y_B}) f(x,y,z)\,dz\wedge dx + \int\int_S (z-\overline{z_B}) f(x,y,z)\,dx\wedge dy\right],\end{aligned}$$

which implies that

$$\frac{3}{4V(B)}\iiint_B f(x,y,z)\,dxdydz \le \frac{1}{4}\frac{1}{V(B)}\left[\int\int_S (x-\overline{x_B})\,f(x,y,z)\,dy\wedge dz \right.$$
$$\left. +\int\int_S (y-\overline{y_B})\,f(x,y,z)\,dz\wedge dx + \int\int_S (z-\overline{z_B})\,f(x,y,z)\,dx\wedge dy\right]$$

that is equivalent to (3.5). □

Corollary 3. *With the assumptions of Theorem 3 we have*

$$\frac{1}{A(S)}\int\int_S f(u,v,w)\,dS + \frac{1}{A(S)}\int\int_S \left[(\overline{x_B}-u)\frac{\partial f(u,v,w)}{\partial x}\right.$$
$$\left. +(\overline{y_B}-v)\frac{\partial f(u,v,w)}{\partial y} + (\overline{z_B}-w)\frac{\partial f(u,v,w)}{\partial z}\right] dS$$
$$\le \frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz$$
$$\le \frac{1}{4}\frac{1}{A(S)}\int\int_S f(u,v,w)\,dS + \frac{1}{4}\frac{1}{V(B)}\left[\int\int_S (x-\overline{x_S})\,f(x,y,z)\,dy\wedge dz\right.$$
$$\left. +\int\int_S (y-\overline{y_S})\,f(x,y,z)\,dz\wedge dx + \int\int_S (z-\overline{z_S})\,f(x,y,z)\,dx\wedge dy\right]. \tag{3.6}$$

Proof. By taking the integral mean $\frac{1}{A(S)}\int\int_S (\cdot)\,dS$ over the variables (u,v,w) in the integral (3.1), we get

$$\frac{1}{A(S)}\int\int_S f(u,v,w)\,dS + \frac{1}{A(S)}\int\int_S \left[(\overline{x_B}-u)\frac{\partial f(u,v,w)}{\partial x}\right.$$
$$\left. +(\overline{y_B}-v)\frac{\partial f(u,v,w)}{\partial y} + (\overline{z_B}-w)\frac{\partial f(u,v,w)}{\partial z}\right] dS$$
$$\le \frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz$$
$$\le \frac{1}{4}\frac{1}{A(S)}\int\int_S f(u,v,w)\,dS$$
$$+\frac{1}{4}\frac{1}{V(B)}\left[\int\int_S\left(x-\frac{1}{A(S)}\int\int_S udS\right)f(x,y,z)\,dy\wedge dz\right.$$
$$+\int\int_S\left(y-\frac{1}{A(S)}\int\int_S vdS\right)f(x,y,z)\,dz\wedge dx$$
$$\left.+\int\int_S\left(z-\frac{1}{A(S)}\int\int_S wdS\right)f(x,y,z)\,dx\wedge dy\right],$$

which is equivalent to (3.6). □

Remark 4. *The second inequality in (3.6) is an equivalent formulation of the second inequality in (1.2).*

4. Applications for Three Dimensional Balls

Now, let us compute the surface integral

$$K(S(C,R),f) := \iint_{S(C,R)} f(x,y,z)\,dS,$$

where

$$S(C,R) := \left\{ (x,y,z) \in \mathbb{R}^3 \middle| (x-a)^2 + (y-b)^2 + (z-c)^2 = R^2 \right\}.$$

If we consider the parametrization of $S(C,R)$ given by:

$$S(C,R) : \begin{cases} x = R\cos\psi\cos\varphi + a \\ y = R\cos\psi\sin\varphi + b \\ z = R\sin\psi + c \end{cases} ; (\psi,\varphi) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [0, 2\pi]$$

and putting

$$A := \begin{vmatrix} \frac{\partial y}{\partial \psi} & \frac{\partial z}{\partial \psi} \\ \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = -R^2\cos^2\psi\cos\varphi,$$

$$B := \begin{vmatrix} \frac{\partial x}{\partial \psi} & \frac{\partial z}{\partial \psi} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = R^2\cos^2\psi\sin\varphi,$$

and

$$C := \begin{vmatrix} \frac{\partial x}{\partial \psi} & \frac{\partial y}{\partial \psi} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = -R^2\sin\psi\cos\psi,$$

we have that

$$A^2 + B^2 + C^2 = R^4\cos^2\psi \text{ for all } (\psi,\varphi) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [0, 2\pi].$$

Thus,

$$\begin{aligned} (4.1) \quad K(S(C,R),f) &= \iint_{S(C,R)} f(x,y,z)\,dS \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \Big[f(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c) \\ &\qquad \times \sqrt{A^2+B^2+C^2} \Big] d\psi d\varphi \\ &= R^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \cos\psi f(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c)\,d\psi d\varphi. \end{aligned}$$

We also have

$$(4.2)\quad L(S(C,R),f) := \int\int_{S(C,R)} (x-a) f(x,y,z)\, dy \wedge dz$$
$$+ \int\int_{S(C,R)} (y-b) f(x,y,z)\, dz \wedge dx + \int\int_{S(C,R)} (z-c) f(x,y,z)\, dx \wedge dy$$
$$= -R^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \cos^3\psi \cos^2\varphi$$
$$\times f(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c)\, d\psi d\varphi$$
$$+ R^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \cos^3\psi \sin^2\varphi$$
$$\times f(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c)\, d\psi d\varphi$$
$$- R^3 \int\int_S \sin^2\psi\cos\psi f(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c)\, d\psi d\varphi.$$

Let us consider the transformation $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ given by:

$$T_2(r,\psi,\varphi) := (r\cos\psi\cos\varphi + a, r\cos\psi\sin\varphi + b, r\sin\psi + c).$$

It is well known that the Jacobian of T_2 is

$$J(T_2) = r^2\cos\psi$$

and T_2 is a one-to-one mapping defined on the interval of $\mathbb{R}^3$, $[0,R] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [0, 2\pi]$, with values in the ball $B(C,R)$ from $\mathbb{R}^3$. Thus we have the change of variable:

$$(4.3)\quad I(B(C,R),f) := \iiint_{B(C,R)} f(x,y,z)\, dxdydz$$
$$= \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} f(r\cos\psi\cos\varphi + a, r\cos\psi\sin\varphi + b, r\sin\psi + c)\, r^2\cos\psi drd\psi d\varphi.$$

Assume that f is convex on the ball $B(C,R)$. From the inequality (3.2) we get

$$(4.4)\quad f(a,b,c) \le \frac{1}{V(B(C,R))} I(B(C,R),f)$$
$$\le \frac{1}{4} f(a,b,c) + \frac{1}{4V(B(C,R))} L(S(C,R),f),$$

where $V(B(C,R)) = \frac{4\pi R^3}{3}$, while from the inequality (3.5) we also have

$$(4.5)\quad \frac{1}{V(B(C,R))} I(B(C,R),f) \le \frac{1}{3V(B(C,R))} L(S(C,R),f).$$

Further, consider

$$J(S(C,R),f):=\int\int_{S(C,R)}\left[(\overline{x_B}-u)\frac{\partial f(u,v,w)}{\partial x}\right.$$
$$\left.+(\overline{y_B}-v)\frac{\partial f(u,v,w)}{\partial y}+(\overline{z_B}-w)\frac{\partial f(u,v,w)}{\partial z}\right]dS$$
$$=-R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\cos\psi\cos\varphi\frac{\partial f(R\cos\psi\cos\varphi+a,R\cos\psi\sin\varphi+b,R\sin\psi+c)}{\partial x}d\psi d\varphi$$
$$-R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\cos\psi\sin\varphi\frac{\partial f(R\cos\psi\cos\varphi+a,R\cos\psi\sin\varphi+b,R\sin\psi+c)}{\partial y}d\psi d\varphi$$
$$-R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\sin\psi\frac{\partial f(R\cos\psi\cos\varphi+a,R\cos\psi\sin\varphi+b,R\sin\psi+c)}{\partial z}d\psi d\varphi.$$

Then from the inequality (3.6) we get the following inequalities of interest:

$$\frac{1}{A(S(C,R))}K(S(C,R),f)+\frac{1}{A(S(C,R))}J(S(C,R),f)$$
$$\leq\frac{1}{V(B(C,R))}I(B(C,R),f)$$
$$\leq\frac{1}{4}\frac{1}{A(S(C,R))}K(S(C,R),f)+\frac{1}{4}\frac{1}{V(B(C,R))}L(S(C,R),f), \tag{4.6}$$

where $A(S(C,R))=4\pi R^2$ is the area of the sphere.

References

[1] Apostol, T. M.; *Calculus Volume II, Multi Variable Calculus and Linear Algebra, with Applications to Differential Equations and Probability*, Second Edition, John Wiley & Sons, New York, London, Sydney, Toronto, 1969.

[2] A. Barani, Hermite–Hadamard and Ostrowski type inequalities on hemispheres, *Mediterr. J. Math.* **13** (2016), 4253–4263

[3] M. Bessenyei, The Hermite–Hadamard inequality on simplices, *Amer. Math. Monthly* **115** (2008), 339–345.

[4] J. de la Cal and J. Cárcamo, Multidimensional Hermite-Hadamard inequalities and the convex order, *Journal of Mathematical Analysis and Applications*, vol. **324**, no. 1, pp. 248–261, 2006.

[5] S. S. Dragomir, On Hadamard's inequality on a disk, *Journal of Ineq. Pure & Appl. Math.,* **1** (2000), No. 1, Article 2. [Online https://www.emis.de/journals/JIPAM/article95.html?sid=95].

[6] S. S. Dragomir, On Hadamard's inequality for the convex mappings defined on a ball in the space and applications, *Math. Ineq. & Appl.*, **3** (2) (2000), 177-187.

[7] S. S. Dragomir, Double integral inequalities of Hermite-Hadamard type for h-convex functions on linear spaces. *Analysis* (Berlin) **37** (2017), no. 1, 13–22.

[8] S. S. Dragomir, Hermite-Hadamard type integral inequalities for double integral on general domains, Preprint *RGMIA Res. Rep. Coll.* **22** (2019), Art. 46, 10 pp. [Online http://rgmia.org/papers/v22/v22a46.pdf.]

[9] B. Gavrea, On Hadamard's inequality for the convex mappings defined a convex domain in the space. Journal of Inequalities in Pure and Applied Mathematics, Volume **1**, Issue 1, Article 9, 2000. [https://www.emis.de/journals/JIPAM/article102.html?sid=102].

[10] M. Matłoka, On Hadamard's inequality for h-convex function on a disk, *Applied Mathematics and Computation* 235 (2014) 118–123

[11] F.-C. Mitroi and E. Symeonidis, The converse of the Hermite-Hadamard inequality on simplices. *Expo. Math.* **30** (2012), 389-396.

[12] E. Neuman, Inequalities involving multivariate convex functions II, *Proc. Amer. Math. Soc.* **109** (1990), 965–974.

[13] E. Neuman, J. Pečarić, Inequalities involvingmultivariate convex functions, *J. Math. Anal. Appl.* **137** (1989), 541–549.

[14] S. Wasowicz and A. Witkowski, On some inequality of Hermite–Hadamard type. *Opusc. Math.* **32** (3)(2012), 591–600

[15] F.-L. Wang, The generalizations to several-dimensions of the classical Hadamard's inequality, *Mathematics in Practice and Theory*, vol. **36**, no. 9, pp. 370–373, 2006 (Chinese).

[16] F.-L. Wang, A family of mappings associated with Hadamard's inequality on a hypercube, *International Scholarly Research Network ISRN Mathematical Analysis* Volume **2011**, Article ID 594758, 9 pages doi:10.5402/2011/594758.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa