APPROXIMATING THE VOLUME INTEGRAL BY A SURFACE INTEGRAL VIA THE DIVERGENCE THEOREM

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Abstract. In this paper, by utilising the famous *Divergence Theorem* for n-dimensional integral, we provide some error estimates in approximating the integral on a body B, a bounded closed subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary ∂B, by an integral on the surface ∂B and some other simple terms. Some examples for 3-dimensional case are also given.

1. Introduction

In the following, consider D a closed and bounded convex subset of $\mathbb{R}^2$. Define

$$A_D := \int\int_D dxdy$$

the *area* of D and $(\overline{x_D}, \overline{y_D})$ the *centre of mass* for D, where

$$\overline{x_D} := \frac{1}{A_D}\int\int_D xdxdy,\ \overline{y_D} := \frac{1}{A_D}\int\int_D ydxdy.$$

Consider the function of two variables $f = f(x,y)$ and denote by $\frac{\partial f}{\partial x}$ the partial derivative with respect to the variable x and $\frac{\partial f}{\partial y}$ the partial derivative with respect to the variable y.

We assume that the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ satisfy the Lipschitz type conditions in the point $(u,v) \in D$

$$\left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(u,v)\right| \leq L_1|x-u| + K_1|y-v| \tag{1.1}$$

and

$$\left|\frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}(u,v)\right| \leq L_2|x-u| + K_2|y-v| \tag{1.2}$$

for any $(x,y) \in D$, where L_1, K_1, L_2 and K_2 are given positive constants.

In the recent paper [7] we established the following result in approximating the double integral by a contour integral:

Theorem 1. *Let ∂D be a simple, closed counterclockwise curve bounding a region D and f defined on an open set containing D and having continuous partial derivatives*

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on D. Assume that $(u,v)\in D$ and $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ satisfy the Lipschitz type conditions (1.1) and (1.2). Then for any α, $\beta\in\mathbb{C}$ we have

$$\begin{aligned}(1.3)\quad &\left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy\right.\\ &\qquad -\frac{1}{2A_D}\oint_{\partial D}\left[(\beta-y)f(x,y)\,dx+(x-\alpha)f(x,y)\,dy\right]\\ &\qquad \left.-\frac{1}{2}\frac{\partial f}{\partial x}(u,v)(\alpha-\overline{x_D})-\frac{1}{2}\frac{\partial f}{\partial y}(u,v)(\beta-\overline{y_D})\right|\\ &\leq\frac{L_1}{2A_D}\int\int_D|\alpha-x|\,|x-u|\,dxdy+\frac{K_1}{2A_D}\int\int_D|\alpha-x|\,|y-v|\,dxdy\\ &\quad+\frac{L_2}{2A_D}\int\int_D|\beta-y|\,|x-u|\,dxdy+\frac{K_2}{2A_D}\int\int_D|\beta-y|\,|y-v|\,dxdy.\end{aligned}$$

In particular,

$$\begin{aligned}(1.4)\quad &\left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy\right.\\ &\qquad\left.-\frac{1}{2A_D}\oint_{\partial D}\left[(\overline{y_D}-y)f(x,y)\,dx+(x-\overline{x_D})f(x,y)\,dy\right]\right|\\ &\leq\frac{L_1}{2A_D}\int\int_D|\overline{x_D}-x|\,|x-u|\,dxdy+\frac{K_1}{2A_D}\int\int_D|\overline{x_D}-x|\,|y-v|\,dxdy\\ &\quad+\frac{L_2}{2A_D}\int\int_D|\overline{y_D}-y|\,|x-u|\,dxdy+\frac{K_2}{2A_D}\int\int_D|\overline{y_D}-y|\,|y-v|\,dxdy\end{aligned}$$

and

$$\begin{aligned}(1.5)\quad &\left|\frac{1}{A_D}\int\int_D f(x,y)\,dxdy\right.\\ &\qquad\left.-\frac{1}{2}\frac{\partial f}{\partial x}(u,v)(x_{f,\partial D}-\overline{x_D})-\frac{1}{2}\frac{\partial f}{\partial y}(u,v)(y_{f,\partial D}-\overline{y_D})\right|\\ &\leq\frac{L_1}{2A_D}\int\int_D|x_{f,\partial D}-x|\,|x-u|\,dxdy+\frac{K_1}{2A_D}\int\int_D|x_{f,\partial D}-x|\,|y-v|\,dxdy\\ &\quad+\frac{L_2}{2A_D}\int\int_D|y_{f,\partial D}-y|\,|x-u|\,dxdy+\frac{K_2}{2A_D}\int\int_D|y_{f,\partial D}-y|\,|y-v|\,dxdy,\end{aligned}$$

where

$$x_{f,\partial D}:=\frac{\oint_{\partial D}xf(x,y)\,dy}{\oint_{\partial D}f(x,y)\,dy}\text{ and }y_{f,\partial D}:=\frac{\oint_{\partial D}yf(x,y)\,dx}{\oint_{\partial D}f(x,y)\,dx}$$

provided the denominators are not zero.

For other integral inequalities for multiple integrals see [3]-[15].

In this paper, motivated by the above results and by utilising the famous *Divergence Theorem* for n-dimensional integral, we provide some error estimates in approximating the integral on a body B, a bounded closed subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary ∂B, by an integral on the surface ∂B and some other simple terms. Some examples for 3-dimensional case are also given.

2. Some Preliminary Facts

Let B be a bounded open subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary ∂B. Let $F = (F_1, ..., F_n)$ be a smooth vector field defined in $\mathbb{R}^n$, or at least in $B \cup \partial B$. Let $\mathbf{n}$ be the unit outward-pointing normal of ∂B. Then the *Divergence Theorem* states, see for instance [16]:

$$\int_B \operatorname{div} F dV = \int_{\partial B} F \cdot n dA, \tag{2.1}$$

where

$$\operatorname{div} F = \nabla \cdot F = \sum_{k=1}^{n} \frac{\partial F_k}{\partial x_k},$$

dV is the element of volume in $\mathbb{R}^n$ and dA is the element of surface area on ∂B.

If $\mathbf{n} = (\mathbf{n}_1, ..., \mathbf{n}_n)$, $x = (x_1, ..., x_n) \in B$ and use the notation dx for dV we can write (2.1) more explicitly as

$$\sum_{k=1}^{n} \int_B \frac{\partial F_k(x)}{\partial x_k} dx = \sum_{k=1}^{n} \int_{\partial B} F_k(x) n_k(x) dA. \tag{2.2}$$

By taking the real and imaginary part, we can extend the above equality for complex valued functions F_k, $k \in \{1, ..., n\}$ defined on B.

If $n = 2$, the normal is obtained by rotating the tangent vector through $90°$ (in the correct direction so that it points out). The quantity tds can be written (dx_1, dx_2) along the surface, so that

$$ndA := nds = (dx_2, -dx_1).$$

Here t is the tangent vector along the boundary curve and ds is the element of arc-length.

From (2.2) we get for $B \subset \mathbb{R}^2$

$$\begin{aligned} &\int_B \frac{\partial F_1(x_1, x_2)}{\partial x_1} dx_1 dx_2 + \int_B \frac{\partial F_2(x_1, x_2)}{\partial x_2} dx_1 dx_2 \\ &= \int_{\partial B} F_1(x_1, x_2) dx_2 - \int_{\partial B} F_2(x_1, x_2) dx_1, \end{aligned} \tag{2.3}$$

which is *Green's theorem* in plane.

If $n = 3$ and if ∂B is described as a level-set of a function of 3 variables i.e. $\partial B = \{x_1, x_2, x_3 \in \mathbb{R}^3 \mid G(x_1, x_2, x_3) = 0\}$, then a vector pointing in the direction of $\mathbf{n}$ is $\operatorname{grad} G$. We shall use the case where $G(x_1, x_2, x_3) = x_3 - g(x_1, x_2)$, $(x_1, x_2) \in D$, a domain in $\mathbb{R}^2$ for some differentiable function g on D and B corresponds to the inequality $x_3 < g(x_1, x_2)$, namely

$$B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 < g(x_1, x_2)\}.$$

Then

$$\mathbf{n} = \frac{(-g_{x_1}, -g_{x_2}, 1)}{\left(1 + g_{x_1}^2 + g_{x_2}^2\right)^{1/2}}, \ dA = \left(1 + g_{x_1}^2 + g_{x_2}^2\right)^{1/2} dx_1 dx_2$$

and

$$\mathbf{n} dA = (-g_{x_1}, -g_{x_2}, 1) dx_1 dx_2.$$

From (2.2) we get

$$\begin{aligned}(2.4)\qquad &\int_B\left(\frac{\partial F_1(x_1,x_2,x_3)}{\partial x_1}+\frac{\partial F_2(x_1,x_2,x_3)}{\partial x_2}+\frac{\partial F_3(x_1,x_2,x_3)}{\partial x_3}\right)dx_1dx_2dx_3\\ &=-\int_D F_1(x_1,x_2,g(x_1,x_2))\,g_{x_1}(x_1,x_2)\,dx_1dx_2\\ &-\int_D F_1(x_1,x_2,g(x_1,x_2))\,g_{x_2}(x_1,x_2)dx_1dx_2\\ &+\int_D F_3(x_1,x_2,g(x_1,x_2))\,dx_1dx_2,\end{aligned}$$

which is the *Gauss-Ostrogradsky theorem* in space.

Following Apostol [1], consider a surface described by the vector equation

$$(2.5)\qquad r(u,v)=x_1(u,v)\,\overrightarrow{i}+x_2(u,v)\,\overrightarrow{j}+x_3(u,v)\,\overrightarrow{k}$$

where $(u,v)\in[a,b]\times[c,d]$.

If x_1, x_2, x_3 are differentiable on $[a,b]\times[c,d]$ we consider the two vectors

$$\frac{\partial r}{\partial u}=\frac{\partial x_1}{\partial u}\overrightarrow{i}+\frac{\partial x_2}{\partial u}\overrightarrow{j}+\frac{\partial x_3}{\partial u}\overrightarrow{k}$$

and

$$\frac{\partial r}{\partial v}=\frac{\partial x_1}{\partial v}\overrightarrow{i}+\frac{\partial x_2}{\partial v}\overrightarrow{j}+\frac{\partial x_3}{\partial v}\overrightarrow{k}.$$

The *cross product* of these two vectors $\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}$ will be referred to as the fundamental vector product of the representation r. Its components can be expressed as *Jacobian determinants*. In fact, we have [1, p. 420]

$$(2.6)\qquad \frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}=\begin{vmatrix}\frac{\partial x_2}{\partial u} & \frac{\partial x_3}{\partial u}\\ \frac{\partial x_2}{\partial v} & \frac{\partial x_3}{\partial v}\end{vmatrix}\overrightarrow{i}+\begin{vmatrix}\frac{\partial x_3}{\partial u} & \frac{\partial x_1}{\partial u}\\ \frac{\partial x_3}{\partial v} & \frac{\partial x_1}{\partial v}\end{vmatrix}\overrightarrow{j}+\begin{vmatrix}\frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u}\\ \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v}\end{vmatrix}\overrightarrow{k}$$

$$=\frac{\partial(x_2,x_3)}{\partial(u,v)}\overrightarrow{i}+\frac{\partial(x_3,x_1)}{\partial(u,v)}\overrightarrow{j}+\frac{\partial(x_1,x_2)}{\partial(u,v)}\overrightarrow{k}.$$

Let $\partial B=r(T)$ be a parametric surface described by a vector-valued function r defined on the box $T=[a,b]\times[c,d]$. The area of ∂B denoted $A_{\partial B}$ is defined by the double integral [1, p. 424-425]

$$\begin{aligned}(2.7)\qquad A_{\partial B}&=\int_a^b\int_c^d\left\|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right\|dudv\\ &=\int_a^b\int_c^d\sqrt{\left(\frac{\partial(x_2,x_3)}{\partial(u,v)}\right)^2+\left(\frac{\partial(x_3,x_1)}{\partial(u,v)}\right)^2+\left(\frac{\partial(x_1,x_2)}{\partial(u,v)}\right)^2}dudv.\end{aligned}$$

We define surface integrals in terms of a parametric representation for the surface. One can prove that under certain general conditions the value of the integral is independent of the representation.

Let $\partial B=r(T)$ be a parametric surface described by a vector-valued differentiable function r defined on the box $T=[a,b]\times[c,d]$ and let $f:\partial B\to\mathbb{C}$ defined and bounded on ∂B. The surface integral of f over ∂B is defined by [1, p. 430]

$$\begin{aligned}(2.8)\qquad \int\int_{\partial B} f dA &= \int_a^b \int_c^d f(x_1,x_2,x_3)\left\|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right\| dudv\\ &= \int_a^b \int_c^d f(x_1(u,v),x_2(u,v),x_3(u,v))\\ &\times\sqrt{\left(\frac{\partial(x_2,x_3)}{\partial(u,v)}\right)^2+\left(\frac{\partial(x_3,x_1)}{\partial(u,v)}\right)^2+\left(\frac{\partial(x_1,x_2)}{\partial(u,v)}\right)^2}dudv.\end{aligned}$$

If $\partial B = r(T)$ is a parametric surface, the fundamental vector product $N = \frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}$ is normal to ∂B at each regular point of the surface. At each such point there are two unit normals, a unit normal $\mathbf{n}_1$, which has the same direction as N, and a unit normal $\mathbf{n}_2$ which has the opposite direction. Thus

$$\mathbf{n}_1 = \frac{N}{\|N\|} \text{ and } \mathbf{n}_2 = -\mathbf{n}_1.$$

Let $\mathbf{n}$ be one of the two normals $\mathbf{n}_1$ or $\mathbf{n}_2$. Let also F be a vector field defined on ∂B and assume that the surface integral,

$$\int\int_{\partial B}(F\cdot\mathbf{n})\,dA,$$

called the flux surface integral, exists. Here $F\cdot\mathbf{n}$ is the dot or inner product.

We can write [1, p. 434]

$$\int\int_{\partial B}(F\cdot\mathbf{n})\,dA = \pm\int_a^b\int_c^d F(r(u,v))\cdot\left(\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right)dudv$$

where the sign " + " is used if $\mathbf{n}=\mathbf{n}_1$ and the " − " sign is used if $\mathbf{n}=\mathbf{n}_2$.

If

$$F(x_1,x_2,x_3) = F_1(x_1,x_2,x_3)\overrightarrow{i} + F_2(x_1,x_2,x_3)\overrightarrow{j} + F_3(x_1,x_2,x_3)\overrightarrow{k}$$

and

$$r(u,v) = x_1(u,v)\overrightarrow{i} + x_2(u,v)\overrightarrow{j} + x_3(u,v)\overrightarrow{k} \text{ where } (u,v)\in[a,b]\times[c,d]$$

then the flux surface integral for $\mathbf{n}=\mathbf{n}_1$ can be explicitly calculated as [1, p. 435]

$$\begin{aligned}(2.9)\quad \int\int_{\partial B}(F\cdot\mathbf{n})\,dA &= \int_a^b\int_c^d F_1(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_2,x_3)}{\partial(u,v)}dudv\\ &+\int_a^b\int_c^d F_2(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_3,x_1)}{\partial(u,v)}dudv\\ &+\int_a^b\int_c^d F_3(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_1,x_2)}{\partial(u,v)}dudv.\end{aligned}$$

The sum of the double integrals on the right is often written more briefly as [1, p. 435]

$$\begin{aligned}&\int\int_{\partial B}F_1(x_1,x_2,x_3)\,dx_2\wedge dx_3 + \int\int_{\partial B}F_2(x_1,x_2,x_3)\,dx_3\wedge dx_1\\ &+\int\int_{\partial B}F_3(x_1,x_2,x_3)\,dx_1\wedge dx_2.\end{aligned}$$

Let $B \subset \mathbb{R}^3$ be a solid in 3-space bounded by an orientable closed surface ∂B, and let $\mathbf{n}$ be the unit outer normal to ∂B. If F is a continuously differentiable vector field defined on B, we have the *Gauss-Ostrogradsky identity*

$$\iiint_B (\operatorname{div} F)\, dV = \int\int_{\partial B} (F \cdot \mathbf{n})\, dA. \tag{GO}$$

If we express

$$F(x_1, x_2, x_3) = F_1(x_1, x_2, x_3)\overrightarrow{i} + F_2(x_1, x_2, x_3)\overrightarrow{j} + F_3(x_1, x_2, x_3)\overrightarrow{k},$$

then (2.4) can be written as

$$\begin{aligned} &\iiint_B \left(\frac{\partial F_1(x_1, x_2, x_3)}{\partial x_1} + \frac{\partial F_2(x_1, x_2, x_3)}{\partial x_2} + \frac{\partial F_3(x_1, x_2, x_3)}{\partial x_3} \right) dx_1 dx_2 dx_3 \\ &= \int\int_{\partial B} F_1(x_1, x_2, x_3)\, dx_2 \wedge dx_3 + \int\int_{\partial B} F_2(x_1, x_2, x_3)\, dx_3 \wedge dx_1 \\ &+ \int\int_{\partial B} F_3(x_1, x_2, x_3)\, dx_1 \wedge dx_2. \end{aligned} \tag{2.10}$$

3. Some Perturbed Identities

For the body B we consider the coordinates for the *centre of gravity*

$$G\left(\overline{x_{B,1}}, ..., \overline{x_{B,n}}\right)$$

defined by

$$\overline{x_{B,k}} := \frac{1}{V(B)} \int_B x_k dx, \ k \in \{1, ..., n\},$$

where

$$V(B) := \int_B dx$$

is the volume of B.

We have the following identity of interest:

Theorem 2. *Let B be a bounded closed subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary ∂B. Let f be a continuously differentiable function defined in $\mathbb{R}^n$, or at least in on open neighborhood of B and with complex values. If α_k, β_k, $\delta_k \in \mathbb{C}$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^n \alpha_k = 1$, then*

$$\begin{aligned} &\frac{1}{V(B)} \int_B f(x)\, dx \\ &= \sum_{k=1}^n \frac{1}{V(B)} \int_B (\beta_k - \alpha_k x_k) \left(\frac{\partial f(x)}{\partial x_k} - \delta_k \right) dx \\ &+ \sum_{k=1}^n \delta_k \left(\beta_k - \alpha_k \overline{x_{B,k}}\right) + \sum_{k=1}^n \frac{1}{V(B)} \int_{\partial B} (\alpha_k x_k - \beta_k)\, f(x)\, n_k(x)\, dA. \end{aligned} \tag{3.1}$$

We also have

$$\begin{aligned}\frac{1}{V(B)}\int_B f(x)\,dx &= \sum_{k=1}^{n}\alpha_k\frac{1}{V(B)}\int_B(\gamma_k - x_k)\left(\frac{\partial f(x)}{\partial x_k}-\delta_k\right)dx \\ &+ \sum_{k=1}^{n}\alpha_k\delta_k\left(\gamma_k-\overline{x_{B,k}}\right)+\sum_{k=1}^{n}\alpha_k\frac{1}{V(B)}\int_{\partial B}(x_k-\gamma_k)\,f(x)\,n_k(x)\,dA\end{aligned} \tag{3.2}$$

for all $\gamma_k \in \mathbb{C}$, *where* $k \in \{1,...,n\}$ *and, in particular,*

$$\begin{aligned}\frac{1}{V(B)}\int_B f(x)\,dx &= \frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_B(\gamma_k - x_k)\left(\frac{\partial f(x)}{\partial x_k}-\delta_k\right)dx \\ &+ \frac{1}{n}\sum_{k=1}^{n}\alpha_k\delta_k\left(\gamma_k-\overline{x_{B,k}}\right)+\frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k-\gamma_k)\,f(x)\,n_k(x)\,dA.\end{aligned} \tag{3.3}$$

Proof. Let $x = (x_1, ..., x_n) \in B$. We consider

$$F_k(x) = (\alpha_k x_k - \beta_k)\,f(x),\ \ k \in \{1,...,n\}$$

and take the partial derivatives $\frac{\partial F_k(x)}{\partial x_k}$ to get

$$\frac{\partial F_k(x)}{\partial x_k} = \alpha_k f(x) + (\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k},\ \ k \in \{1,...,n\}.$$

If we sum this equality over k from 1 to n we get

$$\begin{aligned}\sum_{k=1}^{n}\frac{\partial F_k(x)}{\partial x_k} &= \sum_{k=1}^{n}\alpha_k f(x) + \sum_{k=1}^{n}(\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k} \\ &= f(x) + \sum_{k=1}^{n}(\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k}\end{aligned} \tag{3.4}$$

for all $x = (x_1, ..., x_n) \in B$.

Now, if we take the integral in the equality (3.4) over $(x_1, ..., x_n) \in B$ we get

$$\int_B\left(\sum_{k=1}^{n}\frac{\partial F_k(x)}{\partial x_k}\right)dx = \int_B f(x)\,dx + \sum_{k=1}^{n}\int_B\left[(\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k}\right]dx. \tag{3.5}$$

By the Divergence Theorem (2.2) we also have

$$\int_B\left(\sum_{k=1}^{n}\frac{\partial F_k(x)}{\partial x_k}\right)dx = \sum_{k=1}^{n}\int_{\partial B}(\alpha_k x_k - \beta_k)\,f(x)\,n_k(x)\,dA \tag{3.6}$$

and by making use of (3.5) and (3.6) we derive

$$\begin{aligned}&\int_B f(x)\,dx + \sum_{k=1}^{n}\int_B\left[(\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k}\right]dx \\ &= \sum_{k=1}^{n}\int_{\partial B}(\alpha_k x_k - \beta_k)\,f(x)\,n_k(x)\,dA,\end{aligned}$$

which gives the representation

$$\int_B f(x)\,dx = \sum_{k=1}^{n} \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} dx + \sum_{k=1}^{n} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x)\, n_k(x)\, dA. \tag{3.7}$$

Now, observe that

$$\begin{aligned}
&\int_B (\beta_k - \alpha_k x_k) \left(\frac{\partial f(x)}{\partial x_k} - \delta_k \right) dx \\
&= \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} dx - \delta_k \int_B (\beta_k - \alpha_k x_k)\, dx \\
&= \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} dx - \delta_k \left(\beta_k V(B) - \alpha_k V(B)\, \overline{x_{B,k}} \right) \\
&= \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} dx - V(B)\, \delta_k \left(\beta_k - \alpha_k \overline{x_{B,k}} \right),
\end{aligned}$$

which by summation over $k \in \{1, ..., n\}$ provides

$$\begin{aligned}
&\sum_{k=1}^{n} \int_B (\beta_k - \alpha_k x_k) \left(\frac{\partial f(x)}{\partial x_k} - \delta_k \right) dx \\
&= \sum_{k=1}^{n} \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} dx - V(B) \sum_{k=1}^{n} \delta_k \left(\beta_k - \alpha_k \overline{x_{B,k}} \right)
\end{aligned}$$

namely

$$\begin{aligned}
&\sum_{k=1}^{n} \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} dx \\
&= \sum_{k=1}^{n} \int_B (\beta_k - \alpha_k x_k) \left(\frac{\partial f(x)}{\partial x_k} - \delta_k \right) dx + V(B) \sum_{k=1}^{n} \delta_k \left(\beta_k - \alpha_k \overline{x_{B,k}} \right).
\end{aligned}$$

From (3.7) we then get

$$\begin{aligned}
\int_B f(x)\,dx &= \sum_{k=1}^{n} \int_B (\beta_k - \alpha_k x_k) \left(\frac{\partial f(x)}{\partial x_k} - \delta_k \right) dx \\
&\quad + V(B) \sum_{k=1}^{n} \delta_k \left(\beta_k - \alpha_k \overline{x_{B,k}} \right) + \sum_{k=1}^{n} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x)\, n_k(x)\, dA,
\end{aligned}$$

which by division with $V(B)$ produces the desired result (3.1)

The identity (3.2) follows by (3.1) for $\beta_k = \alpha_k \gamma_k$, $k \in \{1, ..., n\}$. □

The following particular cases are of interest:

Corollary 1. *With the assumptions of Theorem 2 we have*

$$\begin{aligned}
\frac{1}{V(B)} \int_B f(x)\,dx &= \sum_{k=1}^{n} \alpha_k \frac{1}{V(B)} \int_B \left(\overline{x_{B,k}} - x_k \right) \left(\frac{\partial f(x)}{\partial x_k} - \delta_k \right) dx \\
&\quad + \sum_{k=1}^{n} \alpha_k \frac{1}{V(B)} \int_{\partial B} \left(x_k - \overline{x_{B,k}} \right) f(x)\, n_k(x)\, dA
\end{aligned} \tag{3.8}$$

and, in particular,

$$\frac{1}{V(B)}\int_B f(x)\,dx=\frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_B\left(\overline{x_{B,k}}-x_k\right)\left(\frac{\partial f(x)}{\partial x_k}-\delta_k\right)dx \tag{3.9}$$
$$+\frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\left(x_k-\overline{x_{B,k}}\right)f(x)\,n_k(x)\,dA.$$

The proof follows by (3.1) on taking $\beta_k=\alpha_k\overline{x_{B,k}}$, $k\in\{1,...,n\}$.

For a function f as in Theorem 2 above, we define the points

$$x_{\partial B,f,k}:=\frac{\int_{\partial B}x_k f(x)\,n_k(x)\,dA}{\int_{\partial B}f(x)\,n_k(x)\,dA},\ \ k\in\{1,...,n\}$$

provided that all denominators are not zero.

Corollary 2. *With the assumptions of Theorem 2 we have*

$$\frac{1}{V(B)}\int_B f(x)\,dx=\sum_{k=1}^{n}\alpha_k\frac{1}{V(B)}\int_B\left(x_{\partial B,f,k}-x_k\right)\left(\frac{\partial f(x)}{\partial x_k}-\delta_k\right)dx \tag{3.10}$$
$$+\sum_{k=1}^{n}\delta_k\alpha_k\left(x_{\partial B,f,k}-\overline{x_{B,k}}\right)$$

and, in particular

$$\frac{1}{V(B)}\int_B f(x)\,dx=\frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_B\left(x_{\partial B,f,k}-x_k\right)\left(\frac{\partial f(x)}{\partial x_k}-\delta_k\right)dx \tag{3.11}$$
$$+\frac{1}{n}\sum_{k=1}^{n}\delta_k\left(x_{\partial B,f,k}-\overline{x_{B,k}}\right).$$

The proof follows by (3.1) on taking $\beta_k=\alpha_k x_{\partial B,f,k}$, $k\in\{1,...,n\}$ and observing that

$$\sum_{k=1}^{n}\alpha_k\int_{\partial B}\left(x_k-x_{\partial B,f,k}\right)f(x)\,n_k(x)\,dA=0.$$

4. Some Inequalities for Bounded Partial Derivatives

Let B be a bounded closed subset of $\mathbb{R}^n$ $(n\geq 2)$ with smooth (or piecewise smooth) boundary ∂B. Now, for ϕ, $\Phi\in\mathbb{C}$, define the sets of complex-valued functions

$$\bar{U}_B(\phi,\Phi)$$
$$:=\left\{f:B\to\mathbb{C}|\operatorname{Re}\left[(\Phi-f(x))\left(\overline{f(x)}-\overline{\phi}\right)\right]\geq 0\ \text{ for each }\ x\in B\right\}$$

and

$$\bar{\Delta}_B(\phi,\Phi):=\left\{f:B\to\mathbb{C}|\ \left|f(x)-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}\left|\Phi-\phi\right|\ \text{ for each }\ x\in B\right\}.$$

The following representation result may be stated.

Proposition 1. *For any ϕ, $\Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that $\bar{U}_B(\phi,\Phi)$ and $\bar{\Delta}_B(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_B(\phi,\Phi)=\bar{\Delta}_B(\phi,\Phi). \tag{4.1}$$

Proof. We observe that for any $w \in \mathbb{C}$ we have the equivalence

$$\left| w - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2} \left| \Phi - \phi \right|$$

if and only if

$$\operatorname{Re}\left[\left(\Phi - w \right) \left(\overline{w} - \overline{\phi} \right) \right] \geq 0.$$

This follows by the equality

$$\frac{1}{4} \left| \Phi - \phi \right|^2 - \left| w - \frac{\phi + \Phi}{2} \right|^2 = \operatorname{Re}\left[\left(\Phi - w \right) \left(\overline{w} - \overline{\phi} \right) \right]$$

that holds for any $w \in \mathbb{C}$.

The equality (4.1) is thus a simple consequence of this fact. $\square$

On making use of the complex numbers field properties we can also state that:

Corollary 3. *For any ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that*

$$\begin{aligned} \bar{U}_B \left(\phi, \Phi \right) = \{ f : B \to \mathbb{C} \mid \ & \left(\operatorname{Re} \Phi - \operatorname{Re} f \left(x \right) \right) \left(\operatorname{Re} f \left(x \right) - \operatorname{Re} \phi \right) \\ & + \left(\operatorname{Im} \Phi - \operatorname{Im} f \left(x \right) \right) \left(\operatorname{Im} f \left(x \right) - \operatorname{Im} \phi \right) \geq 0 \text{ for each } x \in B \} . \end{aligned} \tag{4.2}$$

Now, if we assume that $\operatorname{Re}(\Phi) \geq \operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi) \geq \operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_B \left(\phi, \Phi \right) := \{ f : B \to \mathbb{C} \mid \ & \operatorname{Re}(\Phi) \geq \operatorname{Re} f(x) \geq \operatorname{Re}(\phi) \\ & \text{and } \operatorname{Im}(\Phi) \geq \operatorname{Im} f(x) \geq \operatorname{Im}(\phi) \text{ for each } \ x \in B \} . \end{aligned} \tag{4.3}$$

One can easily observe that $\bar{S}_B \left(\phi, \Phi \right)$ is closed, convex and

$$\emptyset \neq \bar{S}_B \left(\phi, \Phi \right) \subseteq \bar{U}_B \left(\phi, \Phi \right) . \tag{4.4}$$

Theorem 3. *Let B be a bounded closed subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary ∂B. Let f be a continuously differentiable function defined in $\mathbb{R}^n$, or at least in on open neighborhood of B and with complex values. Assume that there exist ϕ_k, $\Phi_k \in \mathbb{C}$, $\phi_k \neq \Phi_k$ for $k \in \{1, ..., n\}$ and such that $\frac{\partial f}{\partial x_k} \in \bar{\Delta}_B \left(\phi_k, \Phi_k \right)$ for $k \in \{1, ..., n\}$. If α_k, $\beta_k \in \mathbb{C}$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^n \alpha_k = 1$, then*

$$\begin{aligned} & \left| \frac{1}{V(B)} \int_B f(x) \, dx - \sum_{k=1}^n \frac{\phi_k + \Phi_k}{2} \left(\beta_k - \alpha_k \overline{x_{B,k}} \right) \right. \\ & \left. - \sum_{k=1}^n \frac{1}{V(B)} \int_{\partial B} \left(\alpha_k x_k - \beta_k \right) f(x) \, n_k(x) \, dA \right| \\ & \leq \frac{1}{2} \sum_{k=1}^n \left| \Phi_k - \phi_k \right| \frac{1}{V(B)} \int_B \left| \beta_k - \alpha_k x_k \right| dx \end{aligned} \tag{4.5}$$

We also have

$$\begin{aligned}
&\left|\frac{1}{V(B)}\int_B f(x)\,dx-\sum_{k=1}^{n}\frac{\phi_k+\Phi_k}{2}\alpha_k\left(\gamma_k-\overline{x_{B,k}}\right)\right.\\
&\left.-\sum_{k=1}^{n}\alpha_k\frac{1}{V(B)}\int_{\partial B}(x_k-\gamma_k)\,f(x)\,n_k(x)\,dA\right|\\
&\leq\frac{1}{2}\sum_{k=1}^{n}\left|\Phi_k-\phi_k\right|\left|\alpha_k\right|\frac{1}{V(B)}\int_B\left|\gamma_k-x_k\right|dx
\end{aligned}\tag{4.6}$$

for all $\gamma_k\in\mathbb{C}$, *where* $k\in\{1,...,n\}$ *and, in particular,*

$$\begin{aligned}
&\left|\frac{1}{V(B)}\int_B f(x)\,dx-\frac{1}{n}\sum_{k=1}^{n}\frac{\phi_k+\Phi_k}{2}\left(\gamma_k-\overline{x_{B,k}}\right)\right.\\
&\left.-\frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k-\gamma_k)\,f(x)\,n_k(x)\,dA\right|\\
&\leq\frac{1}{2n}\sum_{k=1}^{n}\left|\Phi_k-\phi_k\right|\frac{1}{V(B)}\int_B\left|\gamma_k-x_k\right|dx.
\end{aligned}\tag{4.7}$$

Proof. By using identity (3.1) for $\delta_k:=\frac{\phi_k+\Phi_k}{2}$, $k\in\{1,...,n\}$, we get

$$\begin{aligned}
&\frac{1}{V(B)}\int_B f(x)\,dx\\
&=\sum_{k=1}^{n}\frac{1}{V(B)}\int_B(\beta_k-\alpha_k x_k)\left(\frac{\partial f(x)}{\partial x_k}-\frac{\phi_k+\Phi_k}{2}\right)dx\\
&+\sum_{k=1}^{n}\frac{\phi_k+\Phi_k}{2}\left(\beta_k-\alpha_k\overline{x_{B,k}}\right)+\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(\alpha_k x_k-\beta_k)\,f(x)\,n_k(x)\,dA.
\end{aligned}\tag{4.8}$$

Since $\frac{\partial f}{\partial x_k}\in\bar{\Delta}_B(\phi_k,\Phi_k)$ for $k\in\{1,...,n\}$, hence by (4.8)

$$\begin{aligned}
&\left|\frac{1}{V(B)}\int_B f(x)\,dx-\sum_{k=1}^{n}\frac{\phi_k+\Phi_k}{2}\left(\beta_k-\alpha_k\overline{x_{B,k}}\right)\right.\\
&\left.-\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(\alpha_k x_k-\beta_k)\,f(x)\,n_k(x)\,dA\right|\\
&\leq\sum_{k=1}^{n}\frac{1}{V(B)}\left|\int_B(\beta_k-\alpha_k x_k)\left(\frac{\partial f(x)}{\partial x_k}-\frac{\phi_k+\Phi_k}{2}\right)dx\right|\\
&\leq\sum_{k=1}^{n}\frac{1}{V(B)}\int_B\left|(\beta_k-\alpha_k x_k)\left(\frac{\partial f(x)}{\partial x_k}-\frac{\phi_k+\Phi_k}{2}\right)\right|dx\\
&\leq\sum_{k=1}^{n}\frac{1}{2}\left|\Phi_k-\phi_k\right|\frac{1}{V(B)}\int_B\left|\beta_k-\alpha_k x_k\right|dx,
\end{aligned}$$

which proves (4.5). The rest is obvious. □

Corollary 4. *With the assumptions of Theorem 3 we have*

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx-\sum_{k=1}^{n}\alpha_k\frac{1}{V(B)}\int_{\partial B}\left(x_k-\overline{x_{B,k}}\right)f(x)\,n_k(x)\,dA\right| \le \frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\,|\Phi_k-\phi_k|\,\frac{1}{V(B)}\int_B\left|\overline{x_{B,k}}-x_k\right|dx \tag{4.9}$$

and, in particular,

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx-\frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\left(x_k-\overline{x_{B,k}}\right)f(x)\,n_k(x)\,dA\right| \le \frac{1}{2n}\sum_{k=1}^{n}|\Phi_k-\phi_k|\,\frac{1}{V(B)}\int_B\left|\overline{x_{B,k}}-x_k\right|dx. \tag{4.10}$$

The proof follows from (4.6) by taking $\gamma_k=\overline{x_{B,k}}$, $k\in\{1,...,n\}$.

Corollary 5. *With the assumptions of Theorem 3 we have*

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx-\sum_{k=1}^{n}\frac{\phi_k+\Phi_k}{2}\alpha_k\left(x_{\partial B,f,k}-\overline{x_{B,k}}\right)\right| \le \frac{1}{2}\sum_{k=1}^{n}|\Phi_k-\phi_k|\,|\alpha_k|\,\frac{1}{V(B)}\int_B\left|x_{\partial B,f,k}-x_k\right|dx \tag{4.11}$$

and, in particular,

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx-\frac{1}{n}\sum_{k=1}^{n}\frac{\phi_k+\Phi_k}{2}\left(x_{\partial B,f,k}-\overline{x_{B,k}}\right)\right| \le \frac{1}{2n}\sum_{k=1}^{n}|\Phi_k-\phi_k|\,\frac{1}{V(B)}\int_B\left|x_{\partial B,f,k}-x_k\right|dx. \tag{4.12}$$

The proof follows from (4.6) by taking $\gamma_k=x_{\partial B,f,k}$, $k\in\{1,...,n\}$ and observing that

$$\sum_{k=1}^{n}\alpha_k\frac{1}{V(B)}\int_{\partial B}\left(x_k-x_{\partial B,f,k}\right)f(x)\,n_k(x)\,dA=0.$$

5. Inequalities for Lipschitzian Partial Derivatives

We assume that the partial derivatives $\frac{\partial f}{\partial x_k}$, $k\in\{1,...,n\}$, satisfy the Lipschitz type conditions in the point $u=(u_1,...,u_n)\in D$

$$\left|\frac{\partial f(x)}{\partial x_k}-\frac{\partial f(u)}{\partial x_k}\right|\le\sum_{j=1}^{n}L_{k,j}\left|x_j-u_j\right| \tag{5.1}$$

for any $x=(x_1,...,x_n)\in D$, where $L_{k,j}$, k, $j\in\{1,...,n\}$ are given positive constants.

Theorem 4. *Let B be a bounded closed subset of $\mathbb{R}^n$ $(n\ge 2)$ with smooth (or piecewise smooth) boundary ∂B. Let f be a continuously differentiable function defined in $\mathbb{R}^n$, or at least in on open neighborhood of B and with complex values. Assume that for $u\in B$ there exist $L_{k,j}$, $k,j\in\{1,...,n\}$ and such that the Lipschitz* *condition (5.1) holds for* $k \in \{1,...,n\}$. *If* $\alpha_k,\ \beta_k \in \mathbb{C}$ *for* $k \in \{1,...,n\}$ *with* $\sum_{k=1}^n \alpha_k = 1$, *then*

$$\begin{aligned}
&\left|\frac{1}{V(B)}\int_B f(x)\,dx - \sum_{k=1}^n \frac{\partial f(u)}{\partial x_k}\left(\beta_k - \alpha_k \overline{x_{B,k}}\right)\right. \\
&\left. - \sum_{k=1}^n \frac{1}{V(B)}\int_{\partial B}\left(\alpha_k x_k - \beta_k\right) f(x)\, n_k(x)\, dA\right| \\
&\leq \sum_{k=1}^n \sum_{j=1}^n L_{k,j}\frac{1}{V(B)}\int_B \left|\beta_k - \alpha_k x_k\right| \left|x_j - u_j\right| dx.
\end{aligned} \tag{5.2}$$

We also have

$$\begin{aligned}
&\left|\frac{1}{V(B)}\int_B f(x)\,dx - \sum_{k=1}^n \alpha_k \frac{\partial f(u)}{\partial x_k}\left(\gamma_k - \overline{x_{B,k}}\right)\right. \\
&\left. - \sum_{k=1}^n \frac{1}{V(B)}\int_{\partial B}\alpha_k\left(x_k - \gamma_k\right) f(x)\, n_k(x)\, dA\right| \\
&\leq \sum_{k=1}^n \sum_{j=1}^n L_{k,j}\left|\alpha_k\right| \frac{1}{V(B)}\int_B \left|\gamma_k - x_k\right| \left|x_j - u_j\right| dx
\end{aligned} \tag{5.3}$$

for all $\gamma_k \in \mathbb{C}$, *where* $k \in \{1,...,n\}$ *and, in particular,*

$$\begin{aligned}
&\left|\frac{1}{V(B)}\int_B f(x)\,dx - \frac{1}{n}\sum_{k=1}^n \frac{\partial f(u)}{\partial x_k}\left(\gamma_k - \overline{x_{B,k}}\right)\right. \\
&\left. - \frac{1}{n}\sum_{k=1}^n \frac{1}{V(B)}\int_{\partial B}\alpha_k\left(x_k - \gamma_k\right) f(x)\, n_k(x)\, dA\right| \\
&\leq \frac{1}{n}\sum_{k=1}^n \sum_{j=1}^n L_{k,j}\frac{1}{V(B)}\int_B \left|\gamma_k - x_k\right| \left|x_j - u_j\right| dx.
\end{aligned} \tag{5.4}$$

Proof. If we write the equality (3.1) for $\delta_k = \frac{\partial f(u)}{\partial x_k}$, $k \in \{1,...,n\}$, we get

$$\begin{aligned}
&\frac{1}{V(B)}\int_B f(x)\,dx \\
&= \sum_{k=1}^n \frac{1}{V(B)}\int_B \left(\beta_k - \alpha_k x_k\right)\left(\frac{\partial f(x)}{\partial x_k} - \frac{\partial f(u)}{\partial x_k}\right) dx \\
&+ \sum_{k=1}^n \frac{\partial f(u)}{\partial x_k}\left(\beta_k - \alpha_k \overline{x_{B,k}}\right) + \sum_{k=1}^n \frac{1}{V(B)}\int_{\partial B}\left(\alpha_k x_k - \beta_k\right) f(x)\, n_k(x)\, dA.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left|\frac{1}{V(B)}\int_B f(x)\,dx-\sum_{k=1}^{n}\frac{\partial f(u)}{\partial x_k}\left(\beta_k-\alpha_k\overline{x_{B,k}}\right)\right.\\
&\left.-\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(\alpha_k x_k-\beta_k)\,f(x)\,n_k(x)\,dA\right|\\
&\leq\sum_{k=1}^{n}\frac{1}{V(B)}\left|\int_B(\beta_k-\alpha_k x_k)\left(\frac{\partial f(x)}{\partial x_k}-\frac{\partial f(u)}{\partial x_k}\right)dx\right|\\
&\leq\sum_{k=1}^{n}\frac{1}{V(B)}\int_B\left|(\beta_k-\alpha_k x_k)\left(\frac{\partial f(x)}{\partial x_k}-\frac{\partial f(u)}{\partial x_k}\right)\right|dx\\
&=\sum_{k=1}^{n}\frac{1}{V(B)}\int_B|\beta_k-\alpha_k x_k|\left|\frac{\partial f(x)}{\partial x_k}-\frac{\partial f(u)}{\partial x_k}\right|dx\\
&\leq\sum_{k=1}^{n}\frac{1}{V(B)}\int_B|\beta_k-\alpha_k x_k|\sum_{j=1}^{n}L_{k,j}\,|x_j-u_j|\,dx\ \text{(by (5.1))}\\
&=\sum_{k=1}^{n}\sum_{j=1}^{n}L_{k,j}\frac{1}{V(B)}\int_B|\beta_k-\alpha_k x_k|\,|x_j-u_j|\,dx
\end{aligned}$$

and the inequality (5.2) is proved. □

Corollary 6. *With the assumptions of Theorem 4 we have*

$$\begin{aligned}
&\left|\frac{1}{V(B)}\int_B f(x)\,dx-\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\alpha_k\left(x_k-\overline{x_{B,k}}\right)f(x)\,n_k(x)\,dA\right|\\
&\leq\sum_{k=1}^{n}\sum_{j=1}^{n}L_{k,j}\,|\alpha_k|\,\frac{1}{V(B)}\int_B\left|\overline{x_{B,k}}-x_k\right|\,|x_j-u_j|\,dx
\end{aligned}\tag{5.5}$$

and, in particular,

$$\begin{aligned}
&\left|\frac{1}{V(B)}\int_B f(x)\,dx-\frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\left(x_k-\overline{x_{B,k}}\right)f(x)\,n_k(x)\,dA\right|\\
&\leq\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{n}L_{k,j}\frac{1}{V(B)}\int_B\left|\overline{x_{B,k}}-x_k\right|\,|x_j-u_j|\,dx.
\end{aligned}\tag{5.6}$$

Corollary 7. *With the assumptions of Theorem 4 we have*

$$\begin{aligned}
&\left|\frac{1}{V(B)}\int_B f(x)\,dx-\sum_{k=1}^{n}\alpha_k\frac{\partial f(u)}{\partial x_k}\left(x_{\partial B,f,k}-\overline{x_{B,k}}\right)\right|\\
&\leq\sum_{k=1}^{n}\sum_{j=1}^{n}L_{k,j}\,|\alpha_k|\,\frac{1}{V(B)}\int_B\left|x_{\partial B,f,k}-x_k\right|\,|x_j-u_j|\,dx
\end{aligned}\tag{5.7}$$

and, in particular,

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx-\frac{1}{n}\sum_{k=1}^{n}\frac{\partial f(u)}{\partial x_k}\left(x_{\partial B,f,k}-\overline{x_{B,k}}\right)\right| \le \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{n}L_{k,j}\frac{1}{V(B)}\int_B\left|x_{\partial B,f,k}-x_k\right|\left|x_j-u_j\right|dx. \tag{5.8}$$

Corollary 8. *With the assumptions of Theorem 4 we have*

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx-\sum_{k=1}^{n}\alpha_k\frac{\partial f(u)}{\partial x_k}\left(u_k-\overline{x_{B,k}}\right)-\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\alpha_k(x_k-u_k)f(x)\,n_k(x)\,dA\right| \le \sum_{k=1}^{n}\sum_{j=1}^{n}L_{k,j}\left|\alpha_k\right|\frac{1}{V(B)}\int_B\left|u_k-x_k\right|\left|x_j-u_j\right|dx \tag{5.9}$$

and, in particular,

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx-\frac{1}{n}\sum_{k=1}^{n}\frac{\partial f(u)}{\partial x_k}\left(u_k-\overline{x_{B,k}}\right)-\frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k-u_k)f(x)\,n_k(x)\,dA\right| \le \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{n}L_{k,j}\frac{1}{V(B)}\int_B\left|u_k-x_k\right|\left|x_j-u_j\right|dx. \tag{5.10}$$

Remark 1. *With the assumptions of Theorem 4 and for* $G=\left(\overline{x_{B,1}},...,\overline{x_{B,n}}\right)\in B$ *there exist* $M_{k,j}>0,\ k,\ j\in\{1,...,n\}$ *such that the Lipschitz conditions*

$$\left|\frac{\partial f(x)}{\partial x_k}-\frac{\partial f(G)}{\partial x_k}\right|\le\sum_{j=1}^{n}M_{k,j}\left|x_j-u_j\right| \tag{5.11}$$

hold for $k\in\{1,...,n\}$, *then*

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx-\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\alpha_k\left(x_k-\overline{x_{B,k}}\right)f(x)\,n_k(x)\,dA\right| \le \sum_{k=1}^{n}\sum_{j=1}^{n}L_{k,j}\left|\alpha_k\right|\frac{1}{V(B)}\int_B\left|\overline{x_{B,k}}-x_k\right|\left|x_j-\overline{x_{B,j}}\right|dx \tag{5.12}$$

and, in particular,

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx-\frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\left(x_k-\overline{x_{B,k}}\right)f(x)\,n_k(x)\,dA\right| \le \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{n}L_{k,j}\frac{1}{V(B)}\int_B\left|\overline{x_{B,k}}-x_k\right|\left|x_j-\overline{x_{B,j}}\right|dx. \tag{5.13}$$

It is well known that if a function g has bounded partial derivatives on B, which is assumed also to be convex, then for all x, $y \in B$ we have the Lipschitz type condition

$$|g(x) - g(y)| \leq \sum_{j=1}^{n} \left\| \frac{\partial g}{\partial x_j} \right\|_{B,\infty} |x_j - y_j|$$

where

$$\left\| \frac{\partial g}{\partial x_j} \right\|_{B,\infty} := \sup_{x \in B} \left| \frac{\partial g(x)}{\partial x_j} \right| < \infty.$$

We can state the following result that is more convenient to apply:

Corollary 9. *Let B be a bounded closed convex subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary ∂B. Let f be a twice differentiable function defined in $\mathbb{R}^n$, or at least in on open neighborhood of B and with complex values and assume that*

$$\left\| \frac{\partial^2 f}{\partial x_k \partial x_j} \right\|_{B,\infty} := \sup_{x \in B} \left| \frac{\partial^2 f(x)}{\partial x_k \partial x_j} \right| < \infty$$

for all k, $j \in \{1, ..., n\}$. For $j = k$ we denote, as usual $\frac{\partial^2 f}{\partial x_k \partial x_k} = \frac{\partial^2 f}{\partial^2 x_k}$, $k \in \{1, ..., n\}$. Then for all $u \in B$ we have

$$\begin{aligned} &\left| \frac{1}{V(B)} \int_B f(x)\, dx - \sum_{k=1}^{n} \alpha_k \frac{\partial f(u)}{\partial x_k} \left(u_k - \overline{x_{B,k}}\right) \right. \\ &\left. - \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} \alpha_k (x_k - u_k) f(x)\, n_k(x)\, dA \right| \\ &\leq \sum_{k=1}^{n} \sum_{j=1}^{n} \left\| \frac{\partial^2 f}{\partial x_k \partial x_j} \right\|_{B,\infty} |\alpha_k| \frac{1}{V(B)} \int_B |u_k - x_k| |x_j - u_j|\, dx \end{aligned} \tag{5.14}$$

and, in particular,

$$\begin{aligned} &\left| \frac{1}{V(B)} \int_B f(x)\, dx - \frac{1}{n} \sum_{k=1}^{n} \frac{\partial f(u)}{\partial x_k} \left(u_k - \overline{x_{B,k}}\right) \right. \\ &\left. - \frac{1}{n} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - u_k) f(x)\, n_k(x)\, dA \right| \\ &\leq \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{n} \left\| \frac{\partial^2 f}{\partial x_k \partial x_j} \right\|_{B,\infty} \frac{1}{V(B)} \int_B |u_k - x_k| |x_j - u_j|\, dx. \end{aligned} \tag{5.15}$$

We also have the centre of gravity inequality

$$\begin{aligned} &\left| \frac{1}{V(B)} \int_B f(x)\, dx - \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} \alpha_k \left(x_k - \overline{x_{B,k}}\right) f(x)\, n_k(x)\, dA \right| \\ &\leq \sum_{k=1}^{n} \sum_{j=1}^{n} |\alpha_k| \left\| \frac{\partial^2 f}{\partial x_k \partial x_j} \right\|_{B,\infty} \frac{1}{V(B)} \int_B \left|\overline{x_{B,k}} - x_k\right| \left|x_j - \overline{x_{B,k}}\right| dx \end{aligned} \tag{5.16}$$

and, in particular,

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx-\frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\left(x_k-\overline{x_{B,k}}\right)f(x)\,n_k(x)\,dA\right| \tag{5.17}$$
$$\leq\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\left\|\frac{\partial^2 f}{\partial x_k\partial x_j}\right\|_{B,\infty}\frac{1}{V(B)}\int_B\left|\overline{x_{B,k}}-x_k\right|\left|x_j-\overline{x_{B,k}}\right|dx.$$

6. Example for 3-Dimensional Spaces

Let B be a bounded closed convex subset of $\mathbb{R}^3$ with smooth (or piecewise smooth) boundary ∂B. Let f be a twice differentiable function defined in $\mathbb{R}^3$, or at least in on open neighborhood of B and with complex values and assume that

$$\left\|\frac{\partial^2 f}{\partial x_k\partial x_j}\right\|_{B,\infty}:=\sup_{x\in B}\left|\frac{\partial^2 f(x)}{\partial x_k\partial x_j}\right|<\infty$$

for all $k,\ j\in\{1,...,3\}$.

Consider a surface described by the vector equation

$$r(u,v)=x_1(u,v)\overrightarrow{i}+x_2(u,v)\overrightarrow{j}+x_3(u,v)\overrightarrow{k}$$

where $(u,v)\in[a,b]\times[c,d]$. Then, by using the notations from the second section, we have

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx-\sum_{k=1}^{3}\alpha_k\frac{\partial f(y_1,y_2,y_3)}{\partial x_k}\left(y_k-\overline{x_{B,k}}\right)\right. \tag{6.1}$$
$$-\frac{1}{V(B)}\int_a^b\int_c^d\alpha_1\left(x_1(u,v)-y_1\right)f\left(x_1(u,v),x_2(u,v),x_3(u,v)\right)\frac{\partial(x_2,x_3)}{\partial(u,v)}dudv$$
$$-\frac{1}{V(B)}\int_a^b\int_c^d\alpha_2\left(x_2(u,v)-y_2\right)f\left(x_1(u,v),x_2(u,v),x_3(u,v)\right)\frac{\partial(x_3,x_1)}{\partial(u,v)}dudv$$
$$\left.-\frac{1}{V(B)}\int_a^b\int_c^d\alpha_3\left(x_3(u,v)-y_3\right)f\left(x_1(u,v),x_2(u,v),x_3(u,v)\right)\frac{\partial(x_1,x_2)}{\partial(u,v)}dudv\right|$$
$$\leq\sum_{k=1}^{3}\sum_{j=1}^{3}|\alpha_k|\left\|\frac{\partial^2 f}{\partial x_k\partial x_j}\right\|_{B,\infty}\frac{1}{V(B)}\int_B|y_k-x_k|\,|x_j-y_j|\,dx$$

for all $(y_1,y_2,y_3)\in B$ and $\alpha_k\in\mathbb{C},\ k\in\{1,...,3\}$, with $\alpha_1+\alpha_2+\alpha_3=1$.

In particular, we have

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx-\frac{1}{3}\sum_{k=1}^{3}\frac{\partial f(y_1,y_2,y_3)}{\partial x_k}\left(y_k-\overline{x_{B,k}}\right)\right. \tag{6.2}$$
$$-\frac{1}{3V(B)}\int_a^b\int_c^d\left(x_1(u,v)-y_1\right)f\left(x_1(u,v),x_2(u,v),x_3(u,v)\right)\frac{\partial(x_2,x_3)}{\partial(u,v)}dudv$$
$$-\frac{1}{3V(B)}\int_a^b\int_c^d\left(x_2(u,v)-y_2\right)f\left(x_1(u,v),x_2(u,v),x_3(u,v)\right)\frac{\partial(x_3,x_1)}{\partial(u,v)}dudv$$
$$\left.-\frac{1}{3V(B)}\int_a^b\int_c^d\left(x_3(u,v)-y_3\right)f\left(x_1(u,v),x_2(u,v),x_3(u,v)\right)\frac{\partial(x_1,x_2)}{\partial(u,v)}dudv\right|$$
$$\leq\frac{1}{3}\sum_{k=1}^{3}\sum_{j=1}^{3}\left\|\frac{\partial^2 f}{\partial x_k\partial x_j}\right\|_{B,\infty}\frac{1}{V(B)}\int_B|y_k-x_k|\,|x_j-y_j|\,dx$$

for all $(y_1,y_2,y_3)\in B$.

We also have the *centre of gravity inequalities*

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx\right. \tag{6.3}$$
$$-\frac{1}{V(B)}\int_a^b\int_c^d\alpha_1\left(x_1(u,v)-\overline{x_{B,1}}\right)f\left(x_1(u,v),x_2(u,v),x_3(u,v)\right)\frac{\partial(x_2,x_3)}{\partial(u,v)}dudv$$
$$-\frac{1}{V(B)}\int_a^b\int_c^d\alpha_2\left(x_2(u,v)-\overline{x_{B,2}}\right)f\left(x_1(u,v),x_2(u,v),x_3(u,v)\right)\frac{\partial(x_3,x_1)}{\partial(u,v)}dudv$$
$$\left.-\frac{1}{V(B)}\int_a^b\int_c^d\alpha_3\left(x_3(u,v)-\overline{x_{B,3}}\right)f\left(x_1(u,v),x_2(u,v),x_3(u,v)\right)\frac{\partial(x_1,x_2)}{\partial(u,v)}dudv\right|$$
$$\leq\sum_{k=1}^{3}\sum_{j=1}^{3}|\alpha_k|\left\|\frac{\partial^2 f}{\partial x_k\partial x_j}\right\|_{B,\infty}\frac{1}{V(B)}\int_B\left|\overline{x_{B,k}}-x_k\right|\left|x_j-\overline{x_{B,j}}\right|dx$$

or all $\alpha_k\in\mathbb{C}$, $k\in\{1,...,3\}$ with $\alpha_1+\alpha_2+\alpha_3=1$.

In particular,

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx\right. \tag{6.4}$$
$$-\frac{1}{3V(B)}\int_a^b\int_c^d\left(x_1(u,v)-\overline{x_{B,1}}\right)f\left(x_1(u,v),x_2(u,v),x_3(u,v)\right)\frac{\partial(x_2,x_3)}{\partial(u,v)}dudv$$
$$-\frac{1}{3V(B)}\int_a^b\int_c^d\left(x_2(u,v)-\overline{x_{B,2}}\right)f\left(x_1(u,v),x_2(u,v),x_3(u,v)\right)\frac{\partial(x_3,x_1)}{\partial(u,v)}dudv$$
$$\left.-\frac{1}{3V(B)}\int_a^b\int_c^d\left(x_3(u,v)-\overline{x_{B,3}}\right)f\left(x_1(u,v),x_2(u,v),x_3(u,v)\right)\frac{\partial(x_1,x_2)}{\partial(u,v)}dudv\right|$$

$$\leq \frac{1}{3}\frac{1}{V(B)}\left[\left\|\frac{\partial^2 f}{\partial^2 x_1}\right\|_{B,\infty}\int_B \left(\overline{x_{B,1}}-x_1\right)^2 dx\right.$$

$$+\left\|\frac{\partial^2 f}{\partial^2 x_2}\right\|_{B,\infty}\int_B \left(\overline{x_{B,2}}-x_2\right)^2 dx+\left.\left\|\frac{\partial^2 f}{\partial^2 x_3}\right\|_{B,\infty}\int_B \left(\overline{x_{B,3}}-x_3\right)^2 dx\right]$$

$$+\frac{2}{3}\frac{1}{V(B)}\left[\left\|\frac{\partial^2 f}{\partial x_1\partial x_2}\right\|_{B,\infty}\int_B \left|\overline{x_{B,1}}-x_1\right|\left|x_2-\overline{x_{B,2}}\right| dx\right.$$

$$+\left\|\frac{\partial^2 f}{\partial x_2\partial x_3}\right\|_{B,\infty}\int_B \left|\overline{x_{B,2}}-x_2\right|\left|x_3-\overline{x_{B,3}}\right| dx$$

$$\left.+\left\|\frac{\partial^2 f}{\partial x_1\partial x_3}\right\|_{B,\infty}\int_B \left|\overline{x_{B,1}}-x_1\right|\left|x_3-\overline{x_{B,3}}\right| dx\right].$$

7. Example for 3-Dimensional Balls

Consider the 3-dimensional ball centered in $C=(a,b,c)$ and having the radius $R>0$,

$$B(C,R):=\left\{(x,y,z)\in\mathbb{R}^3 \middle| (x-a)^2+(y-b)^2+(z-c)^2\leq R^2\right\}$$

and the sphere

$$S(C,R):=\left\{(x,y,z)\in\mathbb{R}^3 \middle| (x-a)^2+(y-b)^2+(z-c)^2= R^2\right\}.$$

Consider the parametrization of $B(C,R)$ and $S(C,R)$ given by:

$$B(C,R):\begin{cases} x=r\cos\psi\cos\varphi+a\\ y=r\cos\psi\sin\varphi+b\\ z=r\sin\psi+c\end{cases};\quad (r,\psi,\varphi)\in[0,R]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi]$$

and

$$S(C,R):\begin{cases} x=R\cos\psi\cos\varphi+a\\ y=R\cos\psi\sin\varphi+b\\ z=R\sin\psi+c\end{cases};\quad (\psi,\varphi)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi].$$

Observe that

$$\begin{vmatrix}\frac{\partial y}{\partial\psi} & \frac{\partial z}{\partial\psi}\\ \frac{\partial y}{\partial\varphi} & \frac{\partial z}{\partial\varphi}\end{vmatrix}=-R^2\cos^2\psi\cos\varphi,$$

$$\begin{vmatrix}\frac{\partial x}{\partial\psi} & \frac{\partial z}{\partial\psi}\\ \frac{\partial x}{\partial\varphi} & \frac{\partial z}{\partial\varphi}\end{vmatrix}=R^2\cos^2\psi\sin\varphi,$$

and

$$\begin{vmatrix}\frac{\partial x}{\partial\psi} & \frac{\partial y}{\partial\psi}\\ \frac{\partial x}{\partial\varphi} & \frac{\partial y}{\partial\varphi}\end{vmatrix}=-R^2\sin\psi\cos\psi.$$

Let us consider the transformation $T_2:\mathbb{R}^3\to\mathbb{R}^3$ given by:

$$T_2(r,\psi,\varphi):=(r\cos\psi\cos\varphi+a, r\cos\psi\sin\varphi+b, r\sin\psi+c).$$

It is well known that the Jacobian of T_2 is

$$J(T_2)=r^2\cos\psi$$

and T_2 is a one-to-one mapping defined on the interval of $\mathbb{R}^3$, $[0,R]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi]$, with values in the ball $B(C,R)$ from $\mathbb{R}^3$. Thus we have the change of variable:

$$\iiint_{B(C,R)} f(x,y,z)\,dxdydz \tag{7.1}$$
$$=\int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} f(r\cos\psi\cos\varphi+a,r\cos\psi\sin\varphi+b,r\sin\psi+c)\,r^2\cos\psi drd\psi d\varphi.$$

We also have

$$\begin{aligned}&\iiint_{B(C,R)}\left|z-\overline{z_{B(C,R)}}\right|^2 dxdydz\\&=\int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} r^2\sin^2\psi r^2\cos\psi drd\psi d\varphi\\&=\int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} r^4\sin^2\psi\cos\psi drd\psi d\varphi=\frac{4}{15}\pi R^5\end{aligned}$$

and, similarly

$$\iiint_{B(C,R)}\left|x-\overline{x_{B(C,R)}}\right|^2 dxdydz=\iiint_{B(C,R)}\left|y-\overline{y_{B(C,R)}}\right|^2 dxdydz=\frac{4}{15}\pi R^5.$$

Also

$$\begin{aligned}&\iiint_{B(C,R)}\left|x-\overline{x_{B(C,R)}}\right|\left|y-\overline{y_{B(C,R)}}\right| dxdydz\\&=\int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}|r\cos\psi\cos\varphi|\,|r\cos\psi\sin\varphi|\,r^2\cos\psi drd\psi d\varphi\\&=\int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} r^4\cos^3\psi\,|\sin\varphi\cos\varphi|\,drd\psi d\varphi=\frac{8}{15}R^5\end{aligned}$$

and, similarly

$$\begin{aligned}&\iiint_{B(C,R)}\left|x-\overline{x_{B(C,R)}}\right|\left|z-\overline{z_{B(C,R)}}\right| dxdydz\\&=\iiint_{B(C,R)}\left|y-\overline{y_{B(C,R)}}\right|\left|z-\overline{z_{B(C,R)}}\right| dxdydz=\frac{8}{15}R^5.\end{aligned}$$

Since $V(B(C,R))=\frac{4\pi R^3}{3}$, then by (6.4) we get

$$\begin{aligned}(7.2)\quad &\left|\frac{1}{\frac{4\pi R^3}{3}}\int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} f(r\cos\psi\cos\varphi+a,r\cos\psi\sin\varphi+b,r\sin\psi+c)\right.\\
&\qquad\times r^2\cos\psi drd\psi d\varphi\\
&+\frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} f(R\cos\psi\cos\varphi+a,R\cos\psi\sin\varphi+b,R\sin\psi+c)\cos^3\psi\cos^2\varphi d\psi d\varphi\\
&-\frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} f(R\cos\psi\cos\varphi+a,R\cos\psi\sin\varphi+b,R\sin\psi+c)\cos^3\psi\sin^2\varphi d\psi d\varphi\\
&\left.+\frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} f(R\cos\psi\cos\varphi+a,R\cos\psi\sin\varphi+b,R\sin\psi+c)\sin^2\psi\cos\psi d\psi d\varphi\right|\\
&\le\frac{1}{15}R^2\left[\left\|\frac{\partial^2 f}{\partial^2 x}\right\|_{B(C,R),\infty}+\left\|\frac{\partial^2 f}{\partial^2 y}\right\|_{B(C,R),\infty}+\left\|\frac{\partial^2 f}{\partial^2 z}\right\|_{B(C,R),\infty}\right]\\
&\qquad+\frac{4}{15\pi}R^2\left[\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{B(C,R),\infty}+\left\|\frac{\partial^2 f}{\partial y\partial z}\right\|_{B(C,R),\infty}+\left\|\frac{\partial^2 f}{\partial z\partial x}\right\|_{B(C,R),\infty}\right].\end{aligned}$$

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