SOME TRIPLE INTEGRAL INEQUALITIES FOR BOUNDED FUNCTIONS DEFINED ON 3-DIMENSIONAL BODIES

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Abstract. In this paper we provide some bounds for the absolute value of the quantity

$$\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz-\delta - \frac{1}{3V(B)}\iiint_B\left[(\alpha-x)\frac{\partial f(x,y,z)}{\partial x}+(\beta-y)\frac{\partial f(x,y,z)}{\partial y} + (\gamma-z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz$$

for some choices of the parameters α, β, γ, δ and under the general assumption that B is a body in the 3-dimensional space $\mathbb{R}^3$ and $f:B\to\mathbb{C}$ is differentiable on B. For this purpose we use an identity obtained by the well known *Gauss-Ostrogradsky* theorem for the divergence of a continuously differentiable vector field. An example for 3-dimensional ball is also given.

1. Introduction

Recall the following inequalities of Hermite-Hadamard's type for convex functions defined on a ball $B(C,R)$, where $C=(a,b,c)\in\mathbb{R}^3$, $R>0$ and

$$B(C,R):=\left\{(x,y,z)\in\mathbb{R}^3\,\middle|\,(x-a)^2+(y-b)^2+(z-c)^2\le R^2\right\}.$$

The following theorem holds [10].

Theorem 1. *Let $f:B(C,R)\to\mathbb{R}$ be a convex mapping on the ball $B(C,R)$. Then we have the inequality:*

$$f(a,b,c)\le\frac{1}{V(B(C,R))}\iiint_{B(C,R)}f(x,y,z)\,dxdydz \le\frac{1}{\sigma(B(C,R))}\iint_{S(C,R)}f(x,y,z)\,dS, \tag{1.1}$$

where

$$S(C,R):=\left\{(x,y,z)\in\mathbb{R}^3\,\middle|\,(x-a)^2+(y-b)^2+(z-c)^2=R^2\right\}$$

and

$$V(B(C,R))=\frac{4\pi R^3}{3},\ \sigma(B(C,R))=4\pi R^2.$$

1991 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Ostrowski inequality, Hermite-Hadamard inequality, Double integral inequalities, Gauss-Ostrogradsky identity.

RGMIA Res. Rep. Coll. 22 (2019), Art. 62, 17 pp. Received 13/06/19

If the assumption of convexity is dropped, then one can prove the following Ostrowski type inequality for the centre of the ball as well, see [11].

Theorem 2. *Assume that $f : B(C,R) \to \mathbb{C}$ is differentiable on $B(C,R)$. Then*

$$\left| f(a,b,c) - \frac{1}{V(B(C,R))} \iiint_{B(C,R)} f(x,y,z)\, dxdydz \right| \leq \frac{3}{8} R \left[\left\| \frac{\partial f}{\partial x} \right\|_{B(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{B(C,R),\infty} + \left\| \frac{\partial f}{\partial z} \right\|_{B(C,R),\infty} \right], \tag{1.2}$$

provided

$$\left\| \frac{\partial f}{\partial x} \right\|_{B(C,R),\infty} := \sup_{(x,y,z)\in B(C,R)} \left| \frac{\partial f(x,y,z)}{\partial x} \right| < \infty ,$$

$$\left\| \frac{\partial f}{\partial y} \right\|_{B(C,R),\infty} := \sup_{(x,y,z)\in B(C,R)} \left| \frac{\partial f(x,y,z)}{\partial y} \right| < \infty$$

and

$$\left\| \frac{\partial f}{\partial z} \right\|_{B(C,R),\infty} := \sup_{(x,y,z)\in B(C,R)} \left| \frac{\partial f(x,y,z)}{\partial y} \right| < \infty.$$

This fact can be furthermore generalized to the following Ostrowski type inequality for any point in a convex body $B \subset \mathbb{R}^3$, see [11].

Theorem 3. *Assume that $f : B \to \mathbb{C}$ is differentiable on the convex body B and $(u,v,w) \in B$. If $V(B)$ is the volume of B, then*

$$\begin{aligned} &\left| f(u,v,w) - \frac{1}{V(B)} \iiint_B f(x,y,z)\, dxdydz \right| \\ &\leq \frac{1}{V(B)} \iiint_B |x-u| \left(\int_0^1 \left| \frac{\partial f}{\partial x} [t(x,y,z) + (1-t)(u,v,w)] \right| dt \right) dxdydz \\ &+ \frac{1}{V(B)} \iiint_B |y-v| \left(\int_0^1 \left| \frac{\partial f}{\partial y} [t(x,y,z) + (1-t)(u,v,w)] \right| dt \right) dxdydz \\ &+ \frac{1}{V(B)} \iiint_B |z-w| \left(\int_0^1 \left| \frac{\partial f}{\partial y} [t(x,y,z) + (1-t)(u,v,w)] \right| dt \right) dxdydz \\ &\leq \left\| \frac{\partial f}{\partial x} \right\|_{B,\infty} \frac{1}{V(B)} \iiint_B |x-u|\, dxdydz + \left\| \frac{\partial f}{\partial y} \right\|_{B,\infty} \frac{1}{V(B)} \iiint_B |y-v|\, dxdydz \\ &\qquad + \left\| \frac{\partial f}{\partial z} \right\|_{B,\infty} \frac{1}{V(B)} \iiint_B |z-w|\, dxdydz \end{aligned} \tag{1.3}$$

provided

$$\left\| \frac{\partial f}{\partial x} \right\|_{B,\infty}, \left\| \frac{\partial f}{\partial y} \right\|_{B,\infty}, \left\| \frac{\partial f}{\partial z} \right\|_{B,\infty} < \infty.$$

In particular,

$$\left|f\left(\overline{x_B},\overline{y_B},\overline{z_B}\right)-\frac{1}{V\left(B\right)}\iiint_B f\left(x,y,z\right)dxdydz\right| \tag{1.4}$$

$$\leq \frac{1}{V\left(B\right)}\iiint_B \left|x-\overline{x_B}\right|\left(\int_0^1\left|\frac{\partial f}{\partial x}\left[t\left(x,y,z\right)+\left(1-t\right)\left(\overline{x_B},\overline{y_B},\overline{z_B}\right)\right]\right|dt\right)dxdydz$$

$$+\frac{1}{V\left(B\right)}\iiint_B \left|y-\overline{y_B}\right|\left(\int_0^1\left|\frac{\partial f}{\partial y}\left[t\left(x,y,z\right)+\left(1-t\right)\left(\overline{x_B},\overline{y_B},\overline{z_B}\right)\right]\right|dt\right)dxdydz$$

$$+\frac{1}{V\left(B\right)}\iiint_B \left|z-\overline{z_B}\right|\left(\int_0^1\left|\frac{\partial f}{\partial y}\left[t\left(x,y,z\right)+\left(1-t\right)\left(\overline{x_B},\overline{y_B},\overline{z_B}\right)\right]\right|dt\right)dxdydz$$

$$\leq \left\|\frac{\partial f}{\partial x}\right\|_{B,\infty}\frac{1}{V\left(B\right)}\iiint_B\left|x-\overline{x_B}\right|dxdydz$$

$$+\left\|\frac{\partial f}{\partial y}\right\|_{B,\infty}\frac{1}{V\left(B\right)}\iiint_B\left|y-\overline{y_B}\right|dxdydz$$

$$+\left\|\frac{\partial f}{\partial z}\right\|_{B,\infty}\frac{1}{V\left(B\right)}\iiint_B\left|z-\overline{z_B}\right|dxdydz,$$

where

$$\overline{x_B}:=\frac{1}{V\left(B\right)}\iiint_B xdxdydz,\ \ \overline{y_B}:=\frac{1}{V\left(B\right)}\iiint_B ydxdydz,$$

$$\overline{z_B}:=\frac{1}{V\left(B\right)}\iiint_B zdxdydz$$

are the centre of gravity coordinates for the convex body B.

For some Hermite-Hadamard type inequalities for multiple integrals see [2], [6], [8], [9], [10], [17], [18], [19], [20], [25], [26] and [27]. For some Ostrowski type inequalities see [3], [4], [5], [7], [11], [12], [13], [14], [15], [16], [21], [22], [23] and [24].

In this paper we provide some bounds for the absolute value of the quantity

$$\frac{1}{V\left(B\right)}\iiint_B f\left(x,y,z\right)dxdydz-\delta \tag{1.5}$$

$$-\frac{1}{3V\left(B\right)}\iiint_B\left[\left(\alpha-x\right)\frac{\partial f\left(x,y,z\right)}{\partial x}+\left(\beta-y\right)\frac{\partial f\left(x,y,z\right)}{\partial y}\right.$$

$$\left.+\left(\gamma-z\right)\frac{\partial f\left(x,y,z\right)}{\partial z}\right]dxdydz$$

for certain choices of the parameters α, β, γ, δ and under the general assumption that B is a body in the 3-dimensional space $\mathbb{R}^3$ and $f:B\to\mathbb{C}$ is differentiable on B. For this purpose we use an identity obtained via the well known *Gauss-Ostrogradsky* theorem for the divergence of a continuously differentiable vector field. An example for 3-dimensional balls is also given.

We need the following preparations.

2. Some Notations, Definitions and Preliminary Facts

Following Apostol [1], consider a surface described by the vector equation

$$r\left(u,v\right)=x\left(u,v\right)\overrightarrow{i}+y\left(u,v\right)\overrightarrow{j}+z\left(u,v\right)\overrightarrow{k} \tag{2.1}$$

where $(u,v) \in [a,b] \times [c,d]$.

If x, y, z are differentiable on $[a,b] \times [c,d]$ we consider the two vectors

$$\frac{\partial r}{\partial u} = \frac{\partial x}{\partial u}\overrightarrow{i} + \frac{\partial y}{\partial u}\overrightarrow{j} + \frac{\partial z}{\partial u}\overrightarrow{k}$$

and

$$\frac{\partial r}{\partial v} = \frac{\partial x}{\partial v}\overrightarrow{i} + \frac{\partial y}{\partial v}\overrightarrow{j} + \frac{\partial z}{\partial v}\overrightarrow{k}.$$

The *cross product* of these two vectors $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ will be referred to as the fundamental vector product of the representation r. Its components can be expressed as *Jacobian determinants*. In fact, we have [1, p. 420]

$$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \overrightarrow{i} + \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix} \overrightarrow{j} + \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \overrightarrow{k} \tag{2.2}$$

$$= \frac{\partial (y,z)}{\partial (u,v)}\overrightarrow{i} + \frac{\partial (z,x)}{\partial (u,v)}\overrightarrow{j} + \frac{\partial (x,y)}{\partial (u,v)}\overrightarrow{k}.$$

Let $S = r(T)$ be a parametric surface described by a vector-valued function r defined on the box $T = [a,b] \times [c,d]$. The area of S denoted A_S is defined by the double integral [1, p. 424-425]

$$A_S = \int_a^b \int_c^d \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| dudv \tag{2.3}$$

$$= \int_a^b \int_c^d \sqrt{\left(\frac{\partial (y,z)}{\partial (u,v)}\right)^2 + \left(\frac{\partial (z,x)}{\partial (u,v)}\right)^2 + \left(\frac{\partial (x,y)}{\partial (u,v)}\right)^2} dudv.$$

We define surface integrals in terms of a parametric representation for the surface. One can prove that under certain general conditions the value of the integral is independent of the representation.

Let $S = r(T)$ be a parametric surface described by a vector-valued differentiable function r defined on the box $T = [a,b] \times [c,d]$ and let $f : S \to \mathbb{C}$ defined and bounded on S. The surface integral of f over S is defined by [1, p. 430]

$$\int\int_S f dS = \int_a^b \int_c^d f(x,y,z) \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| dudv \tag{2.4}$$

$$= \int_a^b \int_c^d f(x(u,v), y(u,v), z(u,v))$$

$$\times \sqrt{\left(\frac{\partial (y,z)}{\partial (u,v)}\right)^2 + \left(\frac{\partial (z,x)}{\partial (u,v)}\right)^2 + \left(\frac{\partial (x,y)}{\partial (u,v)}\right)^2} dudv.$$

If $S = r(T)$ is a parametric surface, the fundamental vector product $N = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ is normal to S at each regular point of the surface. At each such point there are two unit normals, a unit normal n_1, which has the same direction as N, and a unit normal n_2 which has the opposite direction. Thus

$$n_1 = \frac{N}{\|N\|} \text{ and } n_2 = -n_1.$$

Let n be one of the two normals n_1 or n_2. Let also F be a vector field defined on S and assume that the surface integral,

$$\int\int_S (F \cdot n)\, dS,$$

called the flux surface integral, exists. Here $F \cdot n$ is the dot or inner product.

We can write [1, p. 434]

$$\int\int_S (F \cdot n)\, dS = \pm \int_a^b \int_c^d F(r(u,v)) \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right) dudv$$

where the sign " $+$ " is used if $n = n_1$ and the " $-$ " sign is used if $n = n_2$.

If

$$F(x,y,z) = P(x,y,z)\, \overrightarrow{i} + Q(x,y,z)\, \overrightarrow{j} + R(x,y,z)\, \overrightarrow{k}$$

and

$$r(u,v) = x(u,v)\, \overrightarrow{i} + y(u,v)\, \overrightarrow{j} + z(u,v)\, \overrightarrow{k} \text{ where } (u,v) \in [a,b] \times [c,d]$$

then the flux surface integral for $n = n_1$ can be explicitly calculated as [1, p. 435]

$$\begin{aligned} \int\int_S (F \cdot n)\, dS &= \int_a^b \int_c^d P(x(u,v), y(u,v), z(u,v)) \frac{\partial(y,z)}{\partial(u,v)} dudv \\ &+ \int_a^b \int_c^d Q(x(u,v), y(u,v), z(u,v)) \frac{\partial(z,x)}{\partial(u,v)} dudv \\ &+ \int_a^b \int_c^d R(x(u,v), y(u,v), z(u,v)) \frac{\partial(x,y)}{\partial(u,v)} dudv. \end{aligned} \tag{2.5}$$

The sum of the double integrals on the right is often written more briefly as [1, p. 435]

$$\int\int_S P(x,y,z)\, dy \wedge dz + \int\int_S Q(x,y,z)\, dz \wedge dx + \int\int_S R(x,y,z)\, dx \wedge dy.$$

Let $B \subset \mathbb{R}^3$ be a solid in 3-space bounded by an orientable closed surface S, and let n be the unit outer normal to S. If F is a continuously differentiable vector field defined on B, we have the *Gauss-Ostrogradsky identity*

$$\iiint_B (\operatorname{div} F)\, dV = \int\int_S (F \cdot n)\, dS. \tag{GO}$$

If we express

$$F(x,y,z) = P(x,y,z)\, \overrightarrow{i} + Q(x,y,z)\, \overrightarrow{j} + R(x,y,z)\, \overrightarrow{k},$$

then (GO) can be written as

$$\begin{aligned} &\iiint_B \left(\frac{\partial P(x,y,z)}{\partial x} + \frac{\partial Q(x,y,z)}{\partial y} + \frac{\partial R(x,y,z)}{\partial z} \right) dxdydz \\ &= \int\int_S P(x,y,z)\, dy \wedge dz + \int\int_S Q(x,y,z)\, dz \wedge dx \\ &\qquad + \int\int_S R(x,y,z)\, dx \wedge dy. \end{aligned} \tag{2.6}$$

By taking the real and imaginary part, we can extend the above inequality for complex valued functions P, Q, R defined on B.

3. Some Identities of Interest

We have:

Lemma 1. *Let B be a solid in the three dimensional space $\mathbb{R}^3$ bounded by an orientable closed surface S. If $f : B \to \mathbb{C}$ is a continuously differentiable function defined on a open set containing B, then we have the equality*

$$\begin{aligned}(3.1)\quad & \frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz-\delta \\ &= \frac{1}{3V(B)}\iiint_B\left[(\alpha-x)\frac{\partial f(x,y,z)}{\partial x}+(\beta-y)\frac{\partial f(x,y,z)}{\partial y}\right. \\ &\qquad\left.+(\gamma-z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz \\ &\quad+\frac{1}{3V(B)}\left[\int\int_S (x-\alpha)\left[f(x,y,z)-\delta\right]dy\wedge dz\right. \\ &\qquad+\int\int_S (y-\beta)\left[f(x,y,z)-\delta\right]dz\wedge dx \\ &\qquad\left.+\int\int_S (z-\gamma)\left[f(x,y,z)-\delta\right]dx\wedge dy\right]\end{aligned}$$

for all α, β, γ and δ complex numbers.

In particular, we have

$$\begin{aligned}(3.2)\quad & \frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz-\delta \\ &= \frac{1}{3V(B)}\iiint_B\left[(\overline{x_B}-x)\frac{\partial f(x,y,z)}{\partial x}+(\overline{y_B}-y)\frac{\partial f(x,y,z)}{\partial y}\right. \\ &\qquad\left.+(\overline{z_B}-z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz \\ &\quad+\frac{1}{3V(B)}\left[\int\int_S (x-\overline{x_B})\left[f(x,y,z)-\delta\right]dy\wedge dz\right. \\ &\qquad+\int\int_S (y-\overline{y_B})\left[f(x,y,z)-\delta\right]dz\wedge dx \\ &\qquad\left.+\int\int_S (z-\overline{z_B})\left[f(x,y,z)-\delta\right]dx\wedge dy\right].\end{aligned}$$

Proof. It would suffice to prove the equality (3.1) for $\delta = 0$ since the general case will follows by replacing f with $f - \delta$.

We have

$$\frac{\partial\left[(x-\alpha)f(x,y,z)\right]}{\partial x}=f(x,y,z)+(x-\alpha)\frac{\partial f(x,y,z)}{\partial x},$$
$$\frac{\partial\left[(y-\beta)f(x,y,z)\right]}{\partial y}=f(x,y,z)+(y-\beta)\frac{\partial f(x,y,z)}{\partial y}$$

and

$$\frac{\partial\left[(z-\gamma)f(x,y,z)\right]}{\partial z}=f(x,y,z)+(z-\gamma)\frac{\partial f(x,y,z)}{\partial z}.$$

By adding these three equalities we get

$$\frac{\partial\left[(x-\alpha) f(x,y,z)\right]}{\partial x}+\frac{\partial\left[(y-\beta) f(x,y,z)\right]}{\partial y}+\frac{\partial\left[(z-\gamma) f(x,y,z)\right]}{\partial z} = 3f(x,y,z) + (x-\alpha)\frac{\partial f(x,y,z)}{\partial x}+(y-\beta)\frac{\partial f(x,y,z)}{\partial y}+(z-\gamma)\frac{\partial f(x,y,z)}{\partial z} \tag{3.3}$$

for all $(x,y,z)\in B$.

Integrating this equality on B we get

$$\iiint_B\left(\frac{\partial\left[(x-\alpha) f(x,y,z)\right]}{\partial x}+\frac{\partial\left[(y-\beta) f(x,y,z)\right]}{\partial y} +\frac{\partial\left[(z-\gamma) f(x,y,z)\right]}{\partial z}\right)dxdydz = 3\iiint_B f(x,y,z)\,dxdydz + \iiint_B\left[(x-\alpha)\frac{\partial f(x,y,z)}{\partial x}+(y-\beta)\frac{\partial f(x,y,z)}{\partial y} +(z-\gamma)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz. \tag{3.4}$$

Applying the *Gauss-Ostrogradsky identity (2.6)* for the functions

$$P(x,y,z)=(x-\alpha)f(x,y,z),\ Q(x,y,z)=(y-\beta)f(x,y,z)$$

and

$$R(x,y,z)=(z-\gamma)f(x,y,z)$$

we obtain

$$\iiint_B\left(\frac{\partial\left[(x-\alpha) f(x,y,z)\right]}{\partial x}+\frac{\partial\left[(y-\beta) f(x,y,z)\right]}{\partial y} +\frac{\partial\left[(z-\gamma) f(x,y,z)\right]}{\partial z}\right)dxdydz = \int\int_S (x-\alpha)f(x,y,z)\,dy\wedge dz+\int\int_S (y-\beta)f(x,y,z)\,dz\wedge dx + \int\int_S (z-\gamma)f(x,y,z)\,dx\wedge dy. \tag{3.5}$$

By (3.4) and (3.5) we get

$$3\iiint_B f(x,y,z)\,dxdydz + \iiint_B\left[(x-\alpha)\frac{\partial f(x,y,z)}{\partial x}+(y-\beta)\frac{\partial f(x,y,z)}{\partial y}+(z-\gamma)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz = \int\int_S (x-\alpha)f(x,y,z)\,dy\wedge dz+\int\int_S (y-\beta)f(x,y,z)\,dz\wedge dx + \int\int_S (z-\gamma)f(x,y,z)\,dx\wedge dy,$$

which is equivalent to

$$\begin{aligned}
&\iiint_B f(x,y,z)\,dxdydz \\
&= \frac{1}{3}\iiint_B \left[(\alpha - x)\frac{\partial f(x,y,z)}{\partial x} + (\beta - y)\frac{\partial f(x,y,z)}{\partial y} + (\gamma - z)\frac{\partial f(x,y,z)}{\partial z}\right] dxdydz \\
&\quad + \frac{1}{3}\left[\int\int_S (x-\alpha) f(x,y,z)\,dy\wedge dz + \int\int_S (y-\beta) f(x,y,z)\,dz\wedge dx \right. \\
&\qquad \left. + \int\int_S (z-\gamma) f(x,y,z)\,dx\wedge dy\right]
\end{aligned}$$

that, by division with $V(B)$ proves the claim. □

Remark 1. *For a function f as in Lemma 1 above, we define the points*

$$x_{B,\partial f} := \frac{\iiint_B x\frac{\partial f(x,y,z)}{\partial x}dxdydz}{\iiint_B \frac{\partial f(x,y,z)}{\partial x}dxdydz},\ y_{B,\partial f} := \frac{\iiint_B y\frac{\partial f(x,y,z)}{\partial y}dxdydz}{\iiint_B \frac{\partial f(x,y,z)}{\partial y}dxdydz},$$

and

$$z_{B,\partial f} := \frac{\iiint_B z\frac{\partial f(x,y,z)}{\partial z}dxdydz}{\iiint_B \frac{\partial f(x,y,z)}{\partial z}dxdydz}$$

provided the denominators are not zero.

If we take $\alpha = x_{B,\partial f}$, $\beta = y_{B,\partial f}$ and $\gamma = z_{B,\partial f}$ in (3.1), then we get

$$\begin{aligned}
&\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz - \delta \\
&= \frac{1}{3V(B)}\left[\int\int_S (x - x_{B,\partial f})[f(x,y,z)-\delta]\,dy\wedge dz \right. \\
&\quad + \int\int_S (y - \beta y_{B,\partial f})[f(x,y,z)-\delta]\,dz\wedge dx \\
&\qquad \left. + \int\int_S (z - z_{B,\partial f})[f(x,y,z)-\delta]\,dx\wedge dy\right],
\end{aligned} \tag{3.6}$$

since, obviously,

$$\begin{aligned}
&\iiint_B \left[(x_{B,\partial f} - x)\frac{\partial f(x,y,z)}{\partial x} + (y_{B,\partial f} - y)\frac{\partial f(x,y,z)}{\partial y}\right. \\
&\qquad \left. + (z_{B,\partial f} - z)\frac{\partial f(x,y,z)}{\partial z}\right] dxdydz = 0.
\end{aligned}$$

Remark 2. *Let B be a solid in the three dimensional space $\mathbb{R}^3$ bounded by an orientable closed surface S described by the vector equation*

$$r(u,v) = x(u,v)\overrightarrow{i} + y(u,v)\overrightarrow{j} + z(u,v)\overrightarrow{k},\ \ (u,v)\in[a,b]\times[c,d]$$

where $x(u,v)$, $y(u,v)$, $z(u,v)$ are differentiable.

From the equation (3.1) we get

$$\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz-\delta \tag{3.7}$$
$$-\frac{1}{3V(B)}\iiint_B\left[(\alpha-x)\frac{\partial f(x,y,z)}{\partial x}+(\beta-y)\frac{\partial f(x,y,z)}{\partial y}+(\gamma-z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz$$
$$=\frac{1}{3V(B)}\left[\int_a^b\int_c^d(x(u,v)-\alpha)\left[f(x(u,v),y(u,v),z(u,v))-\delta\right]\frac{\partial(y,z)}{\partial(u,v)}dudv\right.$$
$$+\int_a^b\int_c^d(y(u,v)-\beta)\left[f(x(u,v),y(u,v),z(u,v))-\delta\right]\frac{\partial(z,x)}{\partial(u,v)}dudv$$
$$\left.+\int_a^b\int_c^d(z(u,v)-\gamma)\left[f(x(u,v),y(u,v),z(u,v))-\delta\right]\frac{\partial(x,y)}{\partial(u,v)}dudv\right]$$

for all α, β, γ and δ complex numbers, while from (3.2) we have

$$\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz-\delta \tag{3.8}$$
$$-\frac{1}{3V(B)}\iiint_B\left[(\overline{x_B}-x)\frac{\partial f(x,y,z)}{\partial x}+(\overline{y_B}-y)\frac{\partial f(x,y,z)}{\partial y}+(\overline{z_B}-z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz$$
$$=\frac{1}{3V(B)}\left[\int_a^b\int_c^d(x(u,v)-\overline{x_B})\left[f(x(u,v),y(u,v),z(u,v))-\delta\right]\frac{\partial(y,z)}{\partial(u,v)}dudv\right.$$
$$+\int_a^b\int_c^d(y(u,v)-\overline{y_B})\left[f(x(u,v),y(u,v),z(u,v))-\delta\right]\frac{\partial(z,x)}{\partial(u,v)}dudv$$
$$\left.+\int_a^b\int_c^d(z(u,v)-\overline{z_B})\left[f(x(u,v),y(u,v),z(u,v))-\delta\right]\frac{\partial(x,y)}{\partial(u,v)}dudv\right]$$

for all $\delta\in\mathbb{R}$.

From (3.6) we get

$$\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz-\delta \tag{3.9}$$
$$=\frac{1}{3V(B)}\left[\int_a^b\int_c^d(x(u,v)-x_{B,\partial f})\left[f(x(u,v),y(u,v),z(u,v))-\delta\right]\frac{\partial(y,z)}{\partial(u,v)}dudv\right.$$
$$+\int_a^b\int_c^d(y(u,v)-y_{B,\partial f})\left[f(x(u,v),y(u,v),z(u,v))-\delta\right]\frac{\partial(z,x)}{\partial(u,v)}dudv$$
$$\left.+\int_a^b\int_c^d(z(u,v)-z_{B,\partial f})\left[f(x(u,v),y(u,v),z(u,v))-\delta\right]\frac{\partial(x,y)}{\partial(u,v)}dudv\right]$$

for all $\delta\in\mathbb{R}$.

4. Inequalities for Bounded Functions

Let B be a solid in the three dimensional space $\mathbb{R}^3$ bounded by an orientable closed surface S. Now, for ϕ, $\Phi \in \mathbb{C}$, define the sets of complex-valued functions

$$\begin{aligned}&\bar{U}_S(\phi,\Phi)\\ &:=\left\{f:S\to\mathbb{C}|\operatorname{Re}\left[(\Phi-f(x,y,z))\left(\overline{f(x,y,z)}-\overline{\phi}\right)\right]\geq 0\ \text{ for each }\ (x,y,z)\in S\right\}\end{aligned}$$

and

$$\bar{\Delta}_S(\phi,\Phi):=\left\{f:S\to\mathbb{C}|\ \left|f(x,y,z)-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi|\ \text{ for each }\ (x,y,z)\in S\right\}.$$

The following representation result may be stated.

Proposition 1. *For any ϕ, $\Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that $\bar{U}_S(\phi,\Phi)$ and $\bar{\Delta}_S(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_S(\phi,\Phi)=\bar{\Delta}_S(\phi,\Phi).\tag{4.1}$$

Proof. We observe that for any $w\in\mathbb{C}$ we have the equivalence

$$\left|w-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi|$$

if and only if

$$\operatorname{Re}\left[(\Phi-w)\left(\overline{w}-\overline{\phi}\right)\right]\geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Phi-\phi|^2-\left|w-\frac{\phi+\Phi}{2}\right|^2=\operatorname{Re}\left[(\Phi-w)\left(\overline{w}-\overline{\phi}\right)\right]$$

that holds for any $w\in\mathbb{C}$.

The equality (4.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 1. *For any ϕ, $\Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that*

$$\begin{aligned}\bar{U}_S(\phi,\Phi)=\{f:S\to\mathbb{C}\ |\ &(\operatorname{Re}\Phi-\operatorname{Re}f(x,y,z))(\operatorname{Re}f(x,y,z)-\operatorname{Re}\phi)\\ &+(\operatorname{Im}\Phi-\operatorname{Im}f(x,y,z))(\operatorname{Im}f(x,y,z)-\operatorname{Im}\phi)\geq 0\ \textit{for each}\ (x,y,z)\in S\}.\end{aligned}\tag{4.2}$$

Now, if we assume that $\operatorname{Re}(\Phi)\geq\operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi)\geq\operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_S(\phi,\Phi):=\{f:S\to\mathbb{C}\ |\ &\operatorname{Re}(\Phi)\geq\operatorname{Re}f(x,y,z)\geq\operatorname{Re}(\phi)\\ &\text{and}\operatorname{Im}(\Phi)\geq\operatorname{Im}f(x,y,z)\geq\operatorname{Im}(\phi)\ \text{ for each }\ (x,y,z)\in S\}.\end{aligned}\tag{4.3}$$

One can easily observe that $\bar{S}_S(\phi,\Phi)$ is closed, convex and

$$\emptyset\neq\bar{S}_S(\phi,\Phi)\subseteq\bar{U}_S(\phi,\Phi).\tag{4.4}$$

Theorem 4. *Let B be a solid in the three dimensional space $\mathbb{R}^3$ bounded by an orientable closed surface S described by the vector equation*

$$r(u,v)=x(u,v)\overrightarrow{i}+y(u,v)\overrightarrow{j}+z(u,v)\overrightarrow{k},\ \ (u,v)\in[a,b]\times[c,d]$$

where $x(u,v)$, $y(u,v)$, $z(u,v)$ *are differentiable. If* $f \in \bar{\Delta}_S(\phi,\Phi)$ *for some* ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, *then*

$$\begin{aligned}(4.5)\quad &\left|\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz - \frac{\phi+\Phi}{2}\right.\\ &\quad - \frac{1}{3V(B)}\iiint_B \left[(\alpha - x)\frac{\partial f(x,y,z)}{\partial x} + (\beta - y)\frac{\partial f(x,y,z)}{\partial y}\right.\\ &\qquad \left.\left. + (\gamma - z)\frac{\partial f(x,y,z)}{\partial z}\right] dxdydz\right|\\ &\qquad\qquad \leq \frac{1}{6V(B)}\left|\Phi - \phi\right| M(S,\alpha,\beta,\gamma)\end{aligned}$$

where

$$\begin{aligned}M(S,\alpha,\beta,\gamma) &:= \int_a^b\int_c^d \left|x(u,v)-\alpha\right|\left|\frac{\partial(y,z)}{\partial(u,v)}\right| dudv\\ &+ \int_a^b\int_c^d \left|y(u,v)-\beta\right|\left|\frac{\partial(z,x)}{\partial(u,v)}\right| dudv + \int_a^b\int_c^d \left|z(u,v)-\gamma\right|\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.\end{aligned}$$

Moreover, if we put $\square := [a,b]\times[c,d]$, *then we have the bounds*

$$(4.6)\quad M(S,\alpha,\beta,\gamma) \leq \begin{cases} \left\|\frac{\partial(y,z)}{\partial(\cdot,\cdot)}\right\|_{\square,\infty}\|x-\alpha\|_{\square,1} + \left\|\frac{\partial(z,x)}{\partial(\cdot,\cdot)}\right\|_{\square,\infty}\|y-\beta\|_{\square,1} \\ \quad + \left\|\frac{\partial(x,y)}{\partial(\cdot,\cdot)}\right\|_{\square,\infty}\|z-\gamma\|_{\square,1}, \\[2ex] \left\|\frac{\partial(y,z)}{\partial(\cdot,\cdot)}\right\|_{\square,p}\|x-\alpha\|_{\square,q} + \left\|\frac{\partial(z,x)}{\partial(\cdot,\cdot)}\right\|_{\square,p}\|y-\beta\|_{\square,q} \\ \quad + \left\|\frac{\partial(x,y)}{\partial(\cdot,\cdot)}\right\|_{\square,p}\|z-\gamma\|_{\square,q}, \\[2ex] \left\|\frac{\partial(y,z)}{\partial(\cdot,\cdot)}\right\|_{\square,1}\|x-\alpha\|_{\square,\infty} + \left\|\frac{\partial(z,x)}{\partial(\cdot,\cdot)}\right\|_{\square,p}\|y-\beta\|_{\square,\infty} \\ \quad + \left\|\frac{\partial(x,y)}{\partial(\cdot,\cdot)}\right\|_{\square,1}\|z-\gamma\|_{\square,\infty}. \end{cases}$$

Proof. From (3.7) we have for $\delta = \frac{\phi+\Phi}{2}$ that

$$\begin{aligned}&\left|\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz - \frac{\phi+\Phi}{2}\right.\\ &\quad - \frac{1}{3V(B)}\iiint_B \left[(\alpha - x)\frac{\partial f(x,y,z)}{\partial x} + (\beta - y)\frac{\partial f(x,y,z)}{\partial y}\right.\\ &\qquad \left.\left. + (\gamma - z)\frac{\partial f(x,y,z)}{\partial z}\right] dxdydz\right|\\ &\leq \frac{1}{3V(B)}\left[\int_a^b\int_c^d \left|x(u,v)-\alpha\right|\left|f(x(u,v),y(u,v),z(u,v)) - \frac{\phi+\Phi}{2}\right|\left|\frac{\partial(y,z)}{\partial(u,v)}\right| dudv\right.\\ &\quad + \int_a^b\int_c^d \left|y(u,v)-\beta\right|\left|f(x(u,v),y(u,v),z(u,v)) - \frac{\phi+\Phi}{2}\right|\left|\frac{\partial(z,x)}{\partial(u,v)}\right| dudv\\ &\quad \left. + \int_a^b\int_c^d \left|z(u,v)-\gamma\right|\left|f(x(u,v),y(u,v),z(u,v)) - \frac{\phi+\Phi}{2}\right|\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv\right]\end{aligned}$$

$$\leq \frac{1}{6V(B)}|\Phi-\phi| \left[\int_a^b\int_c^d |x(u,v)-\alpha|\left|\frac{\partial(y,z)}{\partial(u,v)}\right| dudv + \int_a^b\int_c^d |y(u,v)-\beta|\left|\frac{\partial(z,x)}{\partial(u,v)}\right| dudv + \int_a^b\int_c^d |z(u,v)-\gamma|\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv\right]$$
$$= \frac{1}{6V(B)}|\Phi-\phi| \; M(S,\alpha,\beta,\gamma)$$

which proves the inequality (4.5).

The bounds in (4.6) follow by Hölder's inequalities, for which we only mention

$$\int_a^b\int_c^d |x(u,v)-\alpha|\left|\frac{\partial(y,z)}{\partial(u,v)}\right| dudv \leq \begin{cases} \sup_{(u,v)\in[a,b]\times[c,d]}\left|\frac{\partial(y,z)}{\partial(u,v)}\right|\int_a^b\int_c^d |x(u,v)-\alpha|\,dudv, \\ \\ \left(\int_a^b\int_c^d |x(u,v)-\alpha|^q\,dudv\right)^{1/q}\left(\int_a^b\int_c^d \left|\frac{\partial(y,z)}{\partial(u,v)}\right|^p dudv\right)^{1/p} \\ \text{if } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \sup_{(u,v)\in[a,b]\times[c,d]}|x(u,v)-\alpha|\int_a^b\int_c^d\left|\frac{\partial(y,z)}{\partial(u,v)}\right| dudv. \end{cases}$$

□

Corollary 2. *With the assumptions of Theorem 4 we have the inequality*

$$\left|\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz - \frac{\phi+\Phi}{2} - \frac{1}{3V(B)}\iiint_B\left[(\alpha-x)\frac{\partial f(x,y,z)}{\partial x} + (\beta-y)\frac{\partial f(x,y,z)}{\partial y} + (\gamma-z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz\right| \tag{4.7}$$
$$\leq \frac{1}{6V(B)}|\Phi-\phi| \int\int_S\left(|x-\alpha|^2+|y-\beta|^2+|z-\gamma|^2\right)^{1/2}dS$$
$$\leq \frac{A_S}{6V(B)}|\Phi-\phi|\sup_{(x,y,z)\in S}\left(|x-\alpha|^2+|y-\beta|^2+|z-\gamma|^2\right)^{1/2}.$$

Proof. Using the discrete Cauchy-Bunyakovsky-Schwarz inequality we have

$$|x(u,v)-\alpha|\left|\frac{\partial(y,z)}{\partial(u,v)}\right| + |y(u,v)-\beta|\left|\frac{\partial(z,x)}{\partial(u,v)}\right| + |z(u,v)-\gamma|\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \tag{4.8}$$
$$\leq \left(|x(u,v)-\alpha|^2+|y(u,v)-\beta|^2+|z(u,v)-\gamma|^2\right)^{1/2} \times\left(\left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2+\left|\frac{\partial(z,x)}{\partial(u,v)}\right|^2+\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2\right)^{1/2}$$

for all $(u,v)\in[a,b]\times[c,d]$.

By taking the double integral over (u,v) on $[a,b]\times[c,d]$ we get

$$\begin{aligned}M(S,\alpha,\beta,\gamma) &\le \int_a^b\int_c^d \left(\left|x(u,v)-\alpha\right|^2+\left|y(u,v)-\beta\right|^2+\left|z(u,v)-\gamma\right|^2\right)^{1/2}\\ &\times\left(\left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2+\left|\frac{\partial(z,x)}{\partial(u,v)}\right|^2+\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2\right)^{1/2} dudv\\ &= \int\int_S \left(\left|x-\alpha\right|^2+\left|y-\beta\right|^2+\left|z-\gamma\right|^2\right)^{1/2} dS\end{aligned}$$

and by (4.5) we get the desired result (4.7). □

Remark 3. *If $f\in\bar{\Delta}_S(\phi,\Phi)$ for some $\phi,\ \Phi\in\mathbb{C},\ \phi\ne\Phi$, then by taking $(\alpha,\beta,\gamma)=(\overline{x_B},\overline{y_B},\overline{z_B})$ in Theorem 4 we get*

$$\begin{aligned}(4.9)\quad &\left|\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz-\frac{\phi+\Phi}{2}\right.\\ &-\frac{1}{3V(B)}\iiint_B\left[(\overline{x_B}-x)\frac{\partial f(x,y,z)}{\partial x}+(\overline{y_B}-y)\frac{\partial f(x,y,z)}{\partial y}\right.\\ &\left.\left.+(\overline{z_B}-z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz\right|\le\frac{1}{6V(B)}\left|\Phi-\phi\right|\ M(\overline{x_B},\overline{y_B},\overline{z_B})\end{aligned}$$

where

$$\begin{aligned}(4.10)\quad M(\overline{x_B},\overline{y_B},\overline{z_B}) &:= \int_a^b\int_c^d\left|x(u,v)-\overline{x_B}\right|\left|\frac{\partial(y,z)}{\partial(u,v)}\right|dudv\\ &+\int_a^b\int_c^d\left|y(u,v)-\overline{y_B}\right|\left|\frac{\partial(z,x)}{\partial(u,v)}\right|dudv+\int_a^b\int_c^d\left|z(u,v)-\overline{z_B}\right|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv.\end{aligned}$$

Moreover,

$$M(\overline{x_B},\overline{y_B},\overline{z_B})\le\begin{cases}\left\|\frac{\partial(y,z)}{\partial(\cdot,\cdot)}\right\|_{\square,\infty}\left\|x-\overline{x_B}\right\|_{\square,1}+\left\|\frac{\partial(z,x)}{\partial(\cdot,\cdot)}\right\|_{\square,\infty}\left\|y-\overline{y_B}\right\|_{\square,1}\\ \quad+\left\|\frac{\partial(x,y)}{\partial(\cdot,\cdot)}\right\|_{\square,\infty}\left\|z-\overline{z_B}\right\|_{\square,1},\\ \\ \left\|\frac{\partial(y,z)}{\partial(\cdot,\cdot)}\right\|_{\square,p}\left\|x-\overline{x_B}\right\|_{\square,q}+\left\|\frac{\partial(z,x)}{\partial(\cdot,\cdot)}\right\|_{\square,p}\left\|y-\overline{y_B}\right\|_{\square,q}\\ \quad+\left\|\frac{\partial(x,y)}{\partial(\cdot,\cdot)}\right\|_{\square,p}\left\|z-\overline{z_B}\right\|_{\square,q},\\ \\ \left\|\frac{\partial(y,z)}{\partial(\cdot,\cdot)}\right\|_{\square,1}\left\|x-\overline{x_B}\right\|_{\square,\infty}+\left\|\frac{\partial(z,x)}{\partial(\cdot,\cdot)}\right\|_{\square,p}\left\|y-\overline{y_B}\right\|_{\square,\infty}\\ \quad+\left\|\frac{\partial(x,y)}{\partial(\cdot,\cdot)}\right\|_{\square,1}\left\|z-\overline{z_B}\right\|_{\square,\infty}.\end{cases}$$

From (4.7) we also have

$$\begin{aligned}(4.11)\quad & \left|\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz-\frac{\phi+\Phi}{2}\right.\\ & -\frac{1}{3V(B)}\iiint_B\left[(\overline{x_B}-x)\frac{\partial f(x,y,z)}{\partial x}+(\overline{y_B}-y)\frac{\partial f(x,y,z)}{\partial y}\right.\\ & \left.\left.+(\overline{z_B}-z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz\right|\\ & \le\frac{1}{6V(B)}|\Phi-\phi|\int\int_S\left(|x-\overline{x_B}|^2+|y-\overline{y_B}|^2+|z-\overline{z_B}|^2\right)^{1/2}dS\\ & \le\frac{A_S}{6V(B)}|\Phi-\phi|\sup_{(x,y,z)\in S}\left(|x-\overline{x_B}|^2+|y-\overline{y_B}|^2+|z-\overline{z_B}|^2\right)^{1/2}.\end{aligned}$$

If $f\in\bar{\Delta}_S(\phi,\Phi)$ for some $\phi,\Phi\in\mathbb{C}$, $\phi\neq\Phi$, then by taking $\alpha=x_{B,\partial f}$, $\beta=y_{B,\partial f}$ and $\gamma=z_{B,\partial f}$ in Theorem 4 we get

$$\begin{aligned}(4.12)\quad & \left|\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz-\frac{\phi+\Phi}{2}\right|\\ & \le\frac{1}{6V(B)}|\Phi-\phi|\;M(S,x_{B,\partial f},y_{B,\partial f},z_{B,\partial f})\end{aligned}$$

where

$$\begin{aligned}(4.13)\quad M(S,x_{B,\partial f},y_{B,\partial f},z_{B,\partial f}) &:= \int_a^b\int_c^d|x(u,v)-x_{B,\partial f}|\left|\frac{\partial(y,z)}{\partial(u,v)}\right|dudv\\ & +\int_a^b\int_c^d|y(u,v)-y_{B,\partial f}|\left|\frac{\partial(z,x)}{\partial(u,v)}\right|dudv\\ & +\int_a^b\int_c^d|z(u,v)-z_{B,\partial f}|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv.\end{aligned}$$

Moreover, we have the bounds

$$\begin{aligned}(4.14)\quad & M(S,x_{B,\partial f},y_{B,\partial f},z_{B,\partial f})\\ & \le\begin{cases}\left\|\frac{\partial(y,z)}{\partial(\cdot,\cdot)}\right\|_{\square,\infty}\|x-x_{B,\partial f}\|_{\square,1}+\left\|\frac{\partial(z,x)}{\partial(\cdot,\cdot)}\right\|_{\square,\infty}\|y-y_{B,\partial f}\|_{\square,1}\\ \quad+\left\|\frac{\partial(x,y)}{\partial(\cdot,\cdot)}\right\|_{\square,\infty}\|z-z_{B,\partial f}\|_{\square,1},\\ \\ \left\|\frac{\partial(y,z)}{\partial(\cdot,\cdot)}\right\|_{\square,p}\|x-x_{B,\partial f}\|_{\square,q}+\left\|\frac{\partial(z,x)}{\partial(\cdot,\cdot)}\right\|_{\square,p}\|y-y_{B,\partial f}\|_{\square,q}\\ \quad+\left\|\frac{\partial(x,y)}{\partial(\cdot,\cdot)}\right\|_{\square,p}\|z-z_{B,\partial f}\|_{\square,q},\\ \\ \left\|\frac{\partial(y,z)}{\partial(\cdot,\cdot)}\right\|_{\square,1}\|x-x_{B,\partial f}\|_{\square,\infty}+\left\|\frac{\partial(z,x)}{\partial(\cdot,\cdot)}\right\|_{\square,p}\|y-y_{B,\partial f}\|_{\square,\infty}\\ \quad+\left\|\frac{\partial(x,y)}{\partial(\cdot,\cdot)}\right\|_{\square,1}\|z-z_{B,\partial f}\|_{\square,\infty}.\end{cases}\end{aligned}$$

From (4.7) we also have

$$\left|\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz-\frac{\phi+\Phi}{2}\right| \tag{4.15}$$
$$\le \frac{1}{6V(B)}|\Phi-\phi|\int\int_S\left(|x-x_{B,\partial f}|^2+|y-y_{B,\partial f}|^2+|z-z_{B,\partial f}|^2\right)^{1/2}dS$$
$$\le \frac{A_S}{6V(B)}|\Phi-\phi|\sup_{(x,y,z)\in S}\left(|x-x_{B,\partial f}|^2+|y-y_{B,\partial f}|^2+|z-z_{B,\partial f}|^2\right)^{1/2}.$$

5. Some Examples for Sphere

Consider the 3-dimensional ball centered in $C=(a,b,c)$ and having the radius $R>0$,

$$B(C,R):=\left\{(x,y,z)\in\mathbb{R}^3\,\middle|\,(x-a)^2+(y-b)^2+(z-c)^2\le R^2\right\}$$

and the sphere

$$S(C,R):=\left\{(x,y,z)\in\mathbb{R}^3\,\middle|\,(x-a)^2+(y-b)^2+(z-c)^2=R^2\right\}.$$

Consider the parametrization of $B(C,R)$ and $S(C,R)$ given by:

$$B(C,R):\begin{cases}x=r\cos\psi\cos\varphi+a\\ y=r\cos\psi\sin\varphi+b\\ z=r\sin\psi+c\end{cases};\quad (r,\psi,\varphi)\in[0,R]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi]$$

and

$$S(C,R):\begin{cases}x=R\cos\psi\cos\varphi+a\\ y=R\cos\psi\sin\varphi+b\\ z=R\sin\psi+c\end{cases};\quad (\psi,\varphi)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi].$$

By setting

$$A:=\begin{vmatrix}\frac{\partial y}{\partial\psi} & \frac{\partial z}{\partial\psi}\\ \frac{\partial y}{\partial\varphi} & \frac{\partial z}{\partial\varphi}\end{vmatrix}=-R^2\cos^2\psi\cos\varphi,$$

$$B:=\begin{vmatrix}\frac{\partial x}{\partial\psi} & \frac{\partial z}{\partial\psi}\\ \frac{\partial x}{\partial\varphi} & \frac{\partial z}{\partial\varphi}\end{vmatrix}=R^2\cos^2\psi\sin\varphi,$$

and

$$C:=\begin{vmatrix}\frac{\partial x}{\partial\psi} & \frac{\partial y}{\partial\psi}\\ \frac{\partial x}{\partial\varphi} & \frac{\partial y}{\partial\varphi}\end{vmatrix}=-R^2\sin\psi\cos\psi,$$

we have that

$$A^2+B^2+C^2=R^4\cos^2\psi \text{ for all } (\psi,\varphi)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi].$$

Obviously $\overline{x_B}=a$, $\overline{y_B}=b$, $\overline{z_B}=c$ and

$$\int\int_S\left(|x-\overline{x_B}|^2+|y-\overline{y_B}|^2+|z-\overline{z_B}|^2\right)^{1/2}dS$$
$$=\int\int_S\left(|x-a|^2+|y-b|^2+|z-c|^2\right)^{1/2}dS=R^3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\cos\psi\,d\psi\,d\varphi=4\pi R^3.$$

Inequality (4.11) written for $B = B(C, R)$ and $S = S(C, R)$ becomes

$$\left| \frac{1}{V(B(C,R))} \iiint_{B(C,R)} f(x,y,z)\, dxdydz - \frac{\phi + \Phi}{2} \right. \\ - \frac{1}{3V(B(C,R))} \iiint_{B(C,R)} \left[(a-x) \frac{\partial f(x,y,z)}{\partial x} + (b-y) \frac{\partial f(x,y,z)}{\partial y} \right. \\ \left. \left. + (c-z) \frac{\partial f(x,y,z)}{\partial z} \right] dxdydz \right| \leq \frac{1}{2} |\Phi - \phi| \tag{5.1}$$

provided $f \in \bar{\Delta}_{S(C,R)}(\phi, \Phi)$ for some $\phi, \Phi \in \mathbb{C}$, $\phi \neq \Phi$, where $V(B(C,R)) = \frac{4\pi R^3}{3}$.

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