OSTROWSKI TYPE INEQUALITIES FOR MULTIPLE INTEGRALS VIA DIVERGENCE THEOREM

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Abstract. In this paper, by the use of Divergence Theorem, we establish some Ostrowski type inequalities for functions of n-variables defined on closed and bounded bodies of the Euclidean space $\mathbb{R}^n$.

1. Introduction

In paper [2], the authors obtained among others the following results concerning the difference between the double integral on the disk and the values in the center or the path integral on the circle:

Theorem 1. *If $f : D(C,R) \to \mathbb{R}$ has continuous partial derivatives on $D(C,R)$, the disk centered in the point $C = (a,b)$ with the radius $R > 0$, and*

$$\begin{aligned} \left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} &:= \sup_{(x,y)\in D(C,R)} \left| \frac{\partial f(x,y)}{\partial x} \right| < \infty, \\ \left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} &:= \sup_{(x,y)\in D(C,R)} \left| \frac{\partial f(x,y)}{\partial y} \right| < \infty; \end{aligned}$$

then

$$\left| f(C) - \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\,dx\,dy \right| \le \frac{4}{3\pi} R \left[\left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} \right]. \tag{1.1}$$

The constant $\frac{4}{3\pi}$ is sharp.

We also have

$$\left| \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\,dx\,dy - \frac{1}{2\pi R} \int_{\sigma(C,R)} f(\gamma)\,dl(\gamma) \right| \le \frac{2R}{3\pi} \left[\left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} \right], \tag{1.2}$$

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where $\sigma(C,R)$ *is the circle centered in* $C=(a,b)$ *with the radius* $R>0$ *and*

$$\left|f(C)-\frac{1}{2\pi R}\int_{\sigma(C,R)} f(\gamma)\,dl(\gamma)\right| \leq \frac{2R}{\pi}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(C,R),\infty}+\left\|\frac{\partial f}{\partial y}\right\|_{D(C,R),\infty}\right]. \tag{1.3}$$

In the same paper [2] the authors also established the following Ostrowski type inequality:

Theorem 2. *If* f *has bounded partial derivatives on* $D(0,1)$*, the unity disk, then*

$$\left|f(u,v)-\frac{1}{\pi}\iint_{D(0,1)} f(x,y)\,dx\,dy\right| \leq \frac{2}{\pi}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(0,1),\infty}\left(u\arcsin u+\frac{1}{3}\sqrt{1-u^2}\left(2+u^2\right)\right) + \left\|\frac{\partial f}{\partial y}\right\|_{D(0,1),\infty}\left(v\arcsin v+\frac{1}{3}\sqrt{1-v^2}\left(2+v^2\right)\right)\right] \tag{1.4}$$

for any $(u,v)\in D(0,1)$*.*

For other Ostrowski type integral inequalities for multiple integrals see [3]-[15].

Motivated by the above results, in this paper, by the use of Divergence Theorem, we establish some Ostrowski type inequalities for functions of n-variables defined on closed and bounded bodies of the Euclidean space $\mathbb{R}^n$.

2. Some Preliminary Facts

Let B be a bounded open subset of $\mathbb{R}^n$ $(n\geq 2)$ with smooth (or piecewise smooth) boundary ∂B. Let $F=(F_1,...,F_n)$ be a smooth vector field defined in $\mathbb{R}^n$, or at least in $B\cup\partial B$. Let $\mathbf{n}$ be the unit outward-pointing normal of ∂B. Then the *Divergence Theorem* states, see for instance [14]:

$$\int_B \operatorname{div} F dV=\int_{\partial B} F\cdot n dA, \tag{2.1}$$

where

$$\operatorname{div} F=\nabla\cdot F=\sum_{k=1}^{n}\frac{\partial F_i}{\partial x_i},$$

dV is the element of volume in $\mathbb{R}^n$ and dA is the element of surface area on ∂B.

If $\mathbf{n}=(\mathbf{n}_1,...,\mathbf{n}_n)$, $x=(x_1,...,x_n)\in B$ and use the notation dx for dV we can write (2.1) more explicitly as

$$\sum_{k=1}^{n}\int_B \frac{\partial F_k(x)}{\partial x_k}dx=\sum_{k=1}^{n}\int_{\partial B} F_k(x)\,n_k(x)\,dA. \tag{2.2}$$

By taking the real and imaginary part, we can extend the above equality for complex valued functions F_k, $k\in\{1,...,n\}$ defined on B.

If $n = 2$, the normal is obtained by rotating the tangent vector through $90°$ (in the correct direction so that it points out). The quantity tds can be written (dx_1, dx_2) along the surface, so that

$$ndA := nds = (dx_2, -dx_1)$$

Here t is the tangent vector along the boundary curve and ds is the element of arc-length.

From (2.2) we get for $B \subset \mathbb{R}^2$ that

$$\begin{aligned} &\int_B \frac{\partial F_1(x_1,x_2)}{\partial x_1} dx_1 dx_2 + \int_B \frac{\partial F_2(x_1,x_2)}{\partial x_2} dx_1 dx_2 \\ &= \int_{\partial B} F_1(x_1,x_2)\, dx_2 - \int_{\partial B} F_2(x_1,x_2)\, dx_1, \end{aligned} \tag{2.3}$$

which is *Green's theorem* in plane.

If $n = 3$ and if ∂B is described as a level-set of a function of 3 variables i.e. $\partial B = \{x_1, x_2, x_3 \in \mathbb{R}^3 \mid G(x_1,x_2,x_3) = 0\}$, then a vector pointing in the direction of $\mathbf{n}$ is $\operatorname{grad} G$. We shall use the case where $G(x_1,x_2,x_3) = x_3 - g(x_1,x_2)$, $(x_1,x_2) \in D$, a domain in $\mathbb{R}^2$ for some differentiable function g on D and B corresponds to the inequality $x_3 < g(x_1,x_2)$, namely

$$B = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_3 < g(x_1,x_2)\}.$$

Then

$$\mathbf{n} = \frac{(-g_{x_1}, -g_{x_2}, 1)}{\left(1 + g_{x_1}^2 + g_{x_2}^2\right)^{1/2}},\ dA = \left(1 + g_{x_1}^2 + g_{x_2}^2\right)^{1/2} dx_1 dx_2$$

and

$$\mathbf{n}dA = (-g_{x_1}, -g_{x_2}, 1)\, dx_1 dx_2.$$

From (2.2) we get

$$\begin{aligned} &\int_B \left(\frac{\partial F_1(x_1,x_2,x_3)}{\partial x_1} + \frac{\partial F_2(x_1,x_2,x_3)}{\partial x_2} + \frac{\partial F_3(x_1,x_2,x_3)}{\partial x_3} \right) dx_1 dx_2 dx_3 \\ &= -\int_D F_1(x_1,x_2,g(x_1,x_2))\, g_{x_1}(x_1,x_2)\, dx_1 dx_2 \\ &- \int_D F_1(x_1,x_2,g(x_1,x_2))\, g_{x_2}(x_1,x_2) dx_1 dx_2 \\ &+ \int_D F_3(x_1,x_2,g(x_1,x_2))\, dx_1 dx_2 \end{aligned} \tag{2.4}$$

which is the *Gauss-Ostrogradsky theorem* in space.

Following Apostol [1], we can also consider a surface described by the vector equation

$$r(u,v) = x_1(u,v)\, \overrightarrow{i} + x_2(u,v)\, \overrightarrow{j} + x_3(u,v)\, \overrightarrow{k} \tag{2.5}$$

where $(u,v) \in [a,b] \times [c,d]$.

If x_1, x_2, x_3 are differentiable on $[a,b] \times [c,d]$ we consider the two vectors

$$\frac{\partial r}{\partial u} = \frac{\partial x_1}{\partial u}\overrightarrow{i} + \frac{\partial x_2}{\partial u}\overrightarrow{j} + \frac{\partial x_3}{\partial u}\overrightarrow{k}$$

and

$$\frac{\partial r}{\partial v} = \frac{\partial x_1}{\partial v}\overrightarrow{i} + \frac{\partial x_2}{\partial v}\overrightarrow{j} + \frac{\partial x_3}{\partial v}\overrightarrow{k}.$$

The *cross product* of these two vectors $\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}$ will be referred to as the fundamental vector product of the representation r. Its components can be expressed as *Jacobian determinants*. In fact, we have [1, p. 420]

$$\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}=\begin{vmatrix}\frac{\partial x_2}{\partial u} & \frac{\partial x_3}{\partial u}\\ \frac{\partial x_2}{\partial v} & \frac{\partial x_3}{\partial v}\end{vmatrix}\overrightarrow{i}+\begin{vmatrix}\frac{\partial x_3}{\partial u} & \frac{\partial x_1}{\partial u}\\ \frac{\partial x_3}{\partial v} & \frac{\partial x_1}{\partial v}\end{vmatrix}\overrightarrow{j}+\begin{vmatrix}\frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u}\\ \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v}\end{vmatrix}\overrightarrow{k} \tag{2.6}$$

$$=\frac{\partial(x_2,x_3)}{\partial(u,v)}\overrightarrow{i}+\frac{\partial(x_3,x_1)}{\partial(u,v)}\overrightarrow{j}+\frac{\partial(x_1,x_2)}{\partial(u,v)}\overrightarrow{k}.$$

Let $\partial B=r(T)$ be a parametric surface described by a vector-valued function r defined on the box $T=[a,b]\times[c,d]$. The area of ∂B denoted $A_{\partial B}$ is defined by the double integral [1, p. 424-425]

$$A_{\partial B}=\int_a^b\int_c^d\left\|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right\|dudv \tag{2.7}$$

$$=\int_a^b\int_c^d\sqrt{\left(\frac{\partial(x_2,x_3)}{\partial(u,v)}\right)^2+\left(\frac{\partial(x_3,x_1)}{\partial(u,v)}\right)^2+\left(\frac{\partial(x_1,x_2)}{\partial(u,v)}\right)^2}dudv.$$

We define surface integrals in terms of a parametric representation for the surface. One can prove that under certain general conditions the value of the integral is independent of the representation.

Let $\partial B=r(T)$ be a parametric surface described by a vector-valued differentiable function r defined on the box $T=[a,b]\times[c,d]$ and let $f:\partial B\to\mathbb{C}$ defined and bounded on ∂B. The surface integral of f over ∂B is defined by [1, p. 430]

$$\int\int_{\partial B}fdA=\int_a^b\int_c^d f(x_1,x_2,x_3)\left\|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right\|dudv \tag{2.8}$$

$$=\int_a^b\int_c^d f(x_1(u,v),x_2(u,v),x_3(u,v))$$

$$\times\sqrt{\left(\frac{\partial(x_2,x_3)}{\partial(u,v)}\right)^2+\left(\frac{\partial(x_3,x_1)}{\partial(u,v)}\right)^2+\left(\frac{\partial(x_1,x_2)}{\partial(u,v)}\right)^2}dudv.$$

If $\partial B=r(T)$ is a parametric surface, the fundamental vector product $N=\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}$ is normal to ∂B at each regular point of the surface. At each such point there are two unit normals, a unit normal $\mathbf{n}_1$, which has the same direction as N, and a unit normal $\mathbf{n}_2$ which has the opposite direction. Thus

$$\mathbf{n}_1=\frac{N}{\|N\|}\text{ and }\mathbf{n}_2=-\mathbf{n}_1.$$

Let $\mathbf{n}$ be one of the two normals $\mathbf{n}_1$ or $\mathbf{n}_2$. Let also F be a vector field defined on ∂B and assume that the surface integral,

$$\int\int_{\partial B}(F\cdot\mathbf{n})\,dA,$$

called the flux surface integral, exists. Here $F\cdot\mathbf{n}$ is the dot or inner product.

We can write [1, p. 434]

$$\int\int_{\partial B}(F\cdot\mathbf{n})\,dA=\pm\int_a^b\int_c^d F(r(u,v))\cdot\left(\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right)dudv$$

where the sign " $+$ " is used if $\mathbf{n}=\mathbf{n}_1$ and the " $-$ " sign is used if $\mathbf{n}=\mathbf{n}_2$.

If

$$F(x_1,x_2,x_3)=F_1(x_1,x_2,x_3)\overrightarrow{i}+F_2(x_1,x_2,x_3)\overrightarrow{j}+F_3(x_1,x_2,x_3)\overrightarrow{k}$$

and

$$r(u,v)=x_1(u,v)\overrightarrow{i}+x_2(u,v)\overrightarrow{j}+x_3(u,v)\overrightarrow{k}\text{ where }(u,v)\in[a,b]\times[c,d]$$

then the flux surface integral for $\mathbf{n}=\mathbf{n}_1$ can be explicitly calculated as [1, p. 435]

$$\begin{aligned}\int\int_{\partial B}(F\cdot\mathbf{n})\,dA&=\int_a^b\int_c^d F_1(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_2,x_3)}{\partial(u,v)}dudv\\&+\int_a^b\int_c^d F_2(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_3,x_1)}{\partial(u,v)}dudv\\&+\int_a^b\int_c^d F_3(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_1,x_2)}{\partial(u,v)}dudv.\end{aligned}\tag{2.9}$$

The sum of the double integrals on the right is often written more briefly as [1, p. 435]

$$\begin{aligned}&\int\int_{\partial B}F_1(x_1,x_2,x_3)\,dx_2\wedge dx_3+\int\int_{\partial B}F_2(x_1,x_2,x_3)\,dx_3\wedge dx_1\\&+\int\int_{\partial B}F_3(x_1,x_2,x_3)\,dx_1\wedge dx_2\end{aligned}$$

Let $B\subset\mathbb{R}^3$ be a solid in 3-space bounded by an orientable closed surface ∂B, and let $\mathbf{n}$ be the unit outer normal to ∂B. If F is a continuously differentiable vector field defined on B, we have the *Gauss-Ostrogradsky identity*

$$\iiint_B(\operatorname{div}F)\,dV=\int\int_{\partial B}(F\cdot\mathbf{n})\,dA.\tag{GO}$$

If we express

$$F(x_1,x_2,x_3)=F_1(x_1,x_2,x_3)\overrightarrow{i}+F_2(x_1,x_2,x_3)\overrightarrow{j}+F_3(x_1,x_2,x_3)\overrightarrow{k},$$

then (2.4) can be written as

$$\begin{aligned}&\iiint_B\left(\frac{\partial F_1(x_1,x_2,x_3)}{\partial x_1}+\frac{\partial F_2(x_1,x_2,x_3)}{\partial x_2}+\frac{\partial F_3(x_1,x_2,x_3)}{\partial x_3}\right)dx_1dx_2dx_3\\&=\int\int_{\partial B}F_1(x_1,x_2,x_3)\,dx_2\wedge dx_3+\int\int_{\partial B}F_2(x_1,x_2,x_3)\,dx_3\wedge dx_1\\&+\int\int_{\partial B}F_3(x_1,x_2,x_3)\,dx_1\wedge dx_2.\end{aligned}\tag{2.10}$$

3. General Identities

We have the following identity of interest:

Lemma 1. *Let B be a bounded closed subset of $\mathbb{R}^n$ $(n\geq 2)$ with smooth (or piecewise smooth) boundary ∂B. Let f be a continuously differentiable function* *defined in* $\mathbb{R}^n$*, or at least in an open neighborhood of* B *and with complex values. If* $\alpha_k,\ \beta_k \in \mathbb{C}$ *for* $k \in \{1,...,n\}$ *with* $\sum_{k=1}^n \alpha_k = 1$ *and* $y \in B$*, then*

$$\frac{1}{V(B)}\int_B f(x)\,dx = f(y) + \sum_{k=1}^n \frac{1}{V(B)}\int_B (\beta_k - \alpha_k x_k)\frac{\partial f(x)}{\partial x_k}dx + \sum_{k=1}^n \frac{1}{V(B)}\int_{\partial B}(\alpha_k x_k - \beta_k)[f(x)-f(y)]\,n_k(x)\,dA. \tag{3.1}$$

We also have

$$\frac{1}{V(B)}\int_B f(x)\,dx = f(y) + \frac{1}{n}\frac{1}{V(B)}\sum_{k=1}^n \int_B (\gamma_k - x_k)\frac{\partial f(x)}{\partial x_k}dx + \frac{1}{n}\sum_{k=1}^n \frac{1}{V(B)}\int_{\partial B}(x_k - \gamma_k)[f(x)-f(y)]\,n_k(x)\,dA \tag{3.2}$$

for all $\gamma_k \in \mathbb{C}$*, where* $k \in \{1,...,n\}$.

In particular, we have

$$\frac{1}{V(B)}\int_B f(x)\,dx = f(y) + \frac{1}{V(B)}\sum_{k=1}^n \int_B \alpha_k(y_k - x_k)\frac{\partial f(x)}{\partial x_k}dx + \sum_{k=1}^n \frac{1}{V(B)}\int_{\partial B}\alpha_k(x_k - y_k)[f(x)-f(y)]\,n_k(x)\,dA \tag{3.3}$$

and

$$\frac{1}{V(B)}\int_B f(x)\,dx = f(y) + \frac{1}{n}\frac{1}{V(B)}\sum_{k=1}^n \int_B (y_k - x_k)\frac{\partial f(x)}{\partial x_k}dx + \frac{1}{n}\sum_{k=1}^n \frac{1}{V(B)}\int_{\partial B}(x_k - y_k)[f(x)-f(y)]\,n_k(x)\,dA. \tag{3.4}$$

Proof. Let $x=(x_1,...,x_n)\in B$. We consider

$$F_k(x) = (\alpha_k x_k - \beta_k)f(x),\ \ k\in\{1,...,n\}$$

and take the partial derivatives $\frac{\partial F_k(x)}{\partial x_k}$ to get

$$\frac{\partial F_k(x)}{\partial x_k} = \alpha_k f(x) + (\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k},\ \ k\in\{1,...,n\}.$$

If we sum this equality over k from 1 to n we get

$$\sum_{k=1}^n \frac{\partial F_k(x)}{\partial x_k} = \sum_{k=1}^n \alpha_k f(x) + \sum_{k=1}^n (\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k} = f(x) + \sum_{k=1}^n (\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k} \tag{3.5}$$

for all $x=(x_1,...,x_n)\in B$.

Now, if we take the integral in the equality (3.5) over $(x_1,...,x_n)\in B$ we get

$$\int_B\left(\sum_{k=1}^n \frac{\partial F_k(x)}{\partial x_k}\right)dx = \int_B f(x)\,dx + \sum_{k=1}^n\int_B\left[(\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k}\right]dx. \tag{3.6}$$

By the Divergence Theorem (2.2) we also have

$$\int_B \left(\sum_{k=1}^{n} \frac{\partial F_k(x)}{\partial x_k} \right) dx = \sum_{k=1}^{n} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x) \, dA \tag{3.7}$$

and by making use of (3.6) and (3.7) we get

$$\begin{aligned} &\int_B f(x) \, dx + \sum_{k=1}^{n} \int_B \left[(\alpha_k x_k - \beta_k) \frac{\partial f(x)}{\partial x_k} \right] dx \\ &= \sum_{k=1}^{n} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x) \, dA, \end{aligned}$$

which gives, by rearranging the terms and dividing by $V(B)$, that

$$\begin{aligned} \frac{1}{V(B)} \int_B f(x) \, dx &= \sum_{k=1}^{n} \frac{1}{V(B)} \int_B \left[(\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} \right] dx \\ &+ \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x) \, dA. \end{aligned} \tag{3.8}$$

If we write now the equality (3.8) for $f - f(y)$ and take into account that $\frac{\partial [f - f(y)](x)}{\partial x_k} = \frac{\partial f(x)}{\partial x_k}$, $k \in \{1, ..., n\}$ we get the desired identity (3.1).

The identity (3.2) follows by (3.1) for $\alpha_k = \frac{1}{n}$ and $\beta_k = \frac{1}{n}\gamma_k$, $k \in \{1, ..., n\}$.

The other identities go in a similar way. □

For the body B we consider the coordinates for the *centre of gravity*

$$G_B := G\left(\overline{x_{B,1}}, ..., \overline{x_{B,n}}\right)$$

defined by

$$\overline{x_{B,k}} := \frac{1}{V(B)} \int_B x_k dx, \ \ k \in \{1, ..., n\},$$

where

$$V(B) := \int_B dx$$

is the volume of B.

Corollary 1. *With the assumptions of Lemma 1 we have*

$$\begin{aligned} \frac{1}{V(B)} \int_B f(x) \, dx &= f(G_B) + \frac{1}{V(B)} \sum_{k=1}^{n} \int_B \alpha_k \left(\overline{x_{B,k}} - x_k \right) \frac{\partial f(x)}{\partial x_k} dx \\ &+ \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} \alpha_k \left(x_k - \overline{x_{B,k}} \right) [f(x) - f(G_B)] \, n_k(x) \, dA \end{aligned} \tag{3.9}$$

and, in particular,

$$\begin{aligned} \frac{1}{V(B)} \int_B f(x) \, dx &= f(G_B) + \frac{1}{n} \frac{1}{V(B)} \sum_{k=1}^{n} \int_B \left(\overline{x_{B,k}} - x_k \right) \frac{\partial f(x)}{\partial x_k} dx \\ &+ \frac{1}{n} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} \left(x_k - \overline{x_{B,k}} \right) [f(x) - f(G_B)] \, n_k(x) \, dA. \end{aligned} \tag{3.10}$$

The proof follows by (3.1) on taking $\beta_k = \alpha_k \overline{x_{B,k}}$, $k \in \{1, ..., n\}$.

For a function f as in Lemma 1 above, we define the points

$$x_{B,\partial f,k} := \frac{\int_B x_k \frac{\partial f(x)}{\partial x_k} dx}{\int_B \frac{\partial f(x)}{\partial x_k} dx},\ k \in \{1, ..., n\},$$

provided that all denominators are not zero.

Corollary 2. *With the assumptions of Lemma 1 and if* $G_{B,\partial f} := (x_{B,\partial f,1}, ..., x_{B,\partial f,n}) \in B$, *then we have*

$$\frac{1}{V(B)} \int_B f(x)\,dx = f(G_{B,\partial f}) + \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} \alpha_k (x_k - x_{B,\partial f,k}) [f(x) - f(G_{B,\partial f})] n_k(x)\,dA \tag{3.11}$$

and, in particular,

$$\frac{1}{V(B)} \int_B f(x)\,dx = f(G_{B,\partial f}) + \frac{1}{n} \sum_{k=1}^{n} \int_{\partial B} (x_k - x_{B,\partial f,k}) [f(x) - f(G_{B,\partial f})] n_k(x)\,dA. \tag{3.12}$$

The proof follows by (3.1) on taking $\beta_k = \alpha_k x_{B,\partial f,k}$, $k \in \{1, ..., n\}$ and observing that

$$\sum_{k=1}^{n} \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} dx = \sum_{k=1}^{n} \alpha_k \int_B (x_{B,\partial f,k} - x_k) \frac{\partial f(x)}{\partial x_k} dx = 0.$$

For a function f as in Lemma 1 above, we define the points

$$x_{\partial B,f,k} := \frac{\int_{\partial B} x_k f(x) n_k(x)\,dA}{\int_{\partial B} f(x) n_k(x)\,dA},\ k \in \{1, ..., n\}$$

provided that all denominators are not zero.

Corollary 3. *With the assumptions of Lemma 1 and if* $G_{\partial B,f} := (x_{\partial B,f,1}, ..., x_{\partial B,f,n}) \in B$, *then we have*

$$\frac{1}{V(B)} \int_B f(x)\,dx = f(G_{\partial B,f}) + \sum_{k=1}^{n} \frac{1}{V(B)} \int_B \alpha_k (x_{\partial B,f,k} - x_k) \frac{\partial f(x)}{\partial x_k} dx \tag{3.13}$$

and, in particular,

$$\frac{1}{V(B)} \int_B f(x)\,dx = f(G_{\partial B,f}) + \frac{1}{n} \sum_{k=1}^{n} \frac{1}{V(B)} \int_B (x_{\partial B,f,k} - x_k) \frac{\partial f(x)}{\partial x_k} dx. \tag{3.14}$$

The proof follows by (3.1) on taking $\beta_k = \alpha_k x_{\partial B,f,k}$, $k \in \{1, ..., n\}$ and observing that

$$\sum_{k=1}^{n} \alpha_k \int_{\partial B} (x_k - x_{\partial B,f,k}) f(x) n_k(x)\,dA = 0.$$

4. Inequalities for Lipschitzian Functions

Let $y \in B$. We assume that the function $f : B \to \mathbb{C}$ is *surface Lipschitzian in* y if there exists the constants (depending on y) $L_1, ..., L_n > 0$ such that

$$|f(x) - f(y)| \leq \sum_{j=1}^{n} L_j |x_j - y_j| \tag{4.1}$$

for all $x \in \partial B$.

We observe that, if f is differentiable on an open neighborhood of B, a convex subset of $\mathbb{R}^n$, and has bounded partial derivatives on B, then we have

$$|f(x) - f(y)| \leq \sum_{j=1}^{n} \left\| \frac{\partial f}{\partial x_j} \right\|_{B,\infty} |x_j - y_j| \tag{4.2}$$

for all $x, y \in B$, where

$$\left\| \frac{\partial f}{\partial x_j} \right\|_{B,\infty} := \sup_{x \in B} \left| \frac{f(x)}{\partial x_j} \right| < \infty.$$

Indeed, by making use of the *Taylor's representation theorem* we have

$$f(x) = f(y) + \sum_{j=1}^{n} (x_j - y_j) \int_0^1 \frac{\partial f}{\partial x_j} [tx + (1-t) y] \, dt$$

for all $x, y \in B$, which implies that

$$\begin{aligned} |f(x) - f(y)| &\leq \sum_{j=1}^{n} |x_j - y_j| \int_0^1 \left| \frac{\partial f}{\partial x_j} [tx + (1-t) y] \right| dt \\ &\leq \sum_{j=1}^{n} \left\| \frac{\partial f}{\partial x_j} \right\|_{B,\infty} |x_j - y_j| \end{aligned}$$

for all $x, y \in B$.

Theorem 3. *Let B be a bounded closed subset of $\mathbb{R}^n$ ($n \geq 2$) with smooth (or piecewise smooth) boundary ∂B, $\alpha_k \in \mathbb{C}$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} \alpha_k = 1$ and $y \in B$. Let f be a continuously differentiable function defined in $\mathbb{R}^n$, or at least in an open neighborhood of B with complex values and such that the surface Lipschitz condition in y described by (4.1) holds, then*

$$\begin{aligned} &\left| \frac{1}{V(B)} \int_B f(x) \, dx - f(y) - \frac{1}{V(B)} \sum_{k=1}^{n} \int_B \alpha_k (y_k - x_k) \frac{\partial f(x)}{\partial x_k} dx \right| \\ &\leq \sum_{k=1}^{n} L_k |\alpha_k| \frac{1}{V(B)} \int_{\partial B} (x_k - y_k)^2 |n_k(x)| \, dA \\ &\quad + \sum_{1 \leq k \neq j \leq n} L_j |\alpha_k| \frac{1}{V(B)} \int_{\partial B} |x_k - y_k| |x_j - y_j| |n_k(x)| \, dA. \end{aligned} \tag{4.3}$$

In particular,

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx - f(y) - \frac{1}{n}\frac{1}{V(B)}\sum_{k=1}^{n}\int_B (y_k - x_k)\frac{\partial f(x)}{\partial x_k}dx\right| \tag{4.4}$$
$$\leq \frac{1}{n}\sum_{k=1}^{n} L_k \frac{1}{V(B)}\int_{\partial B}(x_k - y_k)^2 |n_k(x)|\,dA$$
$$+\frac{1}{n}\sum_{1\leq k\neq j\leq n} L_j \frac{1}{V(B)}\int_{\partial B}|x_k - y_k|\,|x_j - y_j|\,|n_k(x)|\,dA.$$

Proof. From the representation (3.3) we have

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx - f(y) - \frac{1}{V(B)}\sum_{k=1}^{n}\int_B \alpha_k (y_k - x_k)\frac{\partial f(x)}{\partial x_k}dx\right| \tag{4.5}$$
$$\leq \left|\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\alpha_k (x_k - y_k)[f(x) - f(y)]\,n_k(x)\,dA\right|$$
$$\leq \sum_{k=1}^{n}\frac{1}{V(B)}\left|\int_{\partial B}\alpha_k (x_k - y_k)[f(x) - f(y)]\,n_k(x)\,dA\right|$$
$$\leq \sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}|\alpha_k (x_k - y_k)[f(x) - f(y)]\,n_k(x)|\,dA$$
$$= \sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}|\alpha_k|\,|x_k - y_k|\,|f(x) - f(y)|\,|n_k(x)|\,dA =: M$$

Using the condition (4.1) we get

$$\int_{\partial B}|\alpha_k|\,|x_k - y_k|\,|f(x) - f(y)|\,|n_k(x)|\,dA$$
$$\leq \int_{\partial B}|\alpha_k|\,|x_k - y_k|\sum_{j=1}^{n} L_j |x_j - y_j|\,|n_k(x)|\,dA$$
$$= \sum_{j=1}^{n} L_j \int_{\partial B}|\alpha_k|\,|x_k - y_k|\,|x_j - y_j|\,|n_k(x)|\,dA,$$

which implies that

$$M \leq \sum_{k=1}^{n}\frac{1}{V(B)}\sum_{j=1}^{n} L_j \int_{\partial B}|\alpha_k|\,|x_k - y_k|\,|x_j - y_j|\,|n_k(x)|\,dA$$
$$= \sum_{k=1}^{n}\sum_{j=1}^{n} L_j |\alpha_k| \frac{1}{V(B)}\int_{\partial B}|x_k - y_k|\,|x_j - y_j|\,|n_k(x)|\,dA$$
$$= \sum_{k=1}^{n} L_k |\alpha_k| \frac{1}{V(B)}\int_{\partial B}(x_k - y_k)^2 |n_k(x)|\,dA$$
$$+ \sum_{1\leq k\neq j\leq n} L_j |\alpha_k| \frac{1}{V(B)}\int_{\partial B}|x_k - y_k|\,|x_j - y_j|\,|n_k(x)|\,dA$$

and by (4.5) we get the desired result (4.3). □

A more practical result is incorporated in the following:

Corollary 4. *Let B be a bounded closed convex subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary ∂B and $\alpha_k \in \mathbb{C}$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^n \alpha_k = 1$. If f is differentiable on an open neighborhood of B and has bounded partial derivatives on B, then for all $y \in B$ we have*

$$\left| \frac{1}{V(B)} \int_B f(x)\, dx - f(y) - \frac{1}{V(B)} \sum_{k=1}^n \int_B \alpha_k (y_k - x_k) \frac{\partial f(x)}{\partial x_k} dx \right| \tag{4.6}$$
$$\leq \sum_{k=1}^n \left\| \frac{\partial f}{\partial x_k} \right\|_{B,\infty} |\alpha_k| \frac{1}{V(B)} \int_{\partial B} (x_k - y_k)^2 |n_k(x)|\, dA$$
$$+ \sum_{1 \leq k \neq j \leq n} \left\| \frac{\partial f}{\partial x_j} \right\|_{B,\infty} |\alpha_k| \frac{1}{V(B)} \int_{\partial B} |x_k - y_k| |x_j - y_j| |n_k(x)|\, dA.$$

In particular,

$$\left| \frac{1}{V(B)} \int_B f(x)\, dx - f(y) - \frac{1}{n} \frac{1}{V(B)} \sum_{k=1}^n \int_B (y_k - x_k) \frac{\partial f(x)}{\partial x_k} dx \right| \tag{4.7}$$
$$\leq \frac{1}{n} \sum_{k=1}^n \left\| \frac{\partial f}{\partial x_k} \right\|_{B,\infty} \frac{1}{V(B)} \int_{\partial B} (x_k - y_k)^2 |n_k(x)|\, dA$$
$$+ \frac{1}{n} \sum_{1 \leq k \neq j \leq n} \left\| \frac{\partial f}{\partial x_j} \right\|_{B,\infty} \frac{1}{V(B)} \int_{\partial B} |x_k - y_k| |x_j - y_j| |n_k(x)|\, dA.$$

Remark 1. *If we take $y = G_B$ in Corollary 4, then we get*

$$\left| \frac{1}{V(B)} \int_B f(x)\, dx - f(G_B) - \frac{1}{V(B)} \sum_{k=1}^n \int_B \alpha_k (\overline{x_{B,k}} - x_k) \frac{\partial f(x)}{\partial x_k} dx \right| \tag{4.8}$$
$$\leq \sum_{k=1}^n \left\| \frac{\partial f}{\partial x_k} \right\|_{B,\infty} |\alpha_k| \frac{1}{V(B)} \int_{\partial B} (x_k - \overline{x_{B,k}})^2 |n_k(x)|\, dA$$
$$+ \sum_{1 \leq k \neq j \leq n} \left\| \frac{\partial f}{\partial x_j} \right\|_{B,\infty} |\alpha_k| \frac{1}{V(B)} \int_{\partial B} |x_k - \overline{x_{B,k}}| |x_j - \overline{x_{B,j}}| |n_k(x)|\, dA.$$

In particular,

$$\left| \frac{1}{V(B)} \int_B f(x)\, dx - f(G_B) - \frac{1}{n} \frac{1}{V(B)} \sum_{k=1}^n \int_B (y_k - x_k) \frac{\partial f(x)}{\partial x_k} dx \right| \tag{4.9}$$
$$\leq \frac{1}{n} \sum_{k=1}^n \left\| \frac{\partial f}{\partial x_k} \right\|_{B,\infty} \frac{1}{V(B)} \int_{\partial B} (x_k - \overline{x_{B,k}})^2 |n_k(x)|\, dA$$
$$+ \frac{1}{n} \sum_{1 \leq k \neq j \leq n} \left\| \frac{\partial f}{\partial x_j} \right\|_{B,\infty} \frac{1}{V(B)} \int_{\partial B} |x_k - \overline{x_{B,k}}| |x_j - \overline{x_{B,j}}| |n_k(x)|\, dA.$$

Also, if $G_{B,\partial f}:=(x_{B,\partial f,1},...,x_{B,\partial f,n})\in B$, *then by Corollary 4 for* $y=G_{B,\partial f}$ *we get*

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx-f(G_{B,\partial f})\right| \tag{4.10}$$
$$\leq\sum_{k=1}^{n}\left\|\frac{\partial f}{\partial x_k}\right\|_{B,\infty}|\alpha_k|\frac{1}{V(B)}\int_{\partial B}(x_k-x_{B,\partial f,k})^2|n_k(x)|\,dA$$
$$+\sum_{1\leq k\neq j\leq n}\left\|\frac{\partial f}{\partial x_j}\right\|_{B,\infty}|\alpha_k|\frac{1}{V(B)}\int_{\partial B}|x_k-x_{B,\partial f,k}|\,|x_j-x_{B,\partial f,j}|\,|n_k(x)|\,dA.$$

In particular,

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx-f(G_{B,\partial f})\right| \tag{4.11}$$
$$\leq\frac{1}{n}\sum_{k=1}^{n}\left\|\frac{\partial f}{\partial x_k}\right\|_{B,\infty}\frac{1}{V(B)}\int_{\partial B}(x_k-x_{B,\partial f,k})^2|n_k(x)|\,dA$$
$$+\frac{1}{n}\sum_{1\leq k\neq j\leq n}\left\|\frac{\partial f}{\partial x_j}\right\|_{B,\infty}\frac{1}{V(B)}\int_{\partial B}|x_k-x_{B,\partial f,k}|\,|x_j-x_{B,\partial f,j}|\,|n_k(x)|\,dA.$$

We also have:

Theorem 4. *Let B be a bounded closed convex subset of $\mathbb{R}^n$ $(n\geq 2)$ with smooth (or piecewise smooth) boundary ∂B and $\alpha_k\in\mathbb{C}$ for $k\in\{1,...,n\}$ with $\sum_{k=1}^{n}\alpha_k=1$. If f is differentiable on an open neighborhood of B and has bounded partial derivatives on B, then*

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx-\frac{1}{A(\partial B)}\int_{\partial B}f(y)\,dA\right. \tag{4.12}$$
$$\left.-\frac{1}{V(B)}\sum_{k=1}^{n}\int_B\alpha_k\left(\overline{y_{\partial B,k}}-x_k\right)\frac{\partial f(x)}{\partial x_k}dx\right|$$
$$\leq\sum_{k=1}^{n}\sum_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_{B,\infty}\frac{1}{V(B)}\frac{1}{A(\partial B)}\int_{\partial B}|\alpha_k|\left(\int_{\partial B}|x_k-y_k|\,|x_j-y_j|\,dA\right)|n_k(x)|\,dA,$$

where

$$\overline{y_{\partial B,k}}:=\frac{1}{A(\partial B)}\int_{\partial B}y_k dA,\ \ k\in\{1,...,n\}.$$

In particular, we have

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx-\frac{1}{A(\partial B)}\int_{\partial B}f(y)\,dA\right. \tag{4.13}$$
$$\left.-\frac{1}{n}\frac{1}{V(B)}\sum_{k=1}^{n}\int_B\left(\overline{y_{\partial B,k}}-x_k\right)\frac{\partial f(x)}{\partial x_k}dx\right|$$
$$\leq\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_{B,\infty}\frac{1}{V(B)}\frac{1}{A(\partial B)}\int_{\partial B}\left(\int_{\partial B}|x_k-y_k|\,|x_j-y_j|\,dA\right)|n_k(x)|\,dA.$$

Proof. By taking the integral mean $\frac{1}{A(\partial B)}\int_{\partial B} dA$ over the variable y on ∂B in the identity (3.3) and using Fubini's theorem, we get

$$\begin{aligned}(4.14)\quad \frac{1}{V(B)}\int_B f(x)\,dx &= \frac{1}{A(\partial B)}\int_{\partial B} f(y)\,dA \\ &\quad + \frac{1}{V(B)}\sum_{k=1}^{n}\int_B \alpha_k\left(\frac{1}{A(\partial B)}\int_{\partial B} y_k dA - x_k\right)\frac{\partial f(x)}{\partial x_k}dx \\ &\quad + \sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\alpha_k\frac{1}{A(\partial B)}\left(\int_{\partial B}(x_k-y_k)\left[f(x)-f(y)\right]dA\right)n_k(x)\,dA \\ &= \frac{1}{A(\partial B)}\int_{\partial B} f(y)\,dA + \frac{1}{V(B)}\sum_{k=1}^{n}\int_B \alpha_k\left(\overline{y_{\partial B,k}} - x_k\right)\frac{\partial f(x)}{\partial x_k}dx \\ &\quad + \sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\alpha_k\frac{1}{A(\partial B)}\left(\int_{\partial B}(x_k-y_k)\left[f(x)-f(y)\right]dA\right)n_k(x)\,dA.\end{aligned}$$

From (4.14) we get

$$\begin{aligned}(4.15)\quad &\left|\frac{1}{V(B)}\int_B f(x)\,dx - \frac{1}{A(\partial B)}\int_{\partial B} f(y)\,dA \right. \\ &\qquad \left. -\frac{1}{V(B)}\sum_{k=1}^{n}\int_B \alpha_k\left(\overline{y_{\partial B,k}} - x_k\right)\frac{\partial f(x)}{\partial x_k}dx\right| \\ &\leq \left|\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\alpha_k\frac{1}{A(\partial B)}\left(\int_{\partial B}(x_k-y_k)\left[f(x)-f(y)\right]dA\right)n_k(x)\,dA\right| \\ &\leq \sum_{k=1}^{n}\frac{1}{V(B)}\left|\int_{\partial B}\alpha_k\frac{1}{A(\partial B)}\left(\int_{\partial B}(x_k-y_k)\left[f(x)-f(y)\right]dA\right)n_k(x)\,dA\right| \\ &\leq \sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}|\alpha_k|\left|\frac{1}{A(\partial B)}\left(\int_{\partial B}(x_k-y_k)\left[f(x)-f(y)\right]dA\right)\right||n_k(x)|\,dA \\ &\leq \sum_{k=1}^{n}\frac{1}{V(B)}\frac{1}{A(\partial B)}\int_{\partial B}|\alpha_k|\left(\int_{\partial B}|x_k-y_k|\,|f(x)-f(y)|\,dA\right)|n_k(x)|\,dA =: T.\end{aligned}$$

Using the property (4.2) we get

$$\begin{aligned}\int_{\partial B}|x_k-y_k|\,|f(x)-f(y)|\,dA &\leq \int_{\partial B}|x_k-y_k|\sum_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_{B,\infty}|x_j-y_j|\,dA \\ &= \sum_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_{B,\infty}\int_{\partial B}|x_k-y_k|\,|x_j-y_j|\,dA,\end{aligned}$$

which implies that

$$\begin{aligned} T &\leq \sum_{k=1}^{n} \frac{1}{V(B)} \frac{1}{A(\partial B)} \int_{\partial B} |\alpha_k| \sum_{j=1}^{n} \left\| \frac{\partial f}{\partial x_j} \right\|_{B,\infty} \\ &\times \left(\int_{\partial B} |x_k - y_k| |x_j - y_j| \, dA \right) |n_k(x)| \, dA \\ &= \sum_{k=1}^{n} \sum_{j=1}^{n} \left\| \frac{\partial f}{\partial x_j} \right\|_{B,\infty} \frac{1}{V(B)} \frac{1}{A(\partial B)} \int_{\partial B} |\alpha_k| \\ &\times \left(\int_{\partial B} |x_k - y_k| |x_j - y_j| \, dA \right) |n_k(x)| \, dA. \end{aligned}$$

This inequality together with (4.15) produces the desired result (4.12). □

For various inequalities in the 3-dimensional case that are similar with the above, see the recent paper [7]. We omit the details.

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