OSTROWSKI TYPE TRIPLE INTEGRAL INEQUALITIES FOR FUNCTIONS DEFINED ON 3-DIMENSIONAL BODIES

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Abstract. In this paper we provide some bounds for the absolute value of the quantity

$$\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz - f(a,b,c) - \frac{1}{3V(B)}\iiint_B \left[(a-x)\frac{\partial f(x,y,z)}{\partial x} + (a-y)\frac{\partial f(x,y,z)}{\partial y} + (a-z)\frac{\partial f(x,y,z)}{\partial z}\right] dxdydz$$

under the general assumption that B is a body in the 3-dimensional space $\mathbb{R}^3$, $(a,b,c) \in B$ and $f : B \to \mathbb{C}$ is differentiable on B. For this purpose we use an identity obtained by the well known *Gauss-Ostrogradsky* theorem for the divergence of a continuously differentiable vector field. An example for 3-dimensional ball is also given.

1. Introduction

Recall the following inequalities of Hermite-Hadamard's type for convex functions defined on a ball $B(C,R)$, where $C = (a,b,c) \in \mathbb{R}^3$, $R > 0$ and

$$B(C,R) := \left\{ (x,y,z) \in \mathbb{R}^3 \middle| (x-a)^2 + (y-b)^2 + (z-c)^2 \le R^2 \right\}.$$

The following theorem holds [10].

Theorem 1. *Let $f : B(C,R) \to \mathbb{R}$ be a convex mapping on the ball $B(C,R)$. Then we have the inequality:*

$$f(a,b,c) \le \frac{1}{V(B(C,R))}\iiint_{B(C,R)} f(x,y,z)\,dxdydz \le \frac{1}{\sigma(B(C,R))}\iint_{S(C,R)} f(x,y,z)\,dS, \tag{1.1}$$

where

$$S(C,R) := \left\{ (x,y,z) \in \mathbb{R}^3 \middle| (x-a)^2 + (y-b)^2 + (z-c)^2 = R^2 \right\}$$

and

$$V(B(C,R)) = \frac{4\pi R^3}{3},\ \sigma(B(C,R)) = 4\pi R^2.$$

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If the assumption of convexity is dropped, then one can prove the following Ostrowski type inequality for the centre of the ball as well, see [11].

Theorem 2. *Assume that $f : B(C,R) \to \mathbb{C}$ is differentiable on $B(C,R)$. Then*

$$\left| f(a,b,c) - \frac{1}{V(B(C,R))} \iiint_{B(C,R)} f(x,y,z)\, dxdydz \right| \leq \frac{3}{8} R \left[\left\| \frac{\partial f}{\partial x} \right\|_{B(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{B(C,R),\infty} + \left\| \frac{\partial f}{\partial z} \right\|_{B(C,R),\infty} \right], \tag{1.2}$$

provided

$$\left\| \frac{\partial f}{\partial x} \right\|_{B(C,R),\infty} := \sup_{(x,y,z)\in B(C,R)} \left| \frac{\partial f(x,y,z)}{\partial x} \right| < \infty ,$$

$$\left\| \frac{\partial f}{\partial y} \right\|_{B(C,R),\infty} := \sup_{(x,y,z)\in B(C,R)} \left| \frac{\partial f(x,y,z)}{\partial y} \right| < \infty$$

and

$$\left\| \frac{\partial f}{\partial z} \right\|_{B(C,R),\infty} := \sup_{(x,y,z)\in B(C,R)} \left| \frac{\partial f(x,y,z)}{\partial y} \right| < \infty.$$

This fact can be furthermore generalized to the following Ostrowski type inequality for any point in a convex body $B \subset \mathbb{R}^3$, see [11].

Theorem 3. *Assume that $f : B \to \mathbb{C}$ is differentiable on the convex body B and $(u,v,w) \in B$. If $V(B)$ is the volume of B, then*

$$\begin{aligned} &\left| f(u,v,w) - \frac{1}{V(B)} \iiint_B f(x,y,z)\, dxdydz \right| \\ &\quad \leq \frac{1}{V(B)} \iiint_B |x-u| \left(\int_0^1 \left| \frac{\partial f}{\partial x} [t(x,y,z) + (1-t)(u,v,w)] \right| dt \right) dxdydz \\ &\quad + \frac{1}{V(B)} \iiint_B |y-v| \left(\int_0^1 \left| \frac{\partial f}{\partial y} [t(x,y,z) + (1-t)(u,v,w)] \right| dt \right) dxdydz \\ &\quad + \frac{1}{V(B)} \iiint_B |z-w| \left(\int_0^1 \left| \frac{\partial f}{\partial y} [t(x,y,z) + (1-t)(u,v,w)] \right| dt \right) dxdydz \\ &\leq \left\| \frac{\partial f}{\partial x} \right\|_{B,\infty} \frac{1}{V(B)} \iiint_B |x-u|\, dxdydz + \left\| \frac{\partial f}{\partial y} \right\|_{B,\infty} \frac{1}{V(B)} \iiint_B |y-v|\, dxdydz \\ &\qquad + \left\| \frac{\partial f}{\partial z} \right\|_{B,\infty} \frac{1}{V(B)} \iiint_B |z-w|\, dxdydz \end{aligned} \tag{1.3}$$

provided

$$\left\| \frac{\partial f}{\partial x} \right\|_{B,\infty}, \left\| \frac{\partial f}{\partial y} \right\|_{B,\infty}, \left\| \frac{\partial f}{\partial z} \right\|_{B,\infty} < \infty.$$

In particular,

$$
\begin{aligned}
(1.4)\quad & \left| f\left(\overline{x_B},\overline{y_B},\overline{z_B}\right) - \frac{1}{V(B)} \iiint_B f(x,y,z)\,dxdydz \right| \\
& \leq \frac{1}{V(B)} \iiint_B \left| x-\overline{x_B}\right| \left(\int_0^1 \left| \frac{\partial f}{\partial x}\left[t(x,y,z)+(1-t)\left(\overline{x_B},\overline{y_B},\overline{z_B}\right)\right]\right| dt \right) dxdydz \\
& + \frac{1}{V(B)} \iiint_B \left| y-\overline{y_B}\right| \left(\int_0^1 \left| \frac{\partial f}{\partial y}\left[t(x,y,z)+(1-t)\left(\overline{x_B},\overline{y_B},\overline{z_B}\right)\right]\right| dt \right) dxdydz \\
& + \frac{1}{V(B)} \iiint_B \left| z-\overline{z_B}\right| \left(\int_0^1 \left| \frac{\partial f}{\partial y}\left[t(x,y,z)+(1-t)\left(\overline{x_B},\overline{y_B},\overline{z_B}\right)\right]\right| dt \right) dxdydz \\
& \qquad \leq \left\| \frac{\partial f}{\partial x} \right\|_{B,\infty} \frac{1}{V(B)} \iiint_B \left| x-\overline{x_B}\right| dxdydz \\
& \qquad + \left\| \frac{\partial f}{\partial y} \right\|_{B,\infty} \frac{1}{V(B)} \iiint_B \left| y-\overline{y_B}\right| dxdydz \\
& \qquad\qquad + \left\| \frac{\partial f}{\partial z} \right\|_{B,\infty} \frac{1}{V(B)} \iiint_B \left| z-\overline{z_B}\right| dxdydz,
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{x_B} &:= \frac{1}{V(B)} \iiint_B x dxdydz,\ \ \overline{y_B} := \frac{1}{V(B)} \iiint_B y dxdydz, \\
\overline{z_B} &:= \frac{1}{V(B)} \iiint_B z dxdydz
\end{aligned}
$$

are the centre of gravity coordinates for the convex body B.

For some Hermite-Hadamard type inequalities for multiple integrals see [2], [6], [8], [9], [10], [17], [18], [19], [20], [25], [26] and [27]. For some Ostrowski type inequalities see [3], [4], [5], [7], [11], [12], [13], [14], [15], [16], [21], [22], [23] and [24].

In this paper we provide some bounds for the absolute value of the quantity

$$
\begin{aligned}
& \frac{1}{V(B)} \iiint_B f(x,y,z)\,dxdydz - f(a,b,c) \\
& \qquad - \frac{1}{3V(B)} \iiint_B \left[(a-x)\frac{\partial f(x,y,z)}{\partial x} + (a-y)\frac{\partial f(x,y,z)}{\partial y} \right. \\
& \qquad\qquad \left. + (a-z)\frac{\partial f(x,y,z)}{\partial z} \right] dxdydz
\end{aligned}
$$

under the general assumption that B is a body in the 3-dimensional space $\mathbb{R}^3$, $(a,b,c) \in B$ and $f : B \to \mathbb{C}$ is differentiable on B. For this purpose we use an identity obtained by the well known *Gauss-Ostrogradsky* theorem for the divergence of a continuously differentiable vector field. An example for 3-dimensional ball is also given.

We need the following preparations.

2. Preliminary Facts and Results

Following Apostol [1], consider a surface described by the vector equation

$$
r(u,v) = x(u,v)\,\overrightarrow{i} + y(u,v)\,\overrightarrow{j} + z(u,v)\,\overrightarrow{k} \tag{2.1}
$$

where $(u,v)\in[a,b]\times[c,d]$.

If x, y, z are differentiable on $[a,b]\times[c,d]$ we consider the two vectors

$$\frac{\partial r}{\partial u}=\frac{\partial x}{\partial u}\overrightarrow{i}+\frac{\partial y}{\partial u}\overrightarrow{j}+\frac{\partial z}{\partial u}\overrightarrow{k}$$

and

$$\frac{\partial r}{\partial v}=\frac{\partial x}{\partial v}\overrightarrow{i}+\frac{\partial y}{\partial v}\overrightarrow{j}+\frac{\partial z}{\partial v}\overrightarrow{k}.$$

The *cross product* of these two vectors $\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}$ will be referred to as the fundamental vector product of the representation r. Its components can be expressed as *Jacobian determinants*. In fact, we have [1, p. 420]

$$\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}=\begin{vmatrix}\frac{\partial y}{\partial u} & \frac{\partial z}{\partial u}\\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v}\end{vmatrix}\overrightarrow{i}+\begin{vmatrix}\frac{\partial z}{\partial u} & \frac{\partial x}{\partial u}\\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v}\end{vmatrix}\overrightarrow{j}+\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix}\overrightarrow{k}\tag{2.2}$$

$$=\frac{\partial(y,z)}{\partial(u,v)}\overrightarrow{i}+\frac{\partial(z,x)}{\partial(u,v)}\overrightarrow{j}+\frac{\partial(x,y)}{\partial(u,v)}\overrightarrow{k}.$$

Let $S=r(T)$ be a parametric surface described by a vector-valued function r defined on the box $T=[a,b]\times[c,d]$. The area of S denoted A_S is defined by the double integral [1, p. 424-425]

$$\begin{aligned}A_S&=\int_a^b\int_c^d\left\|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right\|dudv\\&=\int_a^b\int_c^d\sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2+\left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2+\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2}dudv.\end{aligned}\tag{2.3}$$

We define surface integrals in terms of a parametric representation for the surface. One can prove that under certain general conditions the value of the integral is independent of the representation.

Let $S=r(T)$ be a parametric surface described by a vector-valued differentiable function r defined on the box $T=[a,b]\times[c,d]$ and let $f:S\to\mathbb{C}$ defined and bounded on S. The surface integral of f over S is defined by [1, p. 430]

$$\begin{aligned}\int\int_S fdS&=\int_a^b\int_c^d f(x,y,z)\left\|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right\|dudv\\&=\int_a^b\int_c^d f(x(u,v),y(u,v),z(u,v))\\&\quad\times\sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2+\left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2+\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2}dudv.\end{aligned}\tag{2.4}$$

If $S=r(T)$ is a parametric surface, the fundamental vector product $N=\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}$ is normal to S at each regular point of the surface. At each such point there are two unit normals, a unit normal n_1, which has the same direction as N, and a unit normal n_2 which has the opposite direction. Thus

$$n_1=\frac{N}{\|N\|}\text{ and }n_2=-n_1.$$

Let n be one of the two normals n_1 or n_2. Let also F be a vector field defined on S and assume that the surface integral,

$$\int\int_S (F \cdot n)\, dS,$$

called the flux surface integral, exists. Here $F \cdot n$ is the dot or inner product.

We can write [1, p. 434]

$$\int\int_S (F \cdot n)\, dS = \pm \int_a^b \int_c^d F(r(u,v)) \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right) dudv$$

where the sign " $+$ " is used if $n = n_1$ and the " $-$ " sign is used if $n = n_2$.

If

$$F(x,y,z) = P(x,y,z)\overrightarrow{i} + Q(x,y,z)\overrightarrow{j} + R(x,y,z)\overrightarrow{k}$$

and

$$r(u,v) = x(u,v)\overrightarrow{i} + y(u,v)\overrightarrow{j} + z(u,v)\overrightarrow{k} \text{ where } (u,v) \in [a,b] \times [c,d]$$

then the flux surface integral for $n = n_1$ can be explicitly calculated as [1, p. 435]

$$\begin{aligned} \int\int_S (F \cdot n)\, dS &= \int_a^b \int_c^d P(x(u,v), y(u,v), z(u,v)) \frac{\partial(y,z)}{\partial(u,v)} dudv \\ &+ \int_a^b \int_c^d Q(x(u,v), y(u,v), z(u,v)) \frac{\partial(z,x)}{\partial(u,v)} dudv \\ &+ \int_a^b \int_c^d R(x(u,v), y(u,v), z(u,v)) \frac{\partial(x,y)}{\partial(u,v)} dudv. \end{aligned} \tag{2.5}$$

The sum of the double integrals on the right is often written more briefly as [1, p. 435]

$$\int\int_S P(x,y,z)\, dy \wedge dz + \int\int_S Q(x,y,z)\, dz \wedge dx + \int\int_S R(x,y,z)\, dx \wedge dy.$$

Let $B \subset \mathbb{R}^3$ be a solid in 3-space bounded by an orientable closed surface S, and let n be the unit outer normal to S. If F is a continuously differentiable vector field defined on B, we have the *Gauss-Ostrogradsky identity*

$$\iiint_B (\operatorname{div} F)\, dV = \int\int_S (F \cdot n)\, dS. \tag{GO}$$

If we express

$$F(x,y,z) = P(x,y,z)\overrightarrow{i} + Q(x,y,z)\overrightarrow{j} + R(x,y,z)\overrightarrow{k},$$

then (GO) can be written as

$$\begin{aligned} &\iiint_B \left(\frac{\partial P(x,y,z)}{\partial x} + \frac{\partial Q(x,y,z)}{\partial y} + \frac{\partial R(x,y,z)}{\partial z}\right) dxdydz \\ &= \int\int_S P(x,y,z)\, dy \wedge dz + \int\int_S Q(x,y,z)\, dz \wedge dx \\ &\qquad + \int\int_S R(x,y,z)\, dx \wedge dy. \end{aligned} \tag{2.6}$$

By taking the real and imaginary part, we can extend the above inequality for complex valued functions P, Q, R defined on B.

3. Some Identities of Interest

We have:

Lemma 1. *Let B be a solid in the three dimensional space $\mathbb{R}^3$ bounded by an orientable closed surface S. If $f : B \to \mathbb{C}$ is a continuously differentiable function defined on a open set containing B, then we have the equality*

$$\begin{aligned}(3.1)\quad & \frac{1}{V(B)} \iiint_B f(x,y,z)\,dxdydz - f(m,n,p) \\ & = \frac{1}{3V(B)} \iiint_B \left[(m-x)\frac{\partial f(x,y,z)}{\partial x} + (n-y)\frac{\partial f(x,y,z)}{\partial y} \right. \\ & \qquad \left. + (p-z)\frac{\partial f(x,y,z)}{\partial z} \right] dxdydz \\ & \quad + \frac{1}{3V(B)} \left[\int\int_S (x-m)\,[f(x,y,z) - f(m,n,p)]\,dy \wedge dz \right. \\ & \qquad + \int\int_S (y-n)\,[f(x,y,z) - f(m,n,p)]\,dz \wedge dx \\ & \qquad \left. + \int\int_S (z-p)\,[f(x,y,z) - f(m,n,p)]\,dx \wedge dy \right]\end{aligned}$$

for all $(m,n,p) \in B$.

In particular, we have

$$\begin{aligned}(3.2)\quad & \frac{1}{V(B)} \iiint_B f(x,y,z)\,dxdydz - f(\overline{x_B},\overline{y_B},\overline{z_B}) \\ & = \frac{1}{3V(B)} \iiint_B \left[(\overline{x_B}-x)\frac{\partial f(x,y,z)}{\partial x} + (\overline{y_B}-y)\frac{\partial f(x,y,z)}{\partial y} \right. \\ & \qquad \left. + (\overline{z_B}-z)\frac{\partial f(x,y,z)}{\partial z} \right] dxdydz \\ & \quad + \frac{1}{3V(B)} \left[\int\int_S (x-\overline{x_B})\,[f(x,y,z) - f(\overline{x_B},\overline{y_B},\overline{z_B})]\,dy \wedge dz \right. \\ & \qquad + \int\int_S (y-\overline{y_B})\,[f(x,y,z) - f(\overline{x_B},\overline{y_B},\overline{z_B})]\,dz \wedge dx \\ & \qquad \left. + \int\int_S (z-\overline{z_B})\,[f(x,y,z) - f(\overline{x_B},\overline{y_B},\overline{z_B})]\,dx \wedge dy \right].\end{aligned}$$

Proof. It would suffice to prove the equality (3.1) without the value $f(m,n,p)$ since the general case will follows by replacing f with $f - f(m,n,p)$.

We have

$$\frac{\partial\,[(x-m)\,f(x,y,z)]}{\partial x} = f(x,y,z) + (x-m)\frac{\partial f(x,y,z)}{\partial x},$$
$$\frac{\partial\,[(y-n)\,f(x,y,z)]}{\partial y} = f(x,y,z) + (y-n)\frac{\partial f(x,y,z)}{\partial y}$$

and

$$\frac{\partial\,[(z-p)\,f(x,y,z)]}{\partial z} = f(x,y,z) + (z-p)\frac{\partial f(x,y,z)}{\partial z}.$$

By adding these three equalities we get

$$\begin{aligned}(3.3)\quad & \frac{\partial\left[(x-m)f(x,y,z)\right]}{\partial x}+\frac{\partial\left[(y-n)f(x,y,z)\right]}{\partial y}+\frac{\partial\left[(z-p)f(x,y,z)\right]}{\partial z}\\ &=3f(x,y,z)\\ &+(x-m)\frac{\partial f(x,y,z)}{\partial x}+(y-n)\frac{\partial f(x,y,z)}{\partial y}+(z-p)\frac{\partial f(x,y,z)}{\partial z}\end{aligned}$$

for all $(x,y,z)\in B$.

Integrating this equality on B we get

$$\begin{aligned}(3.4)\quad & \iiint_B\left(\frac{\partial\left[(x-m)f(x,y,z)\right]}{\partial x}+\frac{\partial\left[(y-n)f(x,y,z)\right]}{\partial y}\right.\\ &\left.+\frac{\partial\left[(z-p)f(x,y,z)\right]}{\partial z}\right)dxdydz\\ &=3\iiint_B f(x,y,z)\,dxdydz\\ &+\iiint_B\left[(x-m)\frac{\partial f(x,y,z)}{\partial x}+(y-n)\frac{\partial f(x,y,z)}{\partial y}\right.\\ &\left.+(z-p)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz.\end{aligned}$$

Applying the *Gauss-Ostrogradsky identity (2.6)* for the functions

$$P(x,y,z)=(x-m)f(x,y,z),\ Q(x,y,z)=(y-n)f(x,y,z)$$

and

$$R(x,y,z)=(z-p)f(x,y,z)$$

we obtain

$$\begin{aligned}(3.5)\quad & \iiint_B\left(\frac{\partial\left[(x-m)f(x,y,z)\right]}{\partial x}+\frac{\partial\left[(y-n)f(x,y,z)\right]}{\partial y}\right.\\ &\left.+\frac{\partial\left[(z-p)f(x,y,z)\right]}{\partial z}\right)dxdydz\\ &=\int\int_S(x-m)f(x,y,z)\,dy\wedge dz+\int\int_S(y-n)f(x,y,z)\,dz\wedge dx\\ &+\int\int_S(z-p)f(x,y,z)\,dx\wedge dy.\end{aligned}$$

By (3.4) and (3.5) we get

$$\begin{aligned}&3\iiint_B f(x,y,z)\,dxdydz\\ &+\iiint_B\left[(x-m)\frac{\partial f(x,y,z)}{\partial x}+(y-n)\frac{\partial f(x,y,z)}{\partial y}+(z-p)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz\\ &=\int\int_S(x-m)f(x,y,z)\,dy\wedge dz+\int\int_S(y-n)f(x,y,z)\,dz\wedge dx\\ &+\int\int_S(z-p)f(x,y,z)\,dx\wedge dy,\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\iiint_B f(x,y,z)\,dxdydz \\
&= \frac{1}{3}\iiint_B \left[(m-x)\frac{\partial f(x,y,z)}{\partial x} + (n-y)\frac{\partial f(x,y,z)}{\partial y} + (p-z)\frac{\partial f(x,y,z)}{\partial z}\right] dxdydz \\
&\quad + \frac{1}{3}\left[\int\int_S (x-m) f(x,y,z)\,dy\wedge dz + \int\int_S (y-n) f(x,y,z)\,dz\wedge dx \right.\\
&\qquad\qquad \left. + \int\int_S (z-p) f(x,y,z)\,dx\wedge dy\right]
\end{aligned}$$

that, by division with $V(B)$, proves the claim. □

Remark 1. *For a function f as in Lemma 1 above, we define the points*

$$x_{B,\partial f} := \frac{\iiint_B x\frac{\partial f(x,y,z)}{\partial x}dxdydz}{\iiint_B \frac{\partial f(x,y,z)}{\partial x}dxdydz},\ y_{B,\partial f} := \frac{\iiint_B y\frac{\partial f(x,y,z)}{\partial y}dxdydz}{\iiint_B \frac{\partial f(x,y,z)}{\partial y}dxdydz},$$

and

$$z_{B,\partial f} := \frac{\iiint_B z\frac{\partial f(x,y,z)}{\partial z}dxdydz}{\iiint_B \frac{\partial f(x,y,z)}{\partial z}dxdydz}$$

provided the denominators are not zero.

If we take $m = x_{B,\partial f}$, $n = y_{B,\partial f}$ and $p = z_{B,\partial f}$ in (3.1) and assume $(x_{B,\partial f}, y_{B,\partial f}, z_{B,\partial f}) \in B$, then we get

$$\begin{aligned}
(3.6)\quad &\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz - f(x_{B,\partial f}, y_{B,\partial f}, z_{B,\partial f}) \\
&= \frac{1}{3V(B)}\left[\int\int_S (x - x_{B,\partial f})\left[f(x,y,z) - f(x_{B,\partial f}, y_{B,\partial f}, z_{B,\partial f})\right] dy\wedge dz\right. \\
&\quad + \int\int_S (y - ny_{B,\partial f})\left[f(x,y,z) - f(x_{B,\partial f}, y_{B,\partial f}, z_{B,\partial f})\right] dz\wedge dx \\
&\qquad \left. + \int\int_S (z - z_{B,\partial f})\left[f(x,y,z) - f(x_{B,\partial f}, y_{B,\partial f}, z_{B,\partial f})\right] dx\wedge dy\right],
\end{aligned}$$

since, obviously,

$$\begin{aligned}
\iiint_B &\left[(x_{B,\partial f} - x)\frac{\partial f(x,y,z)}{\partial x} + (y_{B,\partial f} - y)\frac{\partial f(x,y,z)}{\partial y}\right. \\
&\qquad \left. + (z_{B,\partial f} - z)\frac{\partial f(x,y,z)}{\partial z}\right] dxdydz = 0.
\end{aligned}$$

Remark 2. *Let B be a solid in the three dimensional space $\mathbb{R}^3$ bounded by an orientable closed surface S described by the vector equation*

$$r(u,v) = x(u,v)\,\overrightarrow{i} + y(u,v)\,\overrightarrow{j} + z(u,v)\,\overrightarrow{k},\ \ (u,v)\in[a,b]\times[c,d]$$

where $x(u,v)$, $y(u,v)$, $z(u,v)$ are differentiable.

From the equation (3.1) we get

$$\begin{aligned}(3.7)\quad &\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz - f(m,n,p)\\ &\quad-\frac{1}{3V(B)}\iiint_B\left[(m-x)\frac{\partial f(x,y,z)}{\partial x}+(n-y)\frac{\partial f(x,y,z)}{\partial y}\right.\\ &\qquad\left.+(p-z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz\\ &=\frac{1}{3V(B)}\left[\int_a^b\int_c^d (x(u,v)-m)\left[f(x(u,v),y(u,v),z(u,v))-f(m,n,p)\right]\frac{\partial(y,z)}{\partial(u,v)}dudv\right.\\ &\quad+\int_a^b\int_c^d (y(u,v)-n)\left[f(x(u,v),y(u,v),z(u,v))-f(m,n,p)\right]\frac{\partial(z,x)}{\partial(u,v)}dudv\\ &\quad\left.+\int_a^b\int_c^d (z(u,v)-p)\left[f(x(u,v),y(u,v),z(u,v))-f(m,n,p)\right]\frac{\partial(x,y)}{\partial(u,v)}dudv\right]\end{aligned}$$

for all $(m,n,p)\in B$, *while from (3.2) we have*

$$\begin{aligned}(3.8)\quad &\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz - f(\overline{x_B},\overline{y_B},\overline{z_B})\\ &\quad-\frac{1}{3V(B)}\iiint_B\left[(\overline{x_B}-x)\frac{\partial f(x,y,z)}{\partial x}+(\overline{y_B}-y)\frac{\partial f(x,y,z)}{\partial y}\right.\\ &\qquad\left.+(\overline{z_B}-z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz\\ &=\frac{1}{3V(B)}\left[\int_a^b\int_c^d (x(u,v)-\overline{x_B})\left[f(x(u,v),y(u,v),z(u,v))-f(m,n,p)\right]\frac{\partial(y,z)}{\partial(u,v)}dudv\right.\\ &\quad+\int_a^b\int_c^d (y(u,v)-\overline{y_B})\left[f(x(u,v),y(u,v),z(u,v))-f(m,n,p)\right]\frac{\partial(z,x)}{\partial(u,v)}dudv\\ &\quad\left.+\int_a^b\int_c^d (z(u,v)-\overline{z_B})\left[f(x(u,v),y(u,v),z(u,v))-f(m,n,p)\right]\frac{\partial(x,y)}{\partial(u,v)}dudv\right]\end{aligned}$$

for all $(m,n,p)\in B$.

From (3.6) we get

$$\begin{aligned}(3.9)\quad &\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz - f(m,n,p)\\ &=\frac{1}{3V(B)}\left[\int_a^b\int_c^d (x(u,v)-x_{B,\partial f})\left[f(x(u,v),y(u,v),z(u,v))-f(m,n,p)\right]\frac{\partial(y,z)}{\partial(u,v)}dudv\right.\\ &\quad+\int_a^b\int_c^d (y(u,v)-y_{B,\partial f})\left[f(x(u,v),y(u,v),z(u,v))-f(m,n,p)\right]\frac{\partial(z,x)}{\partial(u,v)}dudv\\ &\quad\left.+\int_a^b\int_c^d (z(u,v)-z_{B,\partial f})\left[f(x(u,v),y(u,v),z(u,v))-f(m,n,p)\right]\frac{\partial(x,y)}{\partial(u,v)}dudv\right]\end{aligned}$$

for all $(m,n,p)\in B$.

4. Inequalities for Lipschitzian Functions

Let $(m,n,p)\in B$. We assume that the function is *surface Lipschitzian in* (m,n,p) if there exists the constants (depending on (m,n,p)) $L_1,\ L_2,\ L_3>0$ such that

$$|f(x,y,z)-f(m,n,p)|\le L_1|x-m|+L_2|y-n|+L_3|z-p| \tag{4.1}$$

for all $(x,y,z)\in S$.

We observe that, if f is differentiable on an open neighborhood of B and has the partial derivatives bounded on S then we have

$$|f(x,y,z)-f(m,n,p)|\le\left\|\frac{\partial f}{\partial x}\right\|_{S,\infty}|x-m|+\left\|\frac{\partial f}{\partial y}\right\|_{S,\infty}|y-n|+\left\|\frac{\partial f}{\partial z}\right\|_{S,\infty}|z-p|$$

for all $(m,n,p)\in B$ and $(x,y,z)\in S$ where

$$\left\|\frac{\partial f}{\partial x}\right\|_{S,\infty}:=\sup_{(x,y,z)\in S}\left|\frac{f(x,y,z)}{\partial x}\right|<0,\quad \left\|\frac{\partial f}{\partial y}\right\|_{S,\infty}:=\sup_{(x,y,z)\in S}\left|\frac{f(x,y,z)}{\partial y}\right|<\infty$$

and

$$\left\|\frac{\partial f}{\partial z}\right\|_{S,\infty}:=\sup_{(x,y,z)\in S}\left|\frac{f(x,y,z)}{\partial z}\right|<\infty.$$

Theorem 4. *Let B be a solid in the three dimensional space $\mathbb{R}^3$ bounded by an orientable closed surface S. If $f:B\to\mathbb{C}$ is a continuously differentiable function defined on a open set containing B, $(m,n,p)\in B$ and f satisfies the Lipschitz type condition (4.1), then we have*

$$\begin{aligned}&\left|\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz-f(m,n,p)\right.\\&\quad-\frac{1}{3V(B)}\iiint_B\left[(m-x)\frac{\partial f(x,y,z)}{\partial x}+(n-y)\frac{\partial f(x,y,z)}{\partial y}\right.\\&\qquad\left.\left.+(p-z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz\right|\\&\quad\le\frac{1}{3V(B)}\left[L_1B_1(m,n,p)+L_2B_2(m,n,p)+L_3B_3(m,n,p)\right]\end{aligned}\tag{4.2}$$

where

$$\begin{aligned}B_1(m,n,p)&:=\int_a^b\int_c^d\left\{|x(u,v)-m|\left|\frac{\partial(y,z)}{\partial(u,v)}\right|\right.\\&\quad\left.+|y(u,v)-n|\left|\frac{\partial(z,x)}{\partial(u,v)}\right|+|z(u,v)-p|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right\}|x(u,v)-m|\,dudv,\end{aligned}$$

$$\begin{aligned}B_2(m,n,p)&:=\int_a^b\int_c^d\left\{|x(u,v)-m|\left|\frac{\partial(y,z)}{\partial(u,v)}\right|\right.\\&\quad\left.+|y(u,v)-n|\left|\frac{\partial(z,x)}{\partial(u,v)}\right|+|z(u,v)-p|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right\}|y(u,v)-n|\,dudv\end{aligned}$$

and

$$B_3(m,n,p) := \int_a^b \int_c^d \left\{ |x(u,v) - m| \left| \frac{\partial(y,z)}{\partial(u,v)} \right| + |y(u,v) - n| \left| \frac{\partial(z,x)}{\partial(u,v)} \right| + |z(u,v) - p| \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \right\} |z(u,v) - p| \, dudv.$$

Proof. From the equation (3.7) we get

$$\begin{aligned} (4.3)\quad & \Bigg| \frac{1}{V(B)} \iiint_B f(x,y,z)\,dxdydz - f(m,n,p) \\ & - \frac{1}{3V(B)} \iiint_B \left[(m-x) \frac{\partial f(x,y,z)}{\partial x} + (n-y) \frac{\partial f(x,y,z)}{\partial y} + (p-z) \frac{\partial f(x,y,z)}{\partial z} \right] dxdydz \Bigg| \\ & \le \frac{1}{3V(B)} \Bigg[\int_a^b \int_c^d |x(u,v) - m| \, |f(x(u,v), y(u,v), z(u,v)) - f(m,n,p)| \left| \frac{\partial(y,z)}{\partial(u,v)} \right| dudv \\ & + \int_a^b \int_c^d |y(u,v) - n| \, |f(x(u,v), y(u,v), z(u,v)) - f(m,n,p)| \left| \frac{\partial(z,x)}{\partial(u,v)} \right| dudv \\ & + \int_a^b \int_c^d |z(u,v) - p| \, |f(x(u,v), y(u,v), z(u,v)) - f(m,n,p)| \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv \Bigg] \\ & =: \frac{1}{3V(B)} A. \end{aligned}$$

By the Lipschitzian condition (4.1) we have

$$\begin{aligned} & \int_a^b \int_c^d |x(u,v) - m| \, |f(x(u,v), y(u,v), z(u,v)) - f(m,n,p)| \left| \frac{\partial(y,z)}{\partial(u,v)} \right| dudv \\ & \le \int_a^b \int_c^d |x(u,v) - m| \left[L_1 |x(u,v) - m| + L_2 |y(u,v) - n| + L_3 |z(u,v) - p| \right] \left| \frac{\partial(y,z)}{\partial(u,v)} \right| dudv \\ & = L_1 \int_a^b \int_c^d |x(u,v) - m|^2 \left| \frac{\partial(y,z)}{\partial(u,v)} \right| dudv \\ & + L_2 \int_a^b \int_c^d |x(u,v) - m| \, |y(u,v) - n| \left| \frac{\partial(y,z)}{\partial(u,v)} \right| dudv \\ & + L_3 \int_a^b \int_c^d |x(u,v) - m| \, |z(u,v) - p| \left| \frac{\partial(y,z)}{\partial(u,v)} \right| dudv, \end{aligned}$$

$$\begin{aligned}
&\int_a^b \int_c^d |y(u,v)-n| \, |f(x(u,v),y(u,v),z(u,v)) - f(m,n,p)| \left|\frac{\partial(z,x)}{\partial(u,v)}\right| dudv \\
&\leq \int_a^b \int_c^d |y(u,v)-n| \left[L_1 |x(u,v)-m| + L_2 |y(u,v)-n| + L_3 |z(u,v)-p|\right] \left|\frac{\partial(z,x)}{\partial(u,v)}\right| dudv \\
&= L_1 \int_a^b \int_c^d |y(u,v)-n| \, |x(u,v)-m| \left|\frac{\partial(z,x)}{\partial(u,v)}\right| dudv \\
&\quad + L_2 \int_a^b \int_c^d |y(u,v)-n|^2 \left|\frac{\partial(z,x)}{\partial(u,v)}\right| dudv \\
&\quad + L_3 \int_a^b \int_c^d |y(u,v)-n| \, |z(u,v)-p| \left|\frac{\partial(z,x)}{\partial(u,v)}\right| dudv
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b \int_c^d |z(u,v)-p| \, |f(x(u,v),y(u,v),z(u,v)) - f(m,n,p)| \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \\
&\leq \int_a^b \int_c^d |z(u,v)-p| \left[L_1 |x(u,v)-m| + L_2 |y(u,v)-n| + L_3 |z(u,v)-p|\right] \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \\
&= L_1 \int_a^b \int_c^d |z(u,v)-p| \, |x(u,v)-m| \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \\
&\quad + L_2 \int_a^b \int_c^d |z(u,v)-p| \, |y(u,v)-n| \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \\
&\quad + L_3 \int_a^b \int_c^d |z(u,v)-p|^2 \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.
\end{aligned}$$

If we add these three inequalities, we get

$$\begin{aligned}
A &\leq L_1 \int_a^b \int_c^d \left\{ |x(u,v)-m|^2 \left|\frac{\partial(y,z)}{\partial(u,v)}\right| + |y(u,v)-n| \, |x(u,v)-m| \left|\frac{\partial(z,x)}{\partial(u,v)}\right| \right. \\
&\qquad \left. + |z(u,v)-p| \, |x(u,v)-m| \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \right\} dudv \\
&\quad + L_2 \int_a^b \int_c^d \left\{ |x(u,v)-m| \, |y(u,v)-n| \left|\frac{\partial(y,z)}{\partial(u,v)}\right| + |y(u,v)-n|^2 \left|\frac{\partial(z,x)}{\partial(u,v)}\right| \right. \\
&\qquad \left. + |z(u,v)-p| \, |y(u,v)-n| \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \right\} dudv \\
&\quad + L_3 \int_a^b \int_c^d |x(u,v)-m| \, |z(u,v)-p| \left|\frac{\partial(y,z)}{\partial(u,v)}\right| + |y(u,v)-n| \, |z(u,v)-p| \left|\frac{\partial(z,x)}{\partial(u,v)}\right| \\
&\qquad \left. + |z(u,v)-p|^2 \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \right\} dudv \\
&= L_1 B_1(m,n,p) + L_2 B_2(m,n,p) + L_3 B_3(m,n,p),
\end{aligned}$$

which proves the desired result (4.2). □

Remark 3. *If f is differentiable on an open neighborhood of B and has the partial derivatives bounded on S, then we have*

$$\begin{aligned}(4.4)\quad &\left|\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz - f(m,n,p)\right. \\ &\quad - \frac{1}{3V(B)}\iiint_B\left[(m-x)\frac{\partial f(x,y,z)}{\partial x} + (n-y)\frac{\partial f(x,y,z)}{\partial y}\right. \\ &\qquad\left.\left. + (p-z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz\right| \\ &\leq \frac{1}{3V(B)}\left[\left\|\frac{\partial f}{\partial x}\right\|_{S,\infty}B_1(m,n,p) + \left\|\frac{\partial f}{\partial y}\right\|_{S,\infty}B_2(m,n,p) + \left\|\frac{\partial f}{\partial z}\right\|_{S,\infty}B_3(m,n,p)\right]\end{aligned}$$

for all $(m,n,p)\in B$.

Corollary 1. *With the assumptions of Theorem 4 and if $(\overline{x_B},\overline{y_B},\overline{z_B})\in B$ and f satisfies the Lipschitz type condition (4.1) for the point $(\overline{x_B},\overline{y_B},\overline{z_B})$, then we get*

$$\begin{aligned}(4.5)\quad &\left|\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz - f(\overline{x_B},\overline{y_B},\overline{z_B})\right. \\ &\quad - \frac{1}{3V(B)}\iiint_B\left[(\overline{x_B}-x)\frac{\partial f(x,y,z)}{\partial x} + (\overline{y_B}-y)\frac{\partial f(x,y,z)}{\partial y}\right. \\ &\qquad\left.\left. + (\overline{z_B}-z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz\right| \\ &\leq \frac{1}{3V(B)}\left(L_1B_1(\overline{x_B},\overline{y_B},\overline{z_B}) + L_2B_2(\overline{x_B},\overline{y_B},\overline{z_B}) + L_3B_3(\overline{x_B},\overline{y_B},\overline{z_B})\right)\end{aligned}$$

where

$$\begin{aligned}B_1(\overline{x_B},\overline{y_B},\overline{z_B}) := \int_a^b\int_c^d &\left\{|x(u,v)-\overline{x_B}|\left|\frac{\partial(y,z)}{\partial(u,v)}\right|\right. \\ &\left. + |y(u,v)-\overline{y_B}|\left|\frac{\partial(z,x)}{\partial(u,v)}\right| + |z(u,v)-\overline{z_B}|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right\}|x(u,v)-\overline{x_B}|\,dudv,\end{aligned}$$

$$\begin{aligned}B_2(\overline{x_B},\overline{y_B},\overline{z_B}) := \int_a^b\int_c^d &\left\{|x(u,v)-\overline{x_B}|\left|\frac{\partial(y,z)}{\partial(u,v)}\right|\right. \\ &\left. + |y(u,v)-\overline{y_B}|\left|\frac{\partial(z,x)}{\partial(u,v)}\right| + |z(u,v)-\overline{z_B}|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right\}|y(u,v)-\overline{y_B}|\,dudv\end{aligned}$$

and

$$\begin{aligned}B_3(\overline{x_B},\overline{y_B},\overline{z_B}) := \int_a^b\int_c^d &\left\{|x(u,v)-\overline{x_B}|\left|\frac{\partial(y,z)}{\partial(u,v)}\right|\right. \\ &\left. + |y(u,v)-\overline{y_B}|\left|\frac{\partial(z,x)}{\partial(u,v)}\right| + |z(u,v)-\overline{z_B}|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right\}|z(u,v)-\overline{z_B}|\,dudv.\end{aligned}$$

Remark 4. *Observe that*

$$\begin{aligned}&|x(u,v)-m|\left|\frac{\partial(y,z)}{\partial(u,v)}\right|+|y(u,v)-n|\left|\frac{\partial(z,x)}{\partial(u,v)}\right|+|z(u,v)-p|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\\&\quad\le\left(|x(u,v)-m|^2+|y(u,v)-n|^2+|z(u,v)-p|^2\right)^{1/2}\\&\qquad\times\left(\left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2+\left|\frac{\partial(z,x)}{\partial(u,v)}\right|^2+\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right)^{1/2},\end{aligned}$$

which implies that

$$\begin{aligned}B_1(m,n,p)&\le\int_a^b\int_c^d\left(|x(u,v)-m|^2+|y(u,v)-n|^2+|z(u,v)-p|^2\right)^{1/2}\\&\quad\times\left(\left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2+\left|\frac{\partial(z,x)}{\partial(u,v)}\right|^2+\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right)^{1/2}|x(u,v)-m|\,dudv\\&=\int\int_S\left(|x-m|^2+|y-n|^2+|z-p|^2\right)^{1/2}|x-m|\,dS,\end{aligned}$$

$$\begin{aligned}B_2(m,n,p)&\le\int_a^b\int_c^d\left(|x(u,v)-m|^2+|y(u,v)-n|^2+|z(u,v)-p|^2\right)^{1/2}\\&\quad\times\left(\left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2+\left|\frac{\partial(z,x)}{\partial(u,v)}\right|^2+\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right)^{1/2}|y(u,v)-n|\,dudv\\&=\int\int_S\left(|x-m|^2+|y-n|^2+|z-p|^2\right)^{1/2}|y-n|\,dS\end{aligned}$$

and

$$\begin{aligned}B_3(m,n,p)&\le\int_a^b\int_c^d\left(|x(u,v)-m|^2+|y(u,v)-n|^2+|z(u,v)-p|^2\right)^{1/2}\\&\quad\times\left(\left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2+\left|\frac{\partial(z,x)}{\partial(u,v)}\right|^2+\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right)^{1/2}|z(u,v)-p|\,dudv\\&=\int\int_S\left(|x-m|^2+|y-n|^2+|z-p|^2\right)^{1/2}|z-p|\,dS.\end{aligned}$$

Therefore by (4.4) we get

$$\begin{aligned}
(4.6)\quad & \Bigg| \frac{1}{V(B)} \iiint_B f(x,y,z)\,dxdydz - f(m,n,p) \\
& - \frac{1}{3V(B)} \iiint_B \Bigg[(m-x)\frac{\partial f(x,y,z)}{\partial x} + (n-y)\frac{\partial f(x,y,z)}{\partial y} \\
& + (p-z)\frac{\partial f(x,y,z)}{\partial z} \Bigg] dxdydz \Bigg| \\
& \le \frac{1}{3V(B)} \Bigg\{ \left\| \frac{\partial f}{\partial x} \right\|_{S,\infty} \int\int_S \left(|x-m|^2 + |y-n|^2 + |z-p|^2 \right)^{1/2} |x-m|\,dS \\
& + \left\| \frac{\partial f}{\partial y} \right\|_{S,\infty} \int\int_S \left(|x-m|^2 + |y-n|^2 + |z-p|^2 \right)^{1/2} |y-n|\,dS \\
& + \left\| \frac{\partial f}{\partial z} \right\|_{S,\infty} \int\int_S \left(|x-m|^2 + |y-n|^2 + |z-p|^2 \right)^{1/2} |z-p|\,dS \Bigg\}
\end{aligned}$$

for all $(m,n,p) \in B$.

In particular, we have

$$\begin{aligned}
(4.7)\quad & \Bigg| \frac{1}{V(B)} \iiint_B f(x,y,z)\,dxdydz - f(\overline{x_B},\overline{y_B},\overline{z_B}) \\
& - \frac{1}{3V(B)} \iiint_B \Bigg[(\overline{x_B}-x)\frac{\partial f(x,y,z)}{\partial x} + (\overline{y_B}-y)\frac{\partial f(x,y,z)}{\partial y} \\
& + (\overline{z_B}-z)\frac{\partial f(x,y,z)}{\partial z} \Bigg] dxdydz \Bigg| \\
& \le \frac{1}{3V(B)} \Bigg\{ \left\| \frac{\partial f}{\partial x} \right\|_{S,\infty} \int\int_S \left(|x-\overline{x_B}|^2 + |y-\overline{y_B}|^2 + |z-\overline{z_B}|^2 \right)^{1/2} |x-\overline{x_B}|\,dS \\
& + \left\| \frac{\partial f}{\partial y} \right\|_{S,\infty} \int\int_S \left(|x-\overline{x_B}|^2 + |y-\overline{y_B}|^2 + |z-\overline{z_B}|^2 \right)^{1/2} |y-\overline{y_B}|\,dS \\
& + \left\| \frac{\partial f}{\partial z} \right\|_{S,\infty} \int\int_S \left(|x-\overline{x_B}|^2 + |y-\overline{y_B}|^2 + |z-\overline{z_B}|^2 \right)^{1/2} |z-\overline{z_B}|\,dS \Bigg\}
\end{aligned}$$

and

$$\begin{aligned}
(4.8)\quad & \left| \frac{1}{V(B)} \iiint_B f(x,y,z)\,dxdydz - f(x_{B,\partial f}, y_{B,\partial f}, z_{B,\partial f}) \right| \\
& \le \frac{1}{3V(B)} \Bigg\{ \left\| \frac{\partial f}{\partial x} \right\|_{S,\infty} \int\int_S \left(|x-x_{B,\partial f}|^2 + |y-y_{B,\partial f}|^2 + |z-z_{B,\partial f}|^2 \right)^{1/2} |x-x_{B,\partial f}|\,dS \\
& + \left\| \frac{\partial f}{\partial y} \right\|_{S,\infty} \int\int_S \left(|x-x_{B,\partial f}|^2 + |y-y_{B,\partial f}|^2 + |z-z_{B,\partial f}|^2 \right)^{1/2} |y-y_{B,\partial f}|\,dS \\
& + \left\| \frac{\partial f}{\partial z} \right\|_{S,\infty} \int\int_S \left(|x-x_{B,\partial f}|^2 + |y-y_{B,\partial f}|^2 + |z-z_{B,\partial f}|^2 \right)^{1/2} |z-z_{B,\partial f}|\,dS \Bigg\}.
\end{aligned}$$

5. Some Examples for Sphere

Consider the 3-dimensional ball centered in $C=(a,b,c)$ and having the radius $R>0$,

$$B(C,R):=\left\{(x,y,z)\in\mathbb{R}^3\middle|\,(x-a)^2+(y-b)^2+(z-c)^2\le R^2\right\}$$

and the sphere

$$S(C,R):=\left\{(x,y,z)\in\mathbb{R}^3\middle|\,(x-a)^2+(y-b)^2+(z-c)^2=R^2\right\}.$$

Consider the parametrization of $B(C,R)$ and $S(C,R)$ given by:

$$B(C,R):\begin{cases} x=r\cos\psi\cos\varphi+a\\ y=r\cos\psi\sin\varphi+b\\ z=r\sin\psi+c\end{cases};\quad (r,\psi,\varphi)\in[0,R]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi]$$

and

$$S(C,R):\begin{cases} x=R\cos\psi\cos\varphi+a\\ y=R\cos\psi\sin\varphi+b\\ z=R\sin\psi+c\end{cases};\quad (\psi,\varphi)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi].$$

By setting

$$A:=\begin{vmatrix}\frac{\partial y}{\partial\psi} & \frac{\partial z}{\partial\psi}\\ \frac{\partial y}{\partial\varphi} & \frac{\partial z}{\partial\varphi}\end{vmatrix}=-R^2\cos^2\psi\cos\varphi,$$

$$B:=\begin{vmatrix}\frac{\partial x}{\partial\psi} & \frac{\partial z}{\partial\psi}\\ \frac{\partial x}{\partial\varphi} & \frac{\partial z}{\partial\varphi}\end{vmatrix}=R^2\cos^2\psi\sin\varphi,$$

and

$$C:=\begin{vmatrix}\frac{\partial x}{\partial\psi} & \frac{\partial y}{\partial\psi}\\ \frac{\partial x}{\partial\varphi} & \frac{\partial y}{\partial\varphi}\end{vmatrix}=-R^2\sin\psi\cos\psi,$$

we have that

$$A^2+B^2+C^2=R^4\cos^2\psi \text{ for all } (\psi,\varphi)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi].$$

Obviously $\overline{x_B}=a$, $\overline{y_B}=b$, $\overline{z_B}=c$ and

$$\begin{aligned}&\int\int_{S(C,R)}\left(|x-\overline{x_B}|^2+|y-\overline{y_B}|^2+|z-\overline{z_B}|^2\right)^{1/2}|x-\overline{x_B}|\,dS\\ &=\int\int_{S(C,R)}\left(|x-a|^2+|y-b|^2+|z-c|^2\right)^{1/2}|x-\overline{x_B}|\,dS\\ &=R^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\cos^2\psi\,|\cos\varphi|\,d\psi d\varphi=2\pi R^4.\end{aligned}$$

Similarly,

$$\begin{aligned}&\int\int_{S(C,R)}\left(|x-\overline{x_B}|^2+|y-\overline{y_B}|^2+|z-\overline{z_B}|^2\right)^{1/2}|y-\overline{y_B}|\,dS\\ &=\int\int_{S(C,R)}\left(|x-\overline{x_B}|^2+|y-\overline{y_B}|^2+|z-\overline{z_B}|^2\right)^{1/2}|z-\overline{z_B}|\,dS=2\pi R^4\end{aligned}$$

and by (4.7) we get

$$\begin{aligned}(5.1)\quad &\left|\frac{1}{V(B)}\iiint_{B(C,R)} f(x,y,z)\,dxdydz - f(a,b,c)\right.\\ &-\frac{1}{3V(B)}\iiint_{B(C,R)}\left[(a-x)\frac{\partial f(x,y,z)}{\partial x}+(b-y)\frac{\partial f(x,y,z)}{\partial y}\right.\\ &\left.\left.+(c-z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz\right|\\ &\leq \frac{1}{2}R\left[\left\|\frac{\partial f}{\partial x}\right\|_{S(C,R),\infty}+\left\|\frac{\partial f}{\partial y}\right\|_{S(C,R),\infty}+\left\|\frac{\partial f}{\partial z}\right\|_{S(C,R),\infty}\right].\end{aligned}$$

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