SOME MULTIPLE INTEGRAL INEQUALITIES VIA THE DIVERGENCE THEOREM

SILVESTRU SEVER DRAGOMIR[1,2]

ABSTRACT. In this paper, by the use of the divergence theorem, we establish some inequalities for functions defined on closed and bounded subsets of the Euclidean space $\mathbb{R}^n$, $n \geq 2$.

1. INTRODUCTION

Let ∂D be a simple, closed counterclockwise curve bounding a region D and f defined on an open set containing D and having continuous partial derivatives on D. In the recent paper [4], by the use of *Green's identity*, we have shown among others that

$$\left| \int \int_D f(x,y)\, dxdy - \frac{1}{2} \oint_{\partial D} \left[(\beta - y) f(x,y)\, dx + (x-\alpha) f(x,y)\, dy \right] \right| \tag{1.1}$$
$$\leq \frac{1}{2} \int \int_D \left[|\alpha - x| \left| \frac{\partial f(x,y)}{\partial x} \right| + |\beta - y| \left| \frac{\partial f(x,y)}{\partial y} \right| \right] dxdy =: M(\alpha, \beta; f)$$

for all α, $\beta \in \mathbb{C}$ and

$$M(\alpha,\beta;f) \tag{1.2}$$
$$\leq \begin{cases} \left\| \frac{\partial f}{\partial x} \right\|_{D,\infty} \iint_D |\alpha - x|\, dxdy + \left\| \frac{\partial f}{\partial y} \right\|_{D,\infty} \iint_D |\beta - y|\, dxdy; \\ \\ \left\| \frac{\partial f}{\partial x} \right\|_{D,p} \left(\iint_D |\alpha - x|^q\, dxdy \right)^{1/q} + \left\| \frac{\partial f}{\partial y} \right\|_{D,p} \left(\iint_D |\beta - y|^q\, dxdy \right)^{1/q} \\ \text{where } p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \sup_{(x,y)\in D} |\alpha - x| \left\| \frac{\partial f}{\partial x} \right\|_{D,1} + \sup_{(x,y)\in D} |\beta - y| \left\| \frac{\partial f}{\partial y} \right\|_{B,1}, \end{cases}$$

where $\|\cdot\|_{D,p}$ are the usual Lebesgue norms, we recall that

$$\|g\|_{D,p} := \begin{cases} \left(\iint_D |g(x,y)|^p\, dxdy \right)^{1/p},\ p \geq 1; \\ \\ \sup_{(x,y)\in D} |g(x,y)|,\ p = \infty. \end{cases}$$

Applications for rectangles and disks were also provided in [4]. For some recent double integral inequalities see [1], [2] and [3].

1991 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Multiple integral inequalities, Divergence theorem, Green identity, Gauss-Ostrogradsky identity.

We also considered similar inequalities for 3-dimensional bodies as follows, see [5]. Let B be a solid in the three dimensional space $\mathbb{R}^3$ bounded by an orientable closed surface ∂B. If $f : B \to \mathbb{C}$ is a continuously differentiable function defined on a open set containing B, then by making use of the *Gauss-Ostrogradsky identity*, we have obtained the following inequality

$$\begin{aligned} &\left| \iiint_B f(x,y,z)\, dxdydz - \frac{1}{3}\left[\int\int_{\partial B} (x-\alpha) f(x,y,z)\, dy \wedge dz \right.\right. \\ &\quad \left.\left. + \int\int_{\partial B} (y-\beta) f(x,y,z)\, dz \wedge dx + \int\int_{\partial B} (z-\gamma) f(x,y,z)\, dx \wedge dy \right] \right| \\ &\qquad \le \frac{1}{3} \iiint_B \left[|\alpha - x| \left| \frac{\partial f(x,y,z)}{\partial x} \right| + |\beta - y| \left| \frac{\partial f(x,y,z)}{\partial y} \right| \right. \\ &\qquad\qquad \left. + |\gamma - z| \left| \frac{\partial f(x,y,z)}{\partial z} \right| \right] dxdydz =: M(\alpha,\beta,\gamma; f) \end{aligned} \tag{1.3}$$

for all α, β, γ complex numbers. Moreover, we have the bounds

$$M(\alpha,\beta,\gamma; f) \tag{1.4}$$

$$\le \frac{1}{3} \begin{cases} \left\| \frac{\partial f}{\partial x} \right\|_{B,\infty} \iiint_B |\alpha - x|\, dxdydz + \left\| \frac{\partial f}{\partial y} \right\|_{B,\infty} \iiint_B |\beta - y|\, dxdydz \\ + \left\| \frac{\partial f}{\partial z} \right\|_{B,\infty} \iiint_B |\gamma - z|\, dxdydz; \\ \\ \left\| \frac{\partial f}{\partial x} \right\|_{B,p} \left(\iiint_B |\alpha - x|^q\, dxdydz \right)^{1/q} + \left\| \frac{\partial f}{\partial y} \right\|_{B,p} \left(\iiint_B |\beta - y|^q\, dxdydz \right)^{1/q} \\ + \left\| \frac{\partial f}{\partial z} \right\|_{B,p} \left(\iiint_B |\gamma - z|\, dxdydz \right)^{1/q}, \ p,q > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \sup_{(x,y,z)\in B} |\alpha - x| \left\| \frac{\partial f}{\partial x} \right\|_{B,1} + \sup_{(x,y,z)\in B} |\beta - y| \left\| \frac{\partial f}{\partial y} \right\|_{B,1} \\ + \sup_{(x,y,z)\in B} |\gamma - z| \left\| \frac{\partial f}{\partial z} \right\|_{B,1}. \end{cases}$$

Applications for 3-dimensional balls were also given in [5]. For some triple integral inequalities see [6] and [9].

Motivated by the above results, in this paper we establish several similar inequalities for multiple integrals for functions defined on bonded subsets of $\mathbb{R}^n$ $(n \ge 2)$ with smooth (or piecewise smooth) boundary ∂B. To achieve this goal we make use of the well known divergence theorem for multiple integrals as summarized below.

2. Some Preliminary Facts

Let B be a bounded open subset of $\mathbb{R}^n$ $(n \ge 2)$ with smooth (or piecewise smooth) boundary ∂B. Let $F = (F_1, ..., F_n)$ be a smooth vector field defined in $\mathbb{R}^n$, or at least in $B \cup \partial B$. Let $\mathbf{n}$ be the unit outward-pointing normal of ∂B. Then the *Divergence Theorem* states, see for instance [8]:

$$\int_B \operatorname{div} F dV = \int_{\partial B} F \cdot n dA, \tag{2.1}$$

where

$$\operatorname{div} F = \nabla \cdot F = \sum_{k=1}^{n} \frac{\partial F_i}{\partial x_i},$$

dV is the element of volume in $\mathbb{R}^n$ and dA is the element of surface area on ∂B.

If $\mathbf{n} = (\mathbf{n}_1, ..., \mathbf{n}_n)$, $x = (x_1, ..., x_n) \in B$ and use the notation dx for dV we can write (2.1) more explicitly as

$$\sum_{k=1}^{n} \int_B \frac{\partial F_k(x)}{\partial x_k} dx = \sum_{k=1}^{n} \int_{\partial B} F_k(x)\, n_k(x)\, dA. \tag{2.2}$$

By taking the real and imaginary part, we can extend the above equality for complex valued functions F_k, $k \in \{1, ..., n\}$ defined on B.

If $n = 2$, the normal is obtained by rotating the tangent vector through $90°$ (in the correct direction so that it points out). The quantity tds can be written (dx_1, dx_2) along the surface, so that

$$ndA := nds = (dx_2, -dx_1)$$

Here t is the tangent vector along the boundary curve and ds is the element of arc-length.

From (2.2) we get for $B \subset \mathbb{R}^2$ that

$$\begin{aligned} &\int_B \frac{\partial F_1(x_1, x_2)}{\partial x_1} dx_1 dx_2 + \int_B \frac{\partial F_2(x_1, x_2)}{\partial x_2} dx_1 dx_2 \\ &\qquad = \int_{\partial B} F_1(x_1, x_2)\, dx_2 - \int_{\partial B} F_2(x_1, x_2)\, dx_1, \end{aligned} \tag{2.3}$$

which is *Green's theorem* in plane.

If $n = 3$ and if ∂B is described as a level-set of a function of 3 variables i.e. $\partial B = \{x_1, x_2, x_3 \in \mathbb{R}^3 \mid G(x_1, x_2, x_3) = 0\}$, then a vector pointing in the direction of $\mathbf{n}$ is $\operatorname{grad} G$. We shall use the case where $G(x_1, x_2, x_3) = x_3 - g(x_1, x_2)$, $(x_1, x_2) \in D$, a domain in $\mathbb{R}^2$ for some differentiable function g on D and B corresponds to the inequality $x_3 < g(x_1, x_2)$, namely

$$B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 < g(x_1, x_2)\}.$$

Then

$$\mathbf{n} = \frac{(-g_{x_1}, -g_{x_2}, 1)}{\left(1 + g_{x_1}^2 + g_{x_2}^2\right)^{1/2}}, \; dA = \left(1 + g_{x_1}^2 + g_{x_2}^2\right)^{1/2} dx_1 dx_2$$

and

$$\mathbf{n} dA = (-g_{x_1}, -g_{x_2}, 1)\, dx_1 dx_2.$$

From (2.2) we get

$$\begin{aligned} &\int_B \left(\frac{\partial F_1(x_1, x_2, x_3)}{\partial x_1} + \frac{\partial F_2(x_1, x_2, x_3)}{\partial x_2} + \frac{\partial F_3(x_1, x_2, x_3)}{\partial x_3} \right) dx_1 dx_2 dx_3 \\ &\qquad = -\int_D F_1(x_1, x_2, g(x_1, x_2))\, g_{x_1}(x_1, x_2)\, dx_1 dx_2 \\ &\qquad\quad - \int_D F_1(x_1, x_2, g(x_1, x_2))\, g_{x_2}(x_1, x_2) dx_1 dx_2 \\ &\qquad\qquad + \int_D F_3(x_1, x_2, g(x_1, x_2))\, dx_1 dx_2 \end{aligned} \tag{2.4}$$

which is the *Gauss-Ostrogradsky theorem* in space.

3. Identities of Interest

We have the following identity of interest:

Theorem 1. *Let B be a bounded open subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary ∂B. Let f be a continuously differentiable function defined in $\mathbb{R}^n$, or at least in $B \cup \partial B$ and with complex values. If $\alpha_k,\ \beta_k \in \mathbb{C}$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} \alpha_k = 1$, then*

$$\int_B f(x)\,dx = \sum_{k=1}^{n} \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} dx + \sum_{k=1}^{n} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x)\, n_k(x)\, dA. \tag{3.1}$$

We also have

$$\int_B f(x)\,dx = \frac{1}{n} \sum_{k=1}^{n} \int_B (\gamma_k - x_k) \frac{\partial f(x)}{\partial x_k} dx + \frac{1}{n} \sum_{k=1}^{n} \int_{\partial B} (x_k - \gamma_k) f(x)\, n_k(x)\, dA \tag{3.2}$$

for all $\gamma_k \in \mathbb{C}$, where $k \in \{1, ..., n\}$.

Proof. Let $x = (x_1, ..., x_n) \in B$. We consider

$$F_k(x) = (\alpha_k x_k - \beta_k) f(x),\ \ k \in \{1, ..., n\}$$

and take the partial derivatives $\frac{\partial F_k(x)}{\partial x_k}$ to get

$$\frac{\partial F_k(x)}{\partial x_k} = \alpha_k f(x) + (\alpha_k x_k - \beta_k) \frac{\partial f(x)}{\partial x_k},\ \ k \in \{1, ..., n\}.$$

If we sum this equality over k from 1 to n we get

$$\begin{aligned} \sum_{k=1}^{n} \frac{\partial F_k(x)}{\partial x_k} &= \sum_{k=1}^{n} \alpha_k f(x) + \sum_{k=1}^{n} (\alpha_k x_k - \beta_k) \frac{\partial f(x)}{\partial x_k} \\ &= f(x) + \sum_{k=1}^{n} (\alpha_k x_k - \beta_k) \frac{\partial f(x)}{\partial x_k} \end{aligned} \tag{3.3}$$

for all $x = (x_1, ..., x_n) \in B$.

Now, if we take the integral in the equality (3.3) over $(x_1, ..., x_n) \in B$ we get

$$\int_B \left(\sum_{k=1}^{n} \frac{\partial F_k(x)}{\partial x_k} \right) dx = \int_B f(x)\,dx + \sum_{k=1}^{n} \int_B \left[(\alpha_k x_k - \beta_k) \frac{\partial f(x)}{\partial x_k} \right] dx. \tag{3.4}$$

By the Divergence Theorem (2.2) we also have

$$\int_B \left(\sum_{k=1}^{n} \frac{\partial F_k(x)}{\partial x_k} \right) dx = \sum_{k=1}^{n} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x)\, n_k(x)\, dA \tag{3.5}$$

and by making use of (3.4) and (3.5) we get

$$\int_B f(x)\,dx + \sum_{k=1}^{n} \int_B \left[(\alpha_k x_k - \beta_k) \frac{\partial f(x)}{\partial x_k} \right] dx = \sum_{k=1}^{n} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x)\, n_k(x)\, dA$$

which gives the desired representation (3.1).

The identity (3.2) follows by (3.1) for $\alpha_k = \frac{1}{n}$ and $\beta_k = \frac{1}{n}\gamma_k$, $k \in \{1, ..., n\}$. □

For the body B we consider the coordinates for the *centre of gravity*

$$G\left(\overline{x_{B,1}}, ..., \overline{x_{B,n}}\right)$$

defined by

$$\overline{x_{B,k}} := \frac{1}{V(B)} \int_B x_k dx, \ k \in \{1, ..., n\},$$

where

$$V(B) := \int_B x dx$$

is the volume of B.

Corollary 1. *With the assumptions of Theorem 1 we have*

$$\int_B f(x)\,dx = \sum_{k=1}^{n} \int_B \alpha_k \left(\overline{x_{B,k}} - x_k\right) \frac{\partial f(x)}{\partial x_k} dx + \sum_{k=1}^{n} \int_{\partial B} \alpha_k \left(x_k - \overline{x_{B,k}}\right) f(x)\, n_k(x)\, dA \tag{3.6}$$

and, in particular,

$$\int_B f(x)\,dx = \frac{1}{n} \sum_{k=1}^{n} \int_B \left(\overline{x_{B,k}} - x_k\right) \frac{\partial f(x)}{\partial x_k} dx + \frac{1}{n} \sum_{k=1}^{n} \int_{\partial B} \left(x_k - \overline{x_{B,k}}\right) f(x)\, n_k(x)\, dA. \tag{3.7}$$

The proof follows by (3.1) on taking $\beta_k = \alpha_k \overline{x_{B,k}}$, $k \in \{1, ..., n\}$.

For a function f as in Theorem 1 above, we define the points

$$x_{B,\partial f,k} := \frac{\int_B x_k \frac{\partial f(x)}{\partial x_k} dx}{\int_B \frac{\partial f(x)}{\partial x_k} dx}, \ k \in \{1, ..., n\},$$

provided that all denominators are not zero.

Corollary 2. *With the assumptions of Theorem 1 we have*

$$\int_B f(x)\,dx = \sum_{k=1}^{n} \int_{\partial B} \alpha_k \left(x_k - x_{B,\partial f,k}\right) f(x)\, n_k(x)\, dA \tag{3.8}$$

and, in particular,

$$\int_B f(x)\,dx = \frac{1}{n} \sum_{k=1}^{n} \int_{\partial B} \left(x_k - x_{B,\partial f,k}\right) f(x)\, n_k(x)\, dA. \tag{3.9}$$

The proof follows by (3.1) on taking $\beta_k = \alpha_k x_{B,\partial f,k}$, $k \in \{1, ..., n\}$ and observing that

$$\sum_{k=1}^{n} \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} dx = \sum_{k=1}^{n} \alpha_k \int_B (x_{B,\partial f,k} - x_k) \frac{\partial f(x)}{\partial x_k} dx = 0.$$

For a function f as in Theorem 1 above, we define the points

$$x_{\partial B,f,k} := \frac{\int_{\partial B} x_k f(x) n_k(x) dA}{\int_{\partial B} f(x) n_k(x) dA}, \ k \in \{1, ..., n\}$$

provided that all denominators are not zero.

Corollary 3. *With the assumptions of Theorem 1 we have*

$$\int_B f(x) dx = \sum_{k=1}^{n} \int_B \alpha_k (x_{\partial B,f,k} - x_k) \frac{\partial f(x)}{\partial x_k} dx \tag{3.10}$$

and, in particular,

$$\int_B f(x) dx = \frac{1}{n} \sum_{k=1}^{n} \int_B (x_{\partial B,f,k} - x_k) \frac{\partial f(x)}{\partial x_k} dx. \tag{3.11}$$

The proof follows by (3.1) on taking $\beta_k = \alpha_k x_{\partial B,f,k}$, $k \in \{1, ..., n\}$ and observing that

$$\sum_{k=1}^{n} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x) dA = 0.$$

4. Some Integral Inequalities

We have the following result generalizing the inequalities from the introduction:

Theorem 2. *Let B be a bounded open subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary ∂B. Let f be a continuously differentiable function defined in $\mathbb{R}^n$, or at least in $B \cup \partial B$ and with complex values. If $\alpha_k, \beta_k \in \mathbb{C}$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} \alpha_k = 1$, then*

$$\begin{aligned} &\left| \int_B f(x) dx - \sum_{k=1}^{n} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x) dA \right| \\ &\leq \sum_{k=1}^{n} \int_B |\beta_k - \alpha_k x_k| \left| \frac{\partial f(x)}{\partial x_k} \right| dx \\ &\leq \begin{cases} \sum_{k=1}^{n} \int_B |\beta_k - \alpha_k x_k| dx \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B,\infty} \\ \\ \sum_{k=1}^{n} \left(\int_B |\beta_k - \alpha_k x_k|^q dx \right)^{1/q} \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B,p} \\ \text{where } p, \ q > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \sum_{k=1}^{n} \sup_{x \in B} |\beta_k - \alpha_k x_k| \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B,1} \end{cases} \end{aligned} \tag{4.1}$$

We also have

$$\left|\int_B f(x)\,dx - \frac{1}{n}\sum_{k=1}^{n}\int_{\partial B}(x_k-\gamma_k)\,f(x)\,n_k(x)\,dA\right| \tag{4.2}$$

$$\leq \frac{1}{n}\sum_{k=1}^{n}\int_B |\gamma_k - x_k|\left|\frac{\partial f(x)}{\partial x_k}\right|dx$$

$$\leq \frac{1}{n}\begin{cases} \sum_{k=1}^{n}\int_B |\gamma_k - x_k|\,dx \left\|\frac{\partial f(x)}{\partial x_k}\right\|_{B,\infty} \\ \\ \sum_{k=1}^{n}\left(\int_B^q |\gamma_k - x_k|^q\,dx\right)^{1/q}\left\|\frac{\partial f(x)}{\partial x_k}\right\|_{B,p} \\ \textit{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sum_{k=1}^{n}\sup_{x\in B}|\gamma_k - x_k|\left\|\frac{\partial f(x)}{\partial x_k}\right\|_{B,1} \end{cases}$$

for all $\gamma_k \in \mathbb{C}$, *where* $k \in \{1,...,n\}$.

Proof. By the identity (3.1) we have

$$\left|\int_B f(x)\,dx - \sum_{k=1}^{n}\int_{\partial B}(\alpha_k x_k-\beta_k)\,f(x)\,n_k(x)\,dA\right|$$

$$= \left|\sum_{k=1}^{n}\int_B(\beta_k-\alpha_k x_k)\frac{\partial f(x)}{\partial x_k}dx\right| \leq \sum_{k=1}^{n}\left|\int_B(\beta_k-\alpha_k x_k)\frac{\partial f(x)}{\partial x_k}dx\right|$$

$$\leq \sum_{k=1}^{n}\int_B\left|(\beta_k-\alpha_k x_k)\frac{\partial f(x)}{\partial x_k}\right|dx,$$

which proves the first inequality in (4.1).

By Hölder's integral inequality for multiple integrals we have

$$\int_B\left|(\beta_k-\alpha_k x_k)\frac{\partial f(x)}{\partial x_k}\right|dx \leq \begin{cases} \sup_{x\in B}\left|\frac{\partial f(x)}{\partial x_k}\right|\int_B|\beta_k-\alpha_k x_k|\,dx \\ \\ \left(\int_B\left|\frac{\partial f(x)}{\partial x_k}\right|^p\right)^{1/p}\left(\int_B|\beta_k-\alpha_k x_k|^q\,dx\right)^{1/q} \\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sup_{x\in B}|\beta_k-\alpha_k x_k|\int_B\left|\frac{\partial f(x)}{\partial x_k}\right|dx \end{cases}$$

$$= \begin{cases} \int_B|\beta_k-\alpha_k x_k|\,dx\left\|\frac{\partial f}{\partial x_k}\right\|_{B,\infty} \\ \\ \left(\int_B|\beta_k-\alpha_k x_k|^q\,dx\right)^{1/q}\left\|\frac{\partial f}{\partial x_k}\right\|_{B,p} \\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sup_{x\in B}|\beta_k-\alpha_k x_k|\left\|\frac{\partial f}{\partial x_k}\right\|_{B,1} \end{cases},$$

which proves the last part of (4.1). □

Corollary 4. *With the assumptions of Theorem 2 we have*

$$\left|\int_B f(x)\,dx - \frac{1}{n}\sum_{k=1}^n \int_{\partial B}\left(x_k - \overline{x_{B,k}}\right) f(x)\, n_k(x)\, dA\right| \tag{4.3}$$
$$\leq \frac{1}{n}\sum_{k=1}^n \int_B \left|\overline{x_{B,k}} - x_k\right| \left|\frac{\partial f(x)}{\partial x_k}\right| dx$$
$$\leq \frac{1}{n}\begin{cases} \sum_{k=1}^n \int_B \left|\overline{x_{B,k}} - x_k\right| dx \left\|\frac{\partial f(x)}{\partial x_k}\right\|_{B,\infty} \\ \\ \sum_{k=1}^n \left(\int_B^q \left|\overline{x_{B,k}} - x_k\right|^q dx\right)^{1/q} \left\|\frac{\partial f(x)}{\partial x_k}\right\|_{B,p} \\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sum_{k=1}^n \sup_{x\in B}\left|\overline{x_{B,k}} - x_k\right| \left\|\frac{\partial f(x)}{\partial x_k}\right\|_{B,1} \end{cases}$$

and

$$\left|\int_B f(x)\,dx\right| \leq \frac{1}{n}\sum_{k=1}^n \int_B \left|x_{\partial B,f,k} - x_k\right| \left|\frac{\partial f(x)}{\partial x_k}\right| dx \tag{4.4}$$
$$\leq \frac{1}{n}\begin{cases} \sum_{k=1}^n \int_B \left|x_{\partial B,f,k} - x_k\right| dx \left\|\frac{\partial f(x)}{\partial x_k}\right\|_{B,\infty} \\ \\ \sum_{k=1}^n \left(\int_B^q \left|x_{\partial B,f,k} - x_k\right|^q dx\right)^{1/q} \left\|\frac{\partial f(x)}{\partial x_k}\right\|_{B,p} \\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sum_{k=1}^n \sup_{x\in B}\left|x_{\partial B,f,k} - x_k\right| \left\|\frac{\partial f(x)}{\partial x_k}\right\|_{B,1}. \end{cases}$$

We also have the dual result:

Theorem 3. *With the assumption of Theorem 2 we have*

$$\left|\int_B f(x)\,dx - \sum_{k=1}^n \int_B \left(\beta_k - \alpha_k x_k\right) \frac{\partial f(x)}{\partial x_k} dx\right| \tag{4.5}$$
$$\leq \sum_{k=1}^n \int_{\partial B} \left|\alpha_k x_k - \beta_k\right| \left|n_k(x)\right| \left|f(x)\right| dA$$
$$\leq \begin{cases} \|f\|_{\partial B,\infty} \sum_{k=1}^n \int_{\partial B} \left|\alpha_k x_k - \beta_k\right| \left|n_k(x)\right| dA; \\ \\ \|f\|_{\partial B,p} \sum_{k=1}^n \left(\int_{\partial B} \left|\alpha_k x_k - \beta_k\right|^q \left|n_k(x)\right|^q dA\right)^{1/q} \\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \|f\|_{\partial B,1} \sum_{k=1}^n \sup_{x\in\partial B} \left|\left|\alpha_k x_k - \beta_k\right| \left|n_k(x)\right|\right|, \end{cases}$$

where

$$\|f\|_{\partial B,p} := \begin{cases} \left(\int_{\partial B} |f(x)|^p\, dA\right)^{1/p},\ p\geq 1; \\ \\ \sup_{x\in\partial B} |f(x)|,\ p=\infty. \end{cases}$$

In particular,

$$\left|\int_B f(x)\,dx - \frac{1}{n}\sum_{k=1}^{n}\int_B (\gamma_k - x_k)\frac{\partial f(x)}{\partial x_k}dx\right| \tag{4.6}$$
$$\leq \frac{1}{n}\sum_{k=1}^{n}\int_{\partial B}|\gamma_k - x_k|\,|n_k(x)|\,|f(x)|\,dA$$
$$\leq \frac{1}{n}\begin{cases} \|f\|_{\partial B,\infty}\sum_{k=1}^{n}\int_{\partial B}|\gamma_k - x_k|\,|n_k(x)|\,dA; \\ \\ \|f\|_{\partial B,p}\sum_{k=1}^{n}\left(\int_{\partial B}|\gamma_k - x_k|^q\,|n_k(x)|^q\,dA\right)^{1/q} \\ \textit{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \|f\|_{\partial B,1}\sum_{k=1}^{n}\sup_{x\in\partial B}\left[|\gamma_k - x_k|\,|n_k(x)|\right]. \end{cases}$$

Proof. From the identity (3.1) we have

$$\left|\int_B f(x)\,dx - \sum_{k=1}^{n}\int_B (\beta_k - \alpha_k x_k)\frac{\partial f(x)}{\partial x_k}dx\right|$$
$$= \left|\sum_{k=1}^{n}\int_{\partial B}(\alpha_k x_k - \beta_k)\,f(x)\,n_k(x)\,dA\right|$$
$$\leq \sum_{k=1}^{n}\left|\int_{\partial B}(\alpha_k x_k - \beta_k)\,f(x)\,n_k(x)\,dA\right| \leq \sum_{k=1}^{n}\int_{\partial B}|(\alpha_k x_k - \beta_k)\,f(x)\,n_k(x)|\,dA,$$

which proves the first inequality in (4.5).

By Hölder's inequality for functions defined on ∂B we have

$$\int_{\partial B}|\alpha_k x_k - \beta_k|\,|n_k(x)|\,|f(x)|\,dA \leq \begin{cases} \int_{\partial B}|\alpha_k x_k - \beta_k|\,|n_k(x)|\,dA\,\|f\|_{\partial B,\infty}; \\ \\ \left(\int_{\partial B}|\alpha_k x_k - \beta_k|^q\,|n_k(x)|^q\,dA\right)^{1/q}\|f\|_{\partial B,p} \\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sup_{x\in\partial B}\left||\alpha_k x_k - \beta_k|\,|n_k(x)|\right|\,\|f\|_{\partial B,1}, \end{cases}$$

which proves the second part of the inequality (4.5). □

We also have:

Corollary 5. *With the assumptions of Theorem 2 we have*

$$\left|\int_B f(x)\,dx - \frac{1}{n}\sum_{k=1}^{n}\int_B (\overline{x_{B,k}} - x_k)\frac{\partial f(x)}{\partial x_k}dx\right| \tag{4.7}$$
$$\leq \frac{1}{n}\sum_{k=1}^{n}\int_{\partial B}|\overline{x_{B,k}} - x_k|\,|n_k(x)|\,|f(x)|\,dA$$

$$\leq \frac{1}{n}\begin{cases} \|f\|_{\partial B,\infty}\sum_{k=1}^{n}\int_{\partial B}\left|\overline{x_{B,k}}-x_k\right|\left|n_k(x)\right|dA; \\ \\ \|f\|_{\partial B,p}\sum_{k=1}^{n}\left(\int_{\partial B}\left|\overline{x_{B,k}}-x_k\right|^q\left|n_k(x)\right|^q dA\right)^{1/q} \\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \|f\|_{\partial B,1}\sum_{k=1}^{n}\sup_{x\in\partial B}\left[\left|\overline{x_{B,k}}-x_k\right|\left|n_k(x)\right|\right] \end{cases}$$

and

$$\left|\int_B f(x)\,dx\right| \leq \frac{1}{n}\sum_{k=1}^{n}\int_{\partial B}\left|x_{B,\partial f,k}-x_k\right|\left|n_k(x)\right|\left|f(x)\right|dA \tag{4.8}$$

$$\leq \frac{1}{n}\begin{cases} \|f\|_{\partial B,\infty}\sum_{k=1}^{n}\int_{\partial B}\left|x_{B,\partial f,k}-x_k\right|\left|n_k(x)\right|dA; \\ \\ \|f\|_{\partial B,p}\sum_{k=1}^{n}\left(\int_{\partial B}\left|x_{B,\partial f,k}-x_k\right|^q\left|n_k(x)\right|^q dA\right)^{1/q} \\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \|f\|_{\partial B,1}\sum_{k=1}^{n}\sup_{x\in\partial B}\left[\left|x_{B,\partial f,k}-x_k\right|\left|n_k(x)\right|\right]. \end{cases}$$

If we take $n=2$ in Theorem 3, then we get other results from [4], while for $n=3$ we recapture some results from [5].

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa