SOME HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS DEFINED ON CONVEX BODIES IN $\mathbb{R}^n$

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Abstract. In this paper, by the use of Divergence Theorem, we establish some integral inequalities of Hermite-Hadamard type for convex functions of several variables defined on closed and bounded convex bodies in the Euclidean space $\mathbb{R}^n$ for any $n \geq 2$.

1. Introduction

In the following, consider D a closed and bounded convex subset of $\mathbb{R}^2$. Define

$$A_D := \int \int_D dxdy$$

the *area* of D and $(\overline{x_D}, \overline{y_D})$ the *centre of mass* for D, where

$$\overline{x_D} := \frac{1}{A_D} \int \int_D xdxdy, \ \overline{y_D} := \frac{1}{A_D} \int \int_D ydxdy.$$

Consider the function of two variables $f = f(x, y)$ and denote by $\frac{\partial f}{\partial x}$ the partial derivative with respect to the variable x and $\frac{\partial f}{\partial y}$ the partial derivative with respect to the variable y.

In the recent paper [9] we obtained the following Hermite-Hadamard type inequalities:

Theorem 1. *Let $f : D \to \mathbb{R}$ be a differentiable convex function on D, a closed and bounded convex subset of $\mathbb{R}^2$ surrounded by the smooth curve ∂D. Then for all $(u, v) \in D$ we have*

$$\begin{aligned}(1.1) \quad & \frac{\partial f}{\partial x}(u, v)(\overline{x_D} - u) + \frac{\partial f}{\partial y}(u, v)(\overline{y_D} - v) + f(u, v) \\ & \leq \frac{1}{A_D} \int \int_D f(x, y) dxdy \\ & \leq \frac{1}{3} f(u, v) + \frac{1}{3A_D} \oint_{\partial D} [(v - y) f(x, y) dx + (x - u) f(x, y) dy].\end{aligned}$$

1991 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Hermite-Hadamard inequality, Multiple integral inequalities, Green identity, Gauss-Ostrogradsky identity, Divergence theorem.

RGMIA Res. Rep. Coll. 22 (2019), Art. 65, 13 pp. Received 24/06/19

In particular,

$$f\left(\overline{x_D},\overline{y_D}\right)\leq\frac{1}{A_D}\int\int_D f(x,y)\,dxdy \tag{1.2}$$
$$\leq\frac{1}{3}f\left(\overline{x_D},\overline{y_D}\right)+\frac{1}{3A_D}\oint_{\partial D}\left[\left(\overline{y_D}-y\right)f(x,y)\,dx+\left(x-\overline{x_D}\right)f(x,y)\,dy\right].$$

We also have:

Corollary 1. *With the assumptions of Theorem 1 we have*

$$f\left(\overline{x_D},\overline{y_D}\right)\leq\frac{1}{A_D}\int\int_D f(x,y)\,dxdy \tag{1.3}$$
$$\leq\frac{1}{2A_D}\oint_{\partial D}\left[\left(\overline{y_D}-y\right)f(x,y)\,dx+\left(x-\overline{x_D}\right)f(x,y)\,dy\right].$$

Some examples for rectangle and disks on the plane were also provided in [9].

The case of convex function defined on convex body from space was considered in [10] were we obtained the following result:

Theorem 2. *Let B be a convex body in the three dimensional space $\mathbb{R}^3$ bounded by an orientable closed surface ∂B and $f:B\to\mathbb{C}$ a continuously differentiable function defined on a open set containing B. If f is convex on B, then for any $(u,v,w)\in B$ we have*

$$f(u,v,w)+\left(\overline{x_B}-u\right)\frac{\partial f(u,v,w)}{\partial x} \tag{1.4}$$
$$+\left(\overline{y_B}-v\right)\frac{\partial f(u,v,w)}{\partial y}+\left(\overline{z_B}-w\right)\frac{\partial f(u,v,w)}{\partial z}$$
$$\leq\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz$$
$$\leq\frac{1}{4}f(u,v,w)+\frac{1}{4}\frac{1}{V(B)}\left[\int\int_{\partial B}(x-u)\,f(x,y,z)\,dy\wedge dz\right.$$
$$\left.+\int\int_{\partial B}(y-v)\,f(x,y,z)\,dz\wedge dx+\int\int_{\partial B}(z-w)\,f(x,y,z)\,dx\wedge dy\right],$$

where

$$\overline{x_B}:=\frac{1}{V(B)}\iiint_B xdxdydz,\ \ \overline{y_B}:=\frac{1}{V(B)}\iiint_B ydxdydz$$

and

$$\overline{z_B}:=\frac{1}{V(B)}\iiint_B zdxdydz.$$

In particular, we have

$$f\left(\overline{x_B},\overline{y_B},\overline{z_B}\right)\leq\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz \tag{1.5}$$
$$\leq\frac{1}{4}f\left(\overline{x_B},\overline{y_B},\overline{z_B}\right)+\frac{1}{4}\frac{1}{V(B)}\left[\int\int_{\partial B}\left(x-\overline{x_B}\right)f(x,y,z)\,dy\wedge dz\right.$$
$$\left.+\int\int_{\partial B}\left(y-\overline{y_B}\right)f(x,y,z)\,dz\wedge dx+\int\int_{\partial B}\left(z-\overline{z_B}\right)f(x,y,z)\,dx\wedge dy\right].$$

We also have:

Corollary 2. *With the assumptions of Theorem 2,*

$$\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz \leq \frac{1}{3}\frac{1}{V(B)}\left[\int\int_S (x-\overline{x_B})\, f(x,y,z)\,dy\wedge dz \right. \tag{1.6}$$
$$\left. +\int\int_S (y-\overline{y_B})\, f(x,y,z)\,dz\wedge dx+\int\int_S (z-\overline{z_B})\, f(x,y,z)\,dx\wedge dy\right].$$

Examples for 3-dimensional balls and spheres were also considered in [10].

For other Hermite-Hadamard type integral inequalities for multiple integrals, see [2]-[8], [11]-[15] and [17]-[19].

Motivated by the above results, in this paper, by the use of Divergence Theorem, we establish some integral inequalities of Hermite-Hadamard type for convex functions of several variables defined on closed and bounded convex bodies in the Euclidean space $\mathbb{R}^n$ for any $n\geq 2$.

2. Some Preliminary Facts

Let B be a bounded open subset of $\mathbb{R}^n$ ($n\geq 2$) with smooth (or piecewise smooth) boundary ∂B. Let $F=(F_1,...,F_n)$ be a smooth vector field defined in $\mathbb{R}^n$, or at least in $B\cup\partial B$. Let $\mathbf{n}$ be the unit outward-pointing normal of ∂B. Then the *Divergence Theorem* states, see for instance [16]:

$$\int_B \operatorname{div} F dV=\int_{\partial B} F\cdot n dA, \tag{2.1}$$

where

$$\operatorname{div} F=\nabla\cdot F=\sum_{k=1}^n \frac{\partial F_i}{\partial x_i},$$

dV is the element of volume in $\mathbb{R}^n$ and dA is the element of surface area on ∂B.

If $\mathbf{n}=(\mathbf{n}_1,...,\mathbf{n}_n)$, $x=(x_1,...,x_n)\in B$ and use the notation dx for dV we can write (2.1) more explicitly as

$$\sum_{k=1}^n\int_B \frac{\partial F_k(x)}{\partial x_k}dx=\sum_{k=1}^n\int_{\partial B}F_k(x)\,n_k(x)\,dA. \tag{2.2}$$

By taking the real and imaginary part, we can extend the above equality for complex valued functions F_k, $k\in\{1,...,n\}$ defined on B.

If $n=2$, the normal is obtained by rotating the tangent vector through 90° (in the correct direction so that it points out). The quantity tds can be written (dx_1,dx_2) along the surface, so that

$$ndA:=nds=(dx_2,-dx_1)$$

Here t is the tangent vector along the boundary curve and ds is the element of arc-length.

From (2.2) we get for $B\subset\mathbb{R}^2$ that

$$\int_B \frac{\partial F_1(x_1,x_2)}{\partial x_1}dx_1dx_2+\int_B\frac{\partial F_2(x_1,x_2)}{\partial x_2}dx_1dx_2 \tag{2.3}$$
$$=\int_{\partial B}F_1(x_1,x_2)\,dx_2-\int_{\partial B}F_2({}_1,x_2)\,dx_1,$$

which is *Green's theorem* in plane.

If $n=3$ and if ∂B is described as a level-set of a function of 3 variables i.e. $\partial B = \left\{x_1,x_2,x_3 \in \mathbb{R}^3 \mid G(x_1,x_2,x_3)=0\right\}$, then a vector pointing in the direction of $\mathbf{n}$ is $\operatorname{grad} G$. We shall use the case where $G(x_1,x_2,x_3)=x_3-g(x_1,x_2)$, $(x_1,x_2)\in D$, a domain in $\mathbb{R}^2$ for some differentiable function g on D and B corresponds to the inequality $x_3<g(x_1,x_2)$, namely

$$B=\left\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_3<g(x_1,x_2)\right\}.$$

Then

$$\mathbf{n}=\frac{(-g_{x_1},-g_{x_2},1)}{\left(1+g_{x_1}^2+g_{x_2}^2\right)^{1/2}},\ dA=\left(1+g_{x_1}^2+g_{x_2}^2\right)^{1/2}dx_1dx_2$$

and

$$\mathbf{n}dA=(-g_{x_1},-g_{x_2},1)\,dx_1dx_2.$$

From (2.2) we get

$$\begin{aligned}
\int_B\left(\frac{\partial F_1(x_1,x_2,x_3)}{\partial x_1}+\frac{\partial F_2(x_1,x_2,x_3)}{\partial x_2}+\frac{\partial F_3(x_1,x_2,x_3)}{\partial x_3}\right)dx_1dx_2dx_3 \\
=-\int_D F_1(x_1,x_2,g(x_1,x_2))\,g_{x_1}(x_1,x_2)\,dx_1dx_2 \\
-\int_D F_1(x_1,x_2,g(x_1,x_2))\,g_{x_2}(x_1,x_2)dx_1dx_2 \\
+\int_D F_3(x_1,x_2,g(x_1,x_2))\,dx_1dx_2
\end{aligned} \tag{2.4}$$

which is the *Gauss-Ostrogradsky theorem* in space.

Following Apostol [1], we can also consider a surface described by the vector equation

$$r(u,v)=x_1(u,v)\overrightarrow{i}+x_2(u,v)\overrightarrow{j}+x_3(u,v)\overrightarrow{k} \tag{2.5}$$

where $(u,v)\in[a,b]\times[c,d]$.

If x_1, x_2, x_3 are differentiable on $[a,b]\times[c,d]$ we consider the two vectors

$$\frac{\partial r}{\partial u}=\frac{\partial x_1}{\partial u}\overrightarrow{i}+\frac{\partial x_2}{\partial u}\overrightarrow{j}+\frac{\partial x_3}{\partial u}\overrightarrow{k}$$

and

$$\frac{\partial r}{\partial v}=\frac{\partial x_1}{\partial v}\overrightarrow{i}+\frac{\partial x_2}{\partial v}\overrightarrow{j}+\frac{\partial x_3}{\partial v}\overrightarrow{k}.$$

The *cross product* of these two vectors $\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}$ will be referred to as the fundamental vector product of the representation r. Its components can be expressed as *Jacobian determinants*. In fact, we have [1, p. 420]

$$\begin{aligned}
\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}&=\begin{vmatrix}\frac{\partial x_2}{\partial u} & \frac{\partial x_3}{\partial u}\\ \frac{\partial x_2}{\partial v} & \frac{\partial x_3}{\partial v}\end{vmatrix}\overrightarrow{i}+\begin{vmatrix}\frac{\partial x_3}{\partial u} & \frac{\partial x_1}{\partial u}\\ \frac{\partial x_3}{\partial v} & \frac{\partial x_1}{\partial v}\end{vmatrix}\overrightarrow{j}+\begin{vmatrix}\frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u}\\ \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v}\end{vmatrix}\overrightarrow{k} \\
&=\frac{\partial(x_2,x_3)}{\partial(u,v)}\overrightarrow{i}+\frac{\partial(x_3,x_1)}{\partial(u,v)}\overrightarrow{j}+\frac{\partial(x_1,x_2)}{\partial(u,v)}\overrightarrow{k}.
\end{aligned} \tag{2.6}$$

Let $\partial B = r(T)$ be a parametric surface described by a vector-valued function r defined on the box $T = [a,b] \times [c,d]$. The area of ∂B denoted $A_{\partial B}$ is defined by the double integral [1, p. 424-425]

$$A_{\partial B} = \int_a^b \int_c^d \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| dudv \tag{2.7}$$
$$= \int_a^b \int_c^d \sqrt{\left(\frac{\partial (x_2, x_3)}{\partial (u,v)}\right)^2 + \left(\frac{\partial (x_3, x_1)}{\partial (u,v)}\right)^2 + \left(\frac{\partial (x_1, x_2)}{\partial (u,v)}\right)^2} dudv.$$

We define surface integrals in terms of a parametric representation for the surface. One can prove that under certain general conditions the value of the integral is independent of the representation.

Let $\partial B = r(T)$ be a parametric surface described by a vector-valued differentiable function r defined on the box $T = [a,b] \times [c,d]$ and let $f : \partial B \to \mathbb{C}$ defined and bounded on ∂B. The surface integral of f over ∂B is defined by [1, p. 430]

$$\int\int_{\partial B} f dA = \int_a^b \int_c^d f(x_1, x_2, x_3) \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| dudv \tag{2.8}$$
$$= \int_a^b \int_c^d f(x_1(u,v), x_2(u,v), x_3(u,v))$$
$$\times \sqrt{\left(\frac{\partial (x_2, x_3)}{\partial (u,v)}\right)^2 + \left(\frac{\partial (x_3, x_1)}{\partial (u,v)}\right)^2 + \left(\frac{\partial (x_1, x_2)}{\partial (u,v)}\right)^2} dudv.$$

If $\partial B = r(T)$ is a parametric surface, the fundamental vector product $N = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ is normal to ∂B at each regular point of the surface. At each such point there are two unit normals, a unit normal $\mathbf{n}_1$, which has the same direction as N, and a unit normal $\mathbf{n}_2$ which has the opposite direction. Thus

$$\mathbf{n}_1 = \frac{N}{\|N\|} \text{ and } \mathbf{n}_2 = -\mathbf{n}_1.$$

Let $\mathbf{n}$ be one of the two normals $\mathbf{n}_1$ or $\mathbf{n}_2$. Let also F be a vector field defined on ∂B and assume that the surface integral,

$$\int\int_{\partial B} (F \cdot \mathbf{n})\, dA,$$

called the flux surface integral, exists. Here $F \cdot \mathbf{n}$ is the dot or inner product.

We can write [1, p. 434]

$$\int\int_{\partial B} (F \cdot \mathbf{n})\, dA = \pm \int_a^b \int_c^d F(r(u,v)) \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right) dudv$$

where the sign " $+$ " is used if $\mathbf{n} = \mathbf{n}_1$ and the " $-$ " sign is used if $\mathbf{n} = \mathbf{n}_2$.

If

$$F(x_1, x_2, x_3) = F_1(x_1, x_2, x_3) \overrightarrow{i} + F_2(x_1, x_2, x_3) \overrightarrow{j} + F_3(x_1, x_2, x_3) \overrightarrow{k}$$

and

$$r(u,v) = x_1(u,v) \overrightarrow{i} + x_2(u,v) \overrightarrow{j} + x_3(u,v) \overrightarrow{k} \text{ where } (u,v) \in [a,b] \times [c,d]$$

then the flux surface integral for $\mathbf{n}=\mathbf{n}_1$ can be explicitly calculated as [1, p. 435]

$$\begin{aligned}(2.9)\quad \int\int_{\partial B}(F\cdot\mathbf{n})\,dA &= \int_a^b\int_c^d F_1(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_2,x_3)}{\partial(u,v)}dudv\\ &\quad+\int_a^b\int_c^d F_2(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_3,x_1)}{\partial(u,v)}dudv\\ &\quad+\int_a^b\int_c^d F_3(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_1,x_2)}{\partial(u,v)}dudv.\end{aligned}$$

The sum of the double integrals on the right is often written more briefly as [1, p. 435]

$$\begin{aligned}\int\int_{\partial B}F_1(x_1,x_2,x_3)\,dx_2\wedge dx_3+\int\int_{\partial B}F_2(x_1,x_2,x_3)\,dx_3\wedge dx_1&\\ +\int\int_{\partial B}F_3(x_1,x_2,x_3)\,dx_1\wedge dx_2&\end{aligned}$$

Let $B\subset\mathbb{R}^3$ be a solid in 3-space bounded by an orientable closed surface ∂B, and let $\mathbf{n}$ be the unit outer normal to ∂B. If F is a continuously differentiable vector field defined on B, we have the *Gauss-Ostrogradsky identity*

$$\text{(GO)}\qquad \iiint_B(\operatorname{div}F)\,dV=\int\int_{\partial B}(F\cdot\mathbf{n})\,dA.$$

If we express

$$F(x_1,x_2,x_3)=F_1(x_1,x_2,x_3)\overrightarrow{i}+F_2(x_1,x_2,x_3)\overrightarrow{j}+F_3(x_1,x_2,x_3)\overrightarrow{k},$$

then (2.4) can be written as

$$\begin{aligned}(2.10)\quad &\iiint_B\left(\frac{\partial F_1(x_1,x_2,x_3)}{\partial x_1}+\frac{\partial F_2(x_1,x_2,x_3)}{\partial x_2}+\frac{\partial F_3(x_1,x_2,x_3)}{\partial x_3}\right)dx_1dx_2dx_3\\ &=\int\int_{\partial B}F_1(x_1,x_2,x_3)\,dx_2\wedge dx_3+\int\int_{\partial B}F_2(x_1,x_2,x_3)\,dx_3\wedge dx_1\\ &\qquad+\int\int_{\partial B}F_3(x_1,x_2,x_3)\,dx_1\wedge dx_2.\end{aligned}$$

3. General Identities

We have the following identity of interest:

Lemma 1. *Let B be a bounded open subset of $\mathbb{R}^n$ $(n\geq 2)$ with smooth (or piecewise smooth) boundary ∂B. Let f be a continuously differentiable function defined in $\mathbb{R}^n$, or at least in $B\cup\partial B$ and with complex values. If $\alpha_k,\ \beta_k\in\mathbb{C}$ for $k\in\{1,...,n\}$ with $\sum_{k=1}^n\alpha_k=1$, then*

$$\begin{aligned}(3.1)\quad \int_B f(x)\,dx&=\sum_{k=1}^n\int_B(\beta_k-\alpha_kx_k)\frac{\partial f(x)}{\partial x_k}dx\\ &\quad+\sum_{k=1}^n\int_{\partial B}(\alpha_kx_k-\beta_k)\,f(x)\,n_k(x)\,dA.\end{aligned}$$

We also have

$$\int_B f(x)\,dx = \frac{1}{n}\sum_{k=1}^{n}\int_B (\gamma_k - x_k)\frac{\partial f(x)}{\partial x_k}dx + \frac{1}{n}\sum_{k=1}^{n}\int_{\partial B}(x_k-\gamma_k) f(x)\, n_k(x)\, dA \tag{3.2}$$

for all $\gamma_k \in \mathbb{C}$ *where* $k \in \{1, ..., n\}$.

Proof. Let $x = (x_1, ..., x_n) \in B$. We consider

$$F_k(x) = (\alpha_k x_k - \beta_k) f(x), \ k \in \{1, ..., n\}$$

and take the partial derivatives $\frac{\partial F_k(x)}{\partial x_k}$ to get

$$\frac{\partial F_k(x)}{\partial x_k} = \alpha_k f(x) + (\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k}, \ k \in \{1, ..., n\}.$$

If we sum this equality over k from 1 to n we get

$$\begin{aligned}\sum_{k=1}^{n}\frac{\partial F_k(x)}{\partial x_k} &= \sum_{k=1}^{n}\alpha_k f(x) + \sum_{k=1}^{n}(\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k}\\ &= f(x) + \sum_{k=1}^{n}(\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k}\end{aligned} \tag{3.3}$$

for all $x = (x_1, ..., x_n) \in B$.

Now, if we take the integral in the equality (3.3) over $(x_1, ..., x_n) \in B$ we get

$$\int_B\left(\sum_{k=1}^{n}\frac{\partial F_k(x)}{\partial x_k}\right)dx = \int_B f(x)\,dx + \sum_{k=1}^{n}\int_B\left[(\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k}\right]dx. \tag{3.4}$$

By the Divergence Theorem (2.2) we also have

$$\int_B\left(\sum_{k=1}^{n}\frac{\partial F_k(x)}{\partial x_k}\right)dx = \sum_{k=1}^{n}\int_{\partial B}(\alpha_k x_k - \beta_k) f(x)\, n_k(x)\, dA \tag{3.5}$$

and by making use of (3.4) and (3.5) we get

$$\int_B f(x)\,dx + \sum_{k=1}^{n}\int_B\left[(\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k}\right]dx = \sum_{k=1}^{n}\int_{\partial B}(\alpha_k x_k - \beta_k) f(x)\, n_k(x)\, dA$$

which gives the desired representation (3.1).

The identity (3.2) follows by (3.1) for $\alpha_k = \frac{1}{n}$ and $\beta_k = \frac{1}{n}\gamma_k$, $k \in \{1, ..., n\}$. $\square$

For the body B we consider the coordinates for the *centre of gravity*

$$G_B := G\left(\overline{x_{B,1}}, ..., \overline{x_{B,n}}\right)$$

defined by

$$\overline{x_{B,k}} := \frac{1}{V(B)}\int_B x_k dx, \ k \in \{1, ..., n\},$$

where

$$V(B) := \int_B x dx$$

is the volume of B.

Corollary 3. *With the assumptions of Lemma 1 we have*

$$\int_B f(x) dx = \sum_{k=1}^{n} \int_B \alpha_k \left(\overline{x_{B,k}} - x_k\right) \frac{\partial f(x)}{\partial x_k} dx + \sum_{k=1}^{n} \int_{\partial B} \alpha_k \left(x_k - \overline{x_{B,k}}\right) f(x) n_k(x) dA \tag{3.6}$$

and, in particular,

$$\int_B f(x) dx = \frac{1}{n} \sum_{k=1}^{n} \int_B \left(\overline{x_{B,k}} - x_k\right) \frac{\partial f(x)}{\partial x_k} dx + \frac{1}{n} \sum_{k=1}^{n} \int_{\partial B} \left(x_k - \overline{x_{B,k}}\right) f(x) n_k(x) dA. \tag{3.7}$$

The proof follows by (3.1) on taking $\beta_k = \alpha_k \overline{x_{B,k}}$, $k \in \{1, ..., n\}$.

For a function f as in Lemma 1 above, we define the points

$$x_{B,\partial f,k} := \frac{\int_B x_k \frac{\partial f(x)}{\partial x_k} dx}{\int_B \frac{\partial f(x)}{\partial x_k} dx}, \ k \in \{1, ..., n\},$$

provided that all denominators are not zero.

Corollary 4. *With the assumptions of Lemma 1 we have*

$$\int_B f(x) dx = \sum_{k=1}^{n} \int_{\partial B} \alpha_k \left(x_k - x_{B,\partial f,k}\right) f(x) n_k(x) dA \tag{3.8}$$

and, in particular,

$$\int_B f(x) dx = \frac{1}{n} \sum_{k=1}^{n} \int_{\partial B} \left(x_k - x_{B,\partial f,k}\right) f(x) n_k(x) dA. \tag{3.9}$$

The proof follows by (3.1) on taking $\beta_k = \alpha_k x_{B,\partial f,k}$, $k \in \{1, ..., n\}$ and observing that

$$\sum_{k=1}^{n} \int_B \left(\beta_k - \alpha_k x_k\right) \frac{\partial f(x)}{\partial x_k} dx = \sum_{k=1}^{n} \alpha_k \int_B \left(x_{B,\partial f,k} - x_k\right) \frac{\partial f(x)}{\partial x_k} dx = 0.$$

For a function f as in Lemma 1 above, we define the points

$$x_{\partial B,f,k} := \frac{\int_{\partial B} x_k f(x) n_k(x) dA}{\int_{\partial B} f(x) n_k(x) dA}, \ k \in \{1, ..., n\}$$

provided that all denominators are not zero.

Corollary 5. *With the assumptions of Lemma 1 we have*

$$\int_B f(x) dx = \sum_{k=1}^{n} \int_B \alpha_k \left(x_{\partial B,f,k} - x_k\right) \frac{\partial f(x)}{\partial x_k} dx \tag{3.10}$$

and, in particular,

$$\int_B f(x)\,dx = \frac{1}{n}\sum_{k=1}^{n}\int_B \left(x_{\partial B,f,k} - x_k\right)\frac{\partial f(x)}{\partial x_k}dx. \tag{3.11}$$

The proof follows by (3.1) on taking $\beta_k = \alpha_k x_{\partial B,f,k}$, $k \in \{1,...,n\}$ and observing that

$$\sum_{k=1}^{n}\int_{\partial B}(\alpha_k x_k - \beta_k) f(x) n_k(x)\,dA = \sum_{k=1}^{n}\alpha_k \int_{\partial B}\left(x_k - x_{\partial B,f,k}\right) f(x) n_k(x)\,dA = 0.$$

4. Inequalities for Convex Functions

We have the following result that generalizes the inequalities from Introduction:

Theorem 3. *Let B be a bounded convex and closed subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary ∂B. Let f be a continuously differentiable convex function defined on an open neighborhood of B, then for all $y \in B$ we have*

$$\begin{aligned} f(y) + \sum_{k=1}^{n}\frac{\partial f(y)}{\partial x_k}\left(\overline{x_{B,k}} - y_k\right) &\leq \frac{1}{V(B)}\int_B f(x)\,dx \\ &\leq \frac{1}{n+1}f(y) + \frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k - y_k) f(x) n_k(x)\,dA. \end{aligned} \tag{4.1}$$

In particular,

$$\begin{aligned} f(G_B) &\leq \frac{1}{V(B)}\int_B f(x)\,dx \\ &\leq \frac{1}{n+1}f(G_B) + \frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\left(x_k - \overline{x_{B,k}}\right) f(x) n_k(x)\,dA, \end{aligned} \tag{4.2}$$

where $G_B \in B$ is the centre of gravity for B, i.e., $G_B := G\left(\overline{x_{B,1}},...,\overline{x_{B,n}}\right)$.

Proof. Since $f : B \to \mathbb{R}$ is a differentiable convex function on B, then for all $x = (x_1,...,x_n)$, $y = (y_1,...,y_n) \in B$ we have the *gradient inequalities*

$$\sum_{k=1}^{n}\frac{\partial f(y)}{\partial x_k}(x_k - y_k) \leq f(x) - f(y) \leq \sum_{k=1}^{n}\frac{\partial f(x)}{\partial x_k}(x_k - y_k). \tag{4.3}$$

Taking the integral mean $\frac{1}{V(B)}\int_B$ in (4.3) over the variable $x \in B$ we deduce

$$\begin{aligned} \sum_{k=1}^{n}\frac{\partial f(y)}{\partial x_k}\left(\frac{1}{V(B)}\int_B x_k dx - y_k\right) &\leq \frac{1}{V(B)}\int_B f(x)\,dx - f(y) \\ &\leq \sum_{k=1}^{n}\frac{1}{V(B)}\int_B \frac{\partial f(x)}{\partial x_k}(x_k - y_k)\,dx. \end{aligned} \tag{4.4}$$

From the equality (3.2) we get for $\gamma_k = y_k$, $k \in \{1, ..., n\}$ that

$$\int_B f(x)\,dx = \frac{1}{n}\sum_{k=1}^{n}\int_B (y_k - x_k)\frac{\partial f(x)}{\partial x_k}dx + \frac{1}{n}\sum_{k=1}^{n}\int_{\partial B}(x_k - y_k) f(x)\, n_k(x)\,dA$$

namely

$$\sum_{k=1}^{n}\int_B (x_k - y_k)\frac{\partial f(x)}{\partial x_k}dx = \sum_{k=1}^{n}\int_{\partial B}(x_k - y_k) f(x)\, n_k(x)\,dA - n\int_B f(x)\,dx.$$

Since

$$\sum_{k=1}^{n}\frac{\partial f(y)}{\partial x_k}\left(\frac{1}{V(B)}\int_B x_k dx - y_k\right) = \sum_{k=1}^{n}\frac{\partial f(y)}{\partial x_k}\left(\overline{x_{B,k}} - y_k\right)$$

and

$$\sum_{k=1}^{n}\frac{1}{V(B)}\int_B \frac{\partial f(x)}{\partial x_k}(x_k - y_k)\,dx = \sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k - y_k) f(x)\, n_k(x)\,dA - n\frac{1}{V(B)}\int_B f(x)\,dx,$$

hence by (4.4) we get

$$\sum_{k=1}^{n}\frac{\partial f(y)}{\partial x_k}\left(\overline{x_{B,k}} - y_k\right) \le \frac{1}{V(B)}\int_B f(x)\,dx - f(y) \le \sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k - y_k) f(x)\, n_k(x)\,dA - n\frac{1}{V(B)}\int_B f(x)\,dx. \tag{4.5}$$

Now, from the first inequality in (4.5) we get the first inequality in (4.1).

The second inequality in (4.5) can be written as

$$\frac{1}{V(B)}\int_B f(x)\,dx + \frac{n}{V(B)}\int_B f(x)\,dx \le f(y) + \sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k - y_k) f(x)\, n_k(x)\,dA,$$

which is equivalent to the second part of (4.1). □

Corollary 6. *With the assumptions of Theorem 3 we have*

$$\frac{1}{V(B)}\int_B f(x)\,dx \le \frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\left(x_k - \overline{x_{B,k}}\right) f(x)\, n_k(x)\,dA. \tag{4.6}$$

Proof. From (4.2) we have

$$\frac{1}{V(B)}\int_B f(x)\,dx \le \frac{1}{n+1} f(G_B) + \frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\left(x_k - \overline{x_{B,k}}\right) f(x)\, n_k(x)\,dA \tag{4.7}$$

and since

$$f(G_B) \le \frac{1}{V(B)} \int_B f(x)\,dx,$$

hence

$$\frac{1}{n+1} f(G_B) + \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - \overline{x_{B,k}}) f(x) n_k(x)\,dA \tag{4.8}$$
$$\le \frac{1}{n+1} \frac{1}{V(B)} \int_B f(x)\,dx + \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - \overline{x_{B,k}}) f(x) n_k(x)\,dA.$$

By (4.7) and (4.8) we get

$$\frac{1}{V(B)} \int_B f(x)\,dx \le \frac{1}{n+1} \frac{1}{V(B)} \int_B f(x)\,dx$$
$$+ \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - \overline{x_{B,k}}) f(x) n_k(x)\,dA$$

that is equivalent to (4.6). □

Corollary 7. *With the assumptions of Theorem 3 and if the vector* $(x_{\partial B,f,1}, ..., x_{\partial B,f,n}) \in B$, *then*

$$f(x_{\partial B,f,1}, ..., x_{\partial B,f,n}) + \sum_{k=1}^{n} \frac{\partial f(y)}{\partial x_k} (\overline{x_{B,k}} - x_{\partial B,f,k}) \tag{4.9}$$
$$\le \frac{1}{V(B)} \int_B f(x)\,dx \le \frac{1}{n+1} f(x_{\partial B,f,1}, ..., x_{\partial B,f,n}).$$

The proof follows by (4.1) observing that

$$\sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - x_{\partial B,f,k}) f(x) n_k(x)\,dA = 0.$$

We also have the following result:

Corollary 8. *With the assumptions of Theorem 3 and if we define*

$$\overline{s_{\partial B,k}} := \frac{1}{A(\partial B)} \int_{\partial B} y_k dS, \;\; k \in \{1, ..., n\}, \tag{4.10}$$

where $A(\partial B)$ *is the area of the surface* ∂B, *then we have the inequality*

$$\frac{1}{A(\partial B)} \int_{\partial B} f(y)\,dS + \sum_{k=1}^{n} \frac{1}{A(\partial B)} \int_{\partial B} \frac{\partial f(y)}{\partial x_k} (\overline{x_{B,k}} - y_k)\,dS \tag{4.11}$$
$$\le \frac{1}{V(B)} \int_B f(x)\,dx$$
$$\le \frac{1}{n+1} \frac{1}{A(\partial B)} \int_{\partial B} f(y)\,dS + \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - \overline{s_{\partial B,k}}) f(x) n_k(x)\,dA.$$

Proof. If we take the integral mean $\frac{1}{A(\partial B)}\int_{\partial B}(\cdot)\,dS$ over the variable $y\in\partial B$, then we get

$$\begin{aligned}
&\frac{1}{A(\partial B)}\int_{\partial B}f(y)\,dS+\frac{1}{A(\partial B)}\int_{\partial B}\left(\sum_{k=1}^{n}\frac{\partial f(y)}{\partial x_k}\left(\overline{x_{B,k}}-y_k\right)\right)dS \qquad (4.12)\\
&\leq\frac{1}{V(B)}\int_B f(x)\,dx\\
&\leq\frac{1}{n+1}\frac{1}{A(\partial B)}\int_{\partial B}f(y)\,dS\\
&+\frac{1}{n+1}\frac{1}{A(\partial B)}\int_{\partial B}\left(\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k-y_k)\,f(x)\,n_k(x)\,dA\right)dS.
\end{aligned}$$

Now, observe that

$$\frac{1}{A(\partial B)}\int_{\partial B}\left(\sum_{k=1}^{n}\frac{\partial f(y)}{\partial x_k}\left(\overline{x_{B,k}}-y_k\right)\right)dS=\sum_{k=1}^{n}\frac{1}{A(\partial B)}\int_{\partial B}\frac{\partial f(y)}{\partial x_k}\left(\overline{x_{B,k}}-y_k\right)dS$$

and

$$\begin{aligned}
&\frac{1}{A(\partial B)}\int_{\partial B}\left(\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k-y_k)\,f(x)\,n_k(x)\,dA\right)dS\\
&=\sum_{k=1}^{n}\frac{1}{V(B)}\frac{1}{A(\partial B)}\int_{\partial B}\left(\int_{\partial B}(x_k-y_k)\,f(x)\,n_k(x)\,dA\right)dS\\
&=\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\left(x_k-\frac{1}{A(\partial B)}\int_{\partial B}y_k dS\right)f(x)\,n_k(x)\,dA \text{ (by Fubini's theorem)}\\
&=\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\left(x_k-\overline{s_{\partial B,k}}\right)f(x)\,n_k(x)\,dA \text{ (by 4.10).}
\end{aligned}$$

By making use of the inequality (4.12) we then obtain the desired result (4.11). □

Remark 1. *By taking $n=2$ in the above inequalities we recapture some results from* [9] *while for $n=3$ we obtain results from* [10]*. The details are omitted.*

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