PERTURBED OSTROWSKI TYPE INEQUALITIES FOR MULTIPLE INTEGRAL ON GENERAL DOMAINS

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Abstract. In this paper, by making use of Divergence Theorem for multiple integral, we establish some perturbed Ostrowski type inequalities for functions of independent variables defined on closed and bounded convex subsets of the Euclidian space $\mathbb{R}^n$. Some examples for 3-dimensional space are also provided.

1. Introduction

Let $f$ be a complex valued function defined on an open set containing $D \subset \mathbb{R}^2$ and having continuous partial derivatives on $D$. We assume that the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ satisfy the Lipschitz type conditions

$$
\left| \frac{\partial f}{\partial x} (x,y) - \frac{\partial f}{\partial x} (u,v) \right| \leq L_1 |x - u| + K_1 |y - v|
$$

and

$$
\left| \frac{\partial f}{\partial y} (x,y) - \frac{\partial f}{\partial y} (u,v) \right| \leq L_2 |x - u| + K_2 |y - v|
$$

for any $(x,y), (u,v) \in D$, where $L_1, K_1, L_2$ and $K_2$ are given positive constants.

In the recent paper [8] we obtained among others the following result:

Theorem 1. Let $\partial D$ be a simple, closed counterclockwise curve bounding a convex region $D$ and $f$ defined on an open set containing $D$ and having continuous partial derivatives on $D$. If the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ satisfy the Lipschitz type conditions (1.1) and (1.2), then

$$
\left| \frac{1}{A_D} \iint_D f(x,y) \, dx \, dy - \frac{1}{3} f(u,v) - \frac{2}{3} \frac{1}{A_D} \iint_{\partial D} \left[ \frac{(v-y)f(x,y) \, dx + (x-u)f(x,y) \, dy}{2} \right] \right|
$$

$$
\leq \frac{1}{6} \left[ L_1 \frac{1}{A_D} \iint_D (x-u)^2 \, dx \, dy + K_2 \frac{1}{A_D} \iint_D (y-v)^2 \, dx \, dy \right]
$$

$$
+ \frac{K_1 + L_2}{6} \frac{1}{A_D} \iint_D |x-u||y-v| \, dx \, dy
$$

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and

\[ \frac{1}{A_D} \int_D \int_D \left[ (u - x) \frac{\partial f}{\partial x} (x, y) + (v - y) \frac{\partial f}{\partial y} (x, y) \right] dxdy \]

\[ - \frac{2}{3} f(u, v) - \frac{1}{3} \frac{1}{A_D} \int_D \left[ (y - v) f(x, y) dx + (u - x) f(x, y) dy \right] \]

\[ \leq \frac{1}{3} \left[ L_1 \frac{1}{A_D} \int_D (x - u)^2 dxdy + K_2 \frac{1}{A_D} \int_D (y - v)^2 dxdy \right] \]

\[ + \frac{K_1 + L_2}{3} \frac{1}{A_D} \int_D \int_D |x - u| |y - v| dxdy \]

for all \((u, v) \in D\). 

For other integral inequalities for multiple integrals see [3]-[16].

In this paper, by making use of Divergence Theorem for multiple integral, we establish some perturbed Ostrowski type inequalities for functions of independent variables defined on closed and bounded convex subsets of the Euclidean space \(\mathbb{R}^n\). Some examples for 3-dimensional space are also provided.

2. SOME PRELIMINARY FACTS

Let \(B\) be a bounded open subset of \(\mathbb{R}^n\) \((n \geq 2)\) with smooth (or piecewise smooth) boundary \(\partial B\). Let \(F = (F_1, ..., F_n)\) be a smooth vector field defined in \(\mathbb{R}^n\), or at least in \(B \cup \partial B\). Let \(n\) be the unit outward-pointing normal of \(\partial B\). Then the Divergence Theorem states, see for instance [17]:

\[ \int_B \text{div} F dV = \int_{\partial B} F \cdot n dA, \]

where

\[ \text{div} F = \nabla \cdot F = \sum_{k=1}^n \frac{\partial F_k}{\partial x_k}. \]

d\(V\) is the element of volume in \(\mathbb{R}^n\) and \(dA\) is the element of surface area on \(\partial B\).

If \(n = (n_1, ..., n_n), x = (x_1, ..., x_n) \in B\) and use the notation \(dx\) for \(dV\) we can write (2.1) more explicitly as

\[ \sum_{k=1}^n \int_B \frac{\partial F_k(x)}{\partial x_k} dx = \sum_{k=1}^n \int_{\partial B} F_k(x) n_k(x) dA. \]

By taking the real and imaginary part, we can extend the above equality for complex valued functions \(F_k, k \in \{1, ..., n\}\) defined on \(B\).

If \(n = 2\), the normal is obtained by rotating the tangent vector through \(90^\circ\) (in the correct direction so that it points out). The quantity \(tds\) can be written \((dx_1, dx_2)\) along the surface, so that

\[ ndA := nds = (dx_2, -dx_1). \]

Here \(t\) is the tangent vector along the boundary curve and \(ds\) is the element of arc-length.
From (2.2) we get for $B \subset \mathbb{R}^2$ that

\begin{equation}
(2.3) \quad \int_B \frac{\partial F_1(x_1, x_2)}{\partial x_1} dx_1 dx_2 + \int_B \frac{\partial F_2(x_1, x_2)}{\partial x_2} dx_1 dx_2
\end{equation}

\begin{equation}
= \int_{\partial B} F_1(x_1, x_2) dx_2 - \int_{\partial B} F_2(x_1, x_2) dx_1,
\end{equation}

which is Green’s theorem in plane.

If $n = 3$ and if $\partial B$ is described as a level-set of a function of 3 variables i.e. $\partial B = \{x_1, x_2, x_3 \in \mathbb{R}^3 \mid G(x_1, x_2, x_3) = 0\}$, then a vector pointing in the direction of $\mathbf{n}$ is $\nabla G$. We shall use the case where $G(x_1, x_2, x_3) = x_3 - g(x_1, x_2)$, $(x_1, x_2) \in D$, a domain in $\mathbb{R}^2$ for some differentiable function $g$ on $D$ and $B$ corresponds to the inequality $x_3 < g(x_1, x_2)$, namely

$$B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 < g(x_1, x_2)\}.$$

Then

$$\mathbf{n} = \frac{(-g_{x_1}, -g_{x_2}, 1)}{(1 + g_{x_1}^2 + g_{x_2}^2)^{1/2}}, \quad dA = (1 + g_{x_1}^2 + g_{x_2}^2)^{1/2} dx_1 dx_2$$

and

$$\mathbf{n} dA = (-g_{x_1}, -g_{x_2}, 1) dx_1 dx_2.$$

From (2.2) we get

\begin{equation}
(2.4) \quad \int_B \left( \frac{\partial F_1(x_1, x_2, x_3)}{\partial x_1} + \frac{\partial F_2(x_1, x_2, x_3)}{\partial x_2} + \frac{\partial F_3(x_1, x_2, x_3)}{\partial x_3} \right) dx_1 dx_2 dx_3
\end{equation}

\begin{equation}
= - \int_D F_1(x_1, x_2, g(x_1, x_2)) g_{x_1}(x_1, x_2) dx_1 dx_2
\end{equation}

\begin{equation}
- \int_D F_1(x_1, x_2, g(x_1, x_2)) g_{x_2}(x_1, x_2) dx_1 dx_2
\end{equation}

\begin{equation}
+ \int_D F_3(x_1, x_2, g(x_1, x_2)) dx_1 dx_2,
\end{equation}

which is the Gauss-Ostrogradsky theorem in space.

Following Apostol [1], consider a surface described by the vector equation

$$\mathbf{r}(u, v) = x_1(u, v) \mathbf{i} + x_2(u, v) \mathbf{j} + x_3(u, v) \mathbf{k}$$

where $(u, v) \in [a, b] \times [c, d]$.

If $x_1, x_2, x_3$ are differentiable on $[a, b] \times [c, d]$ we consider the two vectors

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x_1}{\partial u} \mathbf{i} + \frac{\partial x_2}{\partial u} \mathbf{j} + \frac{\partial x_3}{\partial u} \mathbf{k}$$

and

$$\frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x_1}{\partial v} \mathbf{i} + \frac{\partial x_2}{\partial v} \mathbf{j} + \frac{\partial x_3}{\partial v} \mathbf{k}.$$

The cross product of these two vectors $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ will be referred to as the fundamental vector product of the representation $\mathbf{r}$. Its components can be expressed as Jacobian
\textit{Determinants.} In fact, we have \cite[1, p. 420]{dragomir}
\[
\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = \begin{vmatrix}
\frac{\partial x_2}{\partial u} & \frac{\partial x_3}{\partial u} \\
\frac{\partial x_2}{\partial v} & \frac{\partial x_3}{\partial v} \\
\frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v}
\end{vmatrix} \hat{i} + \begin{vmatrix}
\frac{\partial x_2}{\partial u} & \frac{\partial x_3}{\partial u} \\
\frac{\partial x_2}{\partial v} & \frac{\partial x_3}{\partial v} \\
\frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v}
\end{vmatrix} \hat{j} + \begin{vmatrix}
\frac{\partial x_2}{\partial u} & \frac{\partial x_3}{\partial u} \\
\frac{\partial x_2}{\partial v} & \frac{\partial x_3}{\partial v} \\
\frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v}
\end{vmatrix} \hat{k}.
\]

Let \( \partial B = r(T) \) be a parametric surface described by a vector-valued function \( r \) defined on the box \( T = [a, b] \times [c, d] \). The area of \( \partial B \) denoted \( A_{\partial B} \) is defined by the double integral \cite[p. 424-425]{dragomir}
\[
A_{\partial B} = \int_a^b \int_c^d \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| dudv
\]
\[
= \int_a^b \int_c^d \sqrt{\left( \frac{\partial (x_2, x_3)}{\partial (u, v)} \right)^2 + \left( \frac{\partial (x_3, x_1)}{\partial (u, v)} \right)^2 + \left( \frac{\partial (x_1, x_2)}{\partial (u, v)} \right)^2} dudv.
\]

We define surface integrals in terms of a parametric representation for the surface. One can prove that under certain general conditions the value of the integral is independent of the representation.

Let \( \partial B = r(T) \) be a parametric surface described by a vector-valued differentiable function \( r \) defined on the box \( T = [a, b] \times [c, d] \) and let \( f : \partial B \to \mathbb{R} \) is defined and bounded on \( \partial B \). The surface integral of \( f \) over \( \partial B \) is defined by \cite[p. 430]{dragomir}
\[
\int \int_{\partial B} f dA = \int_a^b \int_c^d f(x_1, x_2, x_3) \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| dudv
\]
\[
= \int_a^b \int_c^d f(x_1(u, v), x_2(u, v), x_3(u, v)) \times \sqrt{\left( \frac{\partial (x_2, x_3)}{\partial (u, v)} \right)^2 + \left( \frac{\partial (x_3, x_1)}{\partial (u, v)} \right)^2 + \left( \frac{\partial (x_1, x_2)}{\partial (u, v)} \right)^2} dudv.
\]

If \( \partial B = r(T) \) is a parametric surface, the fundamental vector product \( N = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \) is normal to \( \partial B \) at each regular point of the surface. At each such point there are two unit normals, a unit normal \( \mathbf{n}_1 \), which has the same direction as \( N \), and a unit normal \( \mathbf{n}_2 \) which has the opposite direction. Thus
\[
\mathbf{n}_1 = \frac{N}{\| N \|} \quad \text{and} \quad \mathbf{n}_2 = -\mathbf{n}_1.
\]

Let \( \mathbf{n} \) be one of the two normals \( \mathbf{n}_1 \) or \( \mathbf{n}_2 \). Let also \( F \) be a vector field defined on \( \partial B \) and assume that the surface integral,
\[
\int \int_{\partial B} (F \cdot \mathbf{n}) dA,
\]
called the flux surface integral, exists. Here \( F \cdot \mathbf{n} \) is the dot or inner product.

We can write \cite[p. 434]{dragomir}
\[
\int \int_{\partial B} (F \cdot \mathbf{n}) dA = \pm \int_a^b \int_c^d F(r(u, v)) \cdot \left( \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right) dudv
\]
where the sign " + " is used if \( \mathbf{n} = \mathbf{n}_1 \) and the " − " sign is used if \( \mathbf{n} = \mathbf{n}_2 \).
If
\[ F(x_1, x_2, x_3) = F_1(x_1, x_2, x_3) \vec{i} + F_2(x_1, x_2, x_3) \vec{j} + F_3(x_1, x_2, x_3) \vec{k} \]
and
\[ r(u, v) = x_1(u, v) \vec{i} + x_2(u, v) \vec{j} + x_3(u, v) \vec{k} \]
where \((u, v) \in [a, b] \times [c, d]\)
then the flux surface integral for \(n = n_1\) can be explicitly calculated as [1, p. 435]

\[
\begin{align*}
\int_{\partial B} (F \cdot n) dA &= \int_a^b \int_c^d F_1(x_1(u, v), x_2(u, v), x_3(u, v)) \frac{\partial (x_2, x_3)}{\partial (u, v)} dudv \\
&\quad + \int_a^b \int_c^d F_2(x_1(u, v), x_2(u, v), x_3(u, v)) \frac{\partial (x_3, x_1)}{\partial (u, v)} dudv \\
&\quad + \int_a^b \int_c^d F_3(x_1(u, v), x_2(u, v), x_3(u, v)) \frac{\partial (x_1, x_2)}{\partial (u, v)} dudv.
\end{align*}
\]

The sum of the double integrals on the right is often written more briefly as [1, p. 435]

\[
\begin{align*}
\int_{\partial B} F_1(x_1, x_2, x_3) \, dx_2 \wedge dx_3 &+ \int_{\partial B} F_2(x_1, x_2, x_3) \, dx_3 \wedge dx_1 \\
&+ \int_{\partial B} F_3(x_1, x_2, x_3) \, dx_1 \wedge dx_2
\end{align*}
\]

Let \(B \subset \mathbb{R}^3\) be a solid in 3-space bounded by an orientable closed surface \(\partial B\), and let \(n\) be the unit outer normal to \(\partial B\). If \(F\) is a continuously differentiable vector field defined on \(B\), we have the Gauss-Ostrogradsky identity

\[(GO)\quad \iiint_B (\text{div } F) \, dV = \int_{\partial B} (F \cdot n) \, dA.\]

If we express
\[ F(x_1, x_2, x_3) = F_1(x_1, x_2, x_3) \vec{i} + F_2(x_1, x_2, x_3) \vec{j} + F_3(x_1, x_2, x_3) \vec{k}, \]
then (2.4) can be written as

\[
\begin{align*}
\iiint_B \left( \frac{\partial F_1}{\partial x_1}(x_1, x_2, x_3) + \frac{\partial F_2}{\partial x_2}(x_1, x_2, x_3) + \frac{\partial F_3}{\partial x_3}(x_1, x_2, x_3) \right) dx_1 dx_2 dx_3 &= \int_{\partial B} F_1(x_1, x_2, x_3) \, dx_2 \wedge dx_3 + \int_{\partial B} F_2(x_1, x_2, x_3) \, dx_3 \wedge dx_1 \\
&\quad + \int_{\partial B} F_3(x_1, x_2, x_3) \, dx_1 \wedge dx_2.
\end{align*}
\]

3. Identities of Interest

**Lemma 1.** Let \(B\) be a bounded closed subset of \(\mathbb{R}^n \ (n \geq 2)\) with smooth (or piecewise smooth) boundary \(\partial B\). Let \(f\) be a continuously differentiable function
defined in \( \mathbb{R}^n \), or at least in an open neighborhood of \( B \) and with complex values. If \( \alpha_k, \gamma_k \in \mathbb{C} \) for \( k \in \{1, \ldots, n\} \) with \( \sum_{k=1}^{n} \alpha_k = 1 \), then

\[
\int_B f(x) \, dx = \sum_{k=1}^{n} \alpha_k \int_B (\gamma_k - x_k) \frac{\partial f(x)}{\partial x_k} \, dx
\]

(3.1)

\[
+ \sum_{k=1}^{n} \alpha_k \int_{\partial B} (x_k - \gamma_k) \, f(x) \, n_k(x) \, dA.
\]

In particular,

\[
\int_B f(x) \, dx = 1 \sum_{k=1}^{n} \int_B (\gamma_k - x_k) \frac{\partial f(x)}{\partial x_k} \, dx
\]

(3.2)

\[
+ 1 \sum_{k=1}^{n} \int_{\partial B} (x_k - \gamma_k) \, f(x) \, n_k(x) \, dA.
\]

If \( u = (u_1, \ldots, u_n) \in B \), then

\[
1 \frac{1}{V(B)} \int_B f(x) \, dx = 1 \sum_{k=1}^{n} \frac{1}{V(B)} \int_B (u_k - x_k) \frac{\partial f(x)}{\partial x_k} \, dx
\]

(3.3)

\[
+ 1 \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - u_k) \, f(x) \, n_k(x) \, dA.
\]

Proof. Let \( x = (x_1, \ldots, x_n) \in B \). We consider

\[F_k(x) = \alpha_k (x_k - \gamma_k) f(x), \quad k \in \{1, \ldots, n\}\]

and take the partial derivatives \( \frac{\partial F_k(x)}{\partial x_k} \) to get

\[
\frac{\partial F_k(x)}{\partial x_k} = \alpha_k f(x) + \alpha_k (x_k - \gamma_k) \frac{\partial f(x)}{\partial x_k}, \quad k \in \{1, \ldots, n\}.
\]

If we sum this equality over \( k \) from 1 to \( n \) we get

\[
\sum_{k=1}^{n} \frac{\partial F_k(x)}{\partial x_k} = \sum_{k=1}^{n} \alpha_k f(x) + \sum_{k=1}^{n} \alpha_k (x_k - \gamma_k) \frac{\partial f(x)}{\partial x_k}
\]

(3.7)

\[
= f(x) + \sum_{k=1}^{n} \alpha_k (x_k - \gamma_k) \frac{\partial f(x)}{\partial x_k}
\]

for all \( x = (x_1, \ldots, x_n) \in B \).

Now, if we take the integral in the equality (3.7) over \( (x_1, \ldots, x_n) \in B \) we get

\[
\int_B \left( \sum_{k=1}^{n} \frac{\partial F_k(x)}{\partial x_k} \right) \, dx = \int_B f(x) \, dx + \sum_{k=1}^{n} \int_B \alpha_k (x_k - \gamma_k) \frac{\partial f(x)}{\partial x_k} \, dx.
\]

(3.8)

By the Divergence Theorem (2.2) we also have

\[
\int_B \left( \sum_{k=1}^{n} \frac{\partial F_k(x)}{\partial x_k} \right) \, dx = \sum_{k=1}^{n} \int_{\partial B} \alpha_k (x_k - \gamma_k) \, f(x) \, n_k(x) \, dA
\]

(3.9)
and by making use of (3.8) and (3.9) we get
\[
\int_B f(x) \, dx + \sum_{k=1}^n \alpha_k \int_B (x_k - \gamma_k) ~ \frac{\partial f(x)}{\partial x_k} \, dx = \sum_{k=1}^n \alpha_k \int_{\partial B} (x_k - \gamma_k) f(x) \, n_k(x) \, dA,
\]
which gives the desired representation (3.1). The identity (3.3) follows by (3.1) for 
\[ k = 1, n; \quad k \in \{1, \ldots, n\}. \]
The identity (3.5) follows by (3.3) for 
\[ k = u \in \{1, \ldots, n\}. \]

For the body \( B \) we consider the coordinates for the centre of gravity 
\[ G = (\bar{x}_B, \bar{y}_B, \ldots, \bar{x}_B, \bar{z}_B) \]
defined by 
\[ \bar{x}_{B,k} := \frac{1}{V(B)} \int_B x_k \, dx, \quad k \in \{1, \ldots, n\}, \]
where \( V(B) := \int_B x \, dx \) is the volume of \( B \).

**Corollary 1.** With the assumptions of Theorem 2 we have
\[
\begin{align*}
\int_B f(x) \, dx &= \sum_{k=1}^n \int_B \alpha_k (\bar{x}_{B,k} - x_k) \frac{\partial f(x)}{\partial x_k} \, dx \\
&\quad + \sum_{k=1}^n \int_{\partial B} \alpha_k (x_k - \bar{x}_{B,k}) f(x) \, n_k(x) \, dA,
\end{align*}
\]
and, in particular,
\[
\begin{align*}
\int_B f(x) \, dx &= \frac{1}{n} \sum_{k=1}^n \int_B (\bar{x}_{B,k} - x_k) \frac{\partial f(x)}{\partial x_k} \, dx \\
&\quad + \frac{1}{n} \sum_{k=1}^n \int_{\partial B} (x_k - \bar{x}_{B,k}) f(x) \, n_k(x) \, dA.
\end{align*}
\]
The proof follows by (3.1) on taking \( \gamma_k = \bar{x}_{B,k}, \quad k \in \{1, \ldots, n\}. \)

For a function \( f \) as in Theorem 2 above, we define the points 
\[ x_{B,\partial f,k} := \frac{\int_B x_k \frac{\partial f(x)}{\partial x_k} \, dx}{\int_B \frac{\partial f(x)}{\partial x_k} \, dx}, \quad k \in \{1, \ldots, n\}, \]
provided that all denominators are not zero.

**Corollary 2.** With the assumptions of Theorem 2 we have
\[
\begin{align*}
\int_B f(x) \, dx &= \sum_{k=1}^n \int_{\partial B} \alpha_k (x_k - x_{B,\partial f,k}) f(x) \, n_k(x) \, dA \\
&\quad + \frac{1}{n} \sum_{k=1}^n \int_{\partial B} (x_k - x_{B,\partial f,k}) f(x) \, n_k(x) \, dA.
\end{align*}
\]
The proof follows by (3.1) on taking \( \gamma_k = x_{B,f,k} \), \( k \in \{1, ..., n\} \) and observing that
\[
\sum_{k=1}^{n} \alpha_k \int_B (x_{B,f,k} - x_k) \frac{\partial f (x)}{\partial x_k} dx = 0.
\]

For a function \( f \) as in Theorem 2 above, we define the points
\[
x_{\partial B,f,k} := \int_{\partial B} f (x) n_k (x) dA, k \in \{1, ..., n\}
\]
provided that all denominators are not zero.

**Corollary 3.** With the assumptions of Theorem 2 we have
\[
(3.16) \quad \int_B f (x) dx = \sum_{k=1}^{n} \int_B \alpha_k (x_{\partial B,f,k} - x_k) \frac{\partial f (x)}{\partial x_k} dx
\]
and, in particular,
\[
(3.17) \quad \int_B f (x) dx = \frac{1}{n} \sum_{k=1}^{n} \int_B (x_{\partial B,f,k} - x_k) \frac{\partial f (x)}{\partial x_k} dx.
\]

The proof follows by (3.1) on taking \( \gamma_k = x_{\partial B,f,k} \), \( k \in \{1, ..., n\} \) and observing that
\[
\sum_{k=1}^{n} \int_{\partial B} \alpha_k (x_k - x_{\partial B,f,k}) f (x) n_k (x) dA = 0.
\]

We have:

**Lemma 2.** If \( f : B \rightarrow \mathbb{C} \) is differentiable on \( B \), a convex subset of \( \mathbb{R}^n \) (\( n \geq 2 \)), then for all \( x = (x_1, ..., x_n) \), \( u = (u_1, ..., u_n) \in B \) and \( \lambda_k \in \mathbb{C} \), \( k \in \{1, ..., n\} \) we have the equality
\[
(3.18) \quad f (u) = f (x) + \sum_{k=1}^{n} (u_k - x_k) \lambda_k
\]
\[
+ \sum_{k=1}^{n} (u_k - x_k) \int_0^1 \left( \frac{\partial f (x)}{\partial x_k} [tu + (1 - t) x] - \lambda_k \right) dt.
\]
In particular,
\[
(3.19) \quad f (u) = f (x) + \sum_{k=1}^{n} (u_k - x_k) \frac{\partial f (x)}{\partial x_k}
\]
\[
+ \sum_{k=1}^{n} (u_k - x_k) \int_0^1 \left( \frac{\partial f (x)}{\partial x_k} [tu + (1 - t) x] - \frac{\partial f (x)}{\partial x_k} \right) dt
\]
and
\[
(3.20) \quad f (u) = \frac{1}{V (B)} \int_B f (x) dx + \sum_{k=1}^{n} \frac{1}{V (B)} \int_B (u_k - x_k) \frac{\partial f (x)}{\partial x_k} dx
\]
\[
+ \sum_{k=1}^{n} \frac{1}{V (B)} \int_B \left( [u_k - x_k] \int_0^1 \left( \frac{\partial f (x)}{\partial x_k} [tu + (1 - t) x] - \frac{\partial f (x)}{\partial x_k} \right) dt \right) dx.
\]
Proof. By Taylor’s multivariate theorem with integral remainder, we have

\begin{equation}
(3.21) \quad f(u) = f(x) + \sum_{k=1}^{n} (u_k - x_k) \int_{0}^{1} \frac{\partial f}{\partial x_k} [tu + (1 - t)x] \, dt
\end{equation}

for all \( x = (x_1, ..., x_n) \), \( u = (u_1, ..., u_n) \) \( \in B \).

If \( \lambda_k \in \mathbb{C}, k \in \{1, ..., n\} \), then

\[
(u_k - x_k) \int_{0}^{1} \left( \frac{\partial f}{\partial x_k} [tu + (1 - t)x] - \lambda_k \right) \, dt
\]

\[=(u_k - x_k) \int_{0}^{1} \frac{\partial f}{\partial x_k} [tu + (1 - t)x] \, dt - (u_k - x_k) \lambda_k
\]

and by (3.21) we get the desired result (3.18).

The identity (3.19) follows by (3.18) on taking \( \lambda_k = \frac{\partial f(x)}{\partial x_k} \), \( k \in \{1, ..., n\} \), while the identity (3.20) follows by (3.19) on taking the integral mean \( \frac{1}{V(B)} \int_{B} \) over the variable \( x \).

We have the following equalities that will be used in the sequel:

**Theorem 2.** Let \( B \) be a bounded closed convex subset of \( \mathbb{R}^n \) \( (n \geq 2) \) with smooth (or piecewise smooth) boundary \( \partial B \). Let \( f \) be a continuously differentiable function defined in \( \mathbb{R}^n \), or at least in an open neighborhood of \( B \) and with complex values. Then for all \( u = (u_1, ..., u_n) \) \( \in B \) we have

\begin{equation}
(3.22) \quad \frac{1}{V(B)} \int_{B} f(x) \, dx
\end{equation}

\[= \frac{1}{n+1} f(u) + \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - u_k) f(x) n_k(x) \, dA
\]

\[+ \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)}
\]

\[\times \int_{B} \left[ (x_k - u_k) \int_{0}^{1} \left( \frac{\partial f}{\partial x_k} [tu + (1 - t)x] - \frac{\partial f(x)}{\partial x_k} \right) \, dt \right] \, dx
\]

and

\begin{equation}
(3.23) \quad \sum_{k=1}^{n} \frac{1}{V(B)} \int_{B} (u_k - x_k) \frac{\partial f(x)}{\partial x_k} \, dx
\end{equation}

\[= \frac{n}{n+1} f(u) + \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (u_k - x_k) f(x) n_k(x) \, dA
\]

\[+ \frac{n}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)}
\]

\[\times \int_{B} \left[ (x_k - u_k) \int_{0}^{1} \left( \frac{\partial f}{\partial x_k} [tu + (1 - t)x] - \frac{\partial f(x)}{\partial x_k} \right) \, dt \right] \, dx.
\]
Proof. From the equality (3.5) we have

\[
\sum_{k=1}^{n} \frac{1}{V(B)} \int_{B} (u_k - x_k) \frac{\partial f(x)}{\partial x_k} dx
\]

\[= \frac{n}{V(B)} \int_{B} f(x) dx - \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - u_k) f(x) n_k(x) dA.
\]

By replacing this into (3.20) we get

\[
f(u) = \frac{1}{V(B)} \int_{B} f(x) dx + \frac{n}{V(B)} \int_{B} f(x) dx
\]

\[-\sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - u_k) f(x) n_k(x) dA
\]

\[+ \sum_{k=1}^{n} \frac{1}{V(B)} \int_{B} \left[(u_k - x_k) \int_{0}^{1} \left( \frac{\partial f}{\partial x_k} [tu + (1-t)x] - \frac{\partial f(x)}{\partial x_k} \right) dt \right] dx
\]

\[= \frac{n+1}{V(B)} \int_{B} f(x) dx - \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - u_k) f(x) n_k(x) dA
\]

\[+ \sum_{k=1}^{n} \frac{1}{V(B)} \int_{B} \left[(u_k - x_k) \int_{0}^{1} \left( \frac{\partial f}{\partial x_k} [tu + (1-t)x] - \frac{\partial f(x)}{\partial x_k} \right) dt \right] dx,
\]

namely

\[
\frac{n+1}{V(B)} \int_{B} f(x) dx
\]

\[= f(u) + \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - u_k) f(x) n_k(x) dA
\]

\[+ \sum_{k=1}^{n} \frac{1}{V(B)} \int_{B} \left[(x_k - u_k) \int_{0}^{1} \left( \frac{\partial f}{\partial x_k} [tu + (1-t)x] - \frac{\partial f(x)}{\partial x_k} \right) dt \right] dx
\]

which is equivalent to (3.22).

Further, if we replace \(\frac{1}{V(B)} \int_{B} f(x) dx\) from (3.5) into (3.20) we get

\[
f(u) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{B} (u_k - x_k) \frac{\partial f(x)}{\partial x_k} dx
\]

\[+ \sum_{k=1}^{n} \frac{1}{V(B)} \int_{B} (u_k - x_k) \frac{\partial f(x)}{\partial x_k} dx
\]

\[+ \frac{1}{n} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - u_k) f(x) n_k(x) dA
\]

\[+ \sum_{k=1}^{n} \frac{1}{V(B)} \int_{B} \left[(u_k - x_k) \int_{0}^{1} \left( \frac{\partial f}{\partial x_k} [tu + (1-t)x] - \frac{\partial f(x)}{\partial x_k} \right) dt \right] dx
\]
\[
= \frac{n + 1}{n} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{B} (u_k - x_k) \frac{\partial f(x)}{\partial x_k} \, dx \\
+ \frac{1}{n} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - u_k) f(x) n_k(x) \, dA \\
+ \sum_{k=1}^{n} \frac{1}{V(B)} \int_{B} \left[ (u_k - x_k) \int_{0}^{1} \left( \frac{\partial f}{\partial x_k} [tu + (1 - t)x] - \frac{\partial f(x)}{\partial x_k} \right) \, dt \right] \, dx,
\]
which gives

\[
\frac{n + 1}{n} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{B} (u_k - x_k) \frac{\partial f(x)}{\partial x_k} \, dx \\
= f(u) + \frac{1}{n} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (u_k - x_k) f(x) n_k(x) \, dA \\
+ \sum_{k=1}^{n} \frac{1}{V(B)} \int_{B} \left[ (x_k - u_k) \int_{0}^{1} \left( \frac{\partial f}{\partial x_k} [tu + (1 - t)x] - \frac{\partial f(x)}{\partial x_k} \right) \, dt \right] \, dx
\]

namely, the desired identity (3.23). \qed

**Corollary 4.** With the assumptions of Theorem 2 we have the following inequalities concerning the centre of gravity \(G = (\bar{x}_{B,1}, \ldots, \bar{x}_{B,n})\)

(3.24) \[
\frac{1}{V(B)} \int_{B} f(x) \, dx \\
= \frac{1}{n + 1} f(G) + \frac{1}{n + 1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (\bar{x}_{B,k} - x_k) f(x) n_k(x) \, dA \\
+ \frac{1}{n + 1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{B} \left[ (x_k - \bar{x}_{B,k}) \int_{0}^{1} \left( \frac{\partial f}{\partial x_k} [tG + (1 - t)x] - \frac{\partial f(x)}{\partial x_k} \right) \, dt \right] \, dx
\]

and

(3.25) \[
\frac{1}{\sum_{k=1}^{n} \frac{1}{V(B)} \int_{B} (\bar{x}_{B,k} - x_k) \frac{\partial f(x)}{\partial x_k} \, dx \\
= \frac{1}{n + 1} f(G) + \frac{1}{n + 1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (\bar{x}_{B,k} - x_k) f(x) n_k(x) \, dA \\
+ \frac{1}{n + 1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{B} \left[ (x_k - \bar{x}_{B,k}) \int_{0}^{1} \left( \frac{\partial f}{\partial x_k} [tG + (1 - t)x] - \frac{\partial f(x)}{\partial x_k} \right) \, dt \right] \, dx.
\]

We consider the point

\(S = (x_{\partial B,1}, \ldots, x_{\partial B,n})\)
where
\[ x_{\partial B,f,k} := \int_{\partial B} x_k f(x) n_k(x) \, dA, \quad k \in \{1,\ldots,n\}. \]

**Corollary 5.** With the assumptions of Theorem 2 and if \( S \supset \subset B \), then we have the equalities

\[
\frac{1}{V(B)} \int_B f(x) \, dx = \frac{1}{n+1} f(S) + \frac{1}{n+1} \sum_{k=1}^n \frac{1}{V(B)} \times \int_B \left[ (x_k - x_{\partial B,f,k}) \int_0^1 \left( \frac{\partial f}{\partial x_k} [tS + (1-t) x] - \frac{\partial f}{\partial x_k} \right) dt \right] dx
\]

and

\[
\sum_{k=1}^n \frac{1}{V(B)} \int_B (x_{\partial B,f,k} - x_k) \frac{\partial f}{\partial x_k} \, dx = \frac{n}{n+1} f(S) + \frac{n}{n+1} \sum_{k=1}^n \frac{1}{V(B)} \times \int_B \left[ (x_k - x_{\partial B,f,k}) \int_0^1 \left( \frac{\partial f}{\partial x_k} [tS + (1-t) x] - \frac{\partial f}{\partial x_k} \right) dt \right] dx.
\]

**4. SOME INEQUALITIES**

We assume that the partial derivatives \( \frac{\partial f}{\partial x_k}, \quad k \in \{1,\ldots,n\} \) satisfy the Lipschitz type conditions

\[
\left| \frac{\partial f(x)}{\partial x_k} - \frac{\partial f(u)}{\partial x_k} \right| \leq \sum_{j=1}^n L_{kj} |x_j - u_j|
\]

for all \( x = (x_1,\ldots,x_n), \quad u = (u_1,\ldots,u_n) \in B \) where \( L_{kj} > 0, \quad k, j \in \{1,\ldots,n\} \) are given positive constants.

We have the following generalization for \( n \)-dimensional spaces of the result from Theorem 1:

**Theorem 3.** Let \( B \) be a bounded closed convex subset of \( \mathbb{R}^n \) (\( n \geq 2 \)) with smooth (or piecewise smooth) boundary \( \partial B \). Let \( f \) be a continuously differentiable function defined in \( \mathbb{R}^n \), or at least in an open neighborhood of \( B \) and with complex values and assume that the partial derivatives \( \frac{\partial f}{\partial x_k}, \quad k \in \{1,\ldots,n\} \) satisfy the Lipschitz type conditions (4.1). Then for all \( u = (u_1,\ldots,u_n) \in B \) we have the inequalities

\[
\left| \frac{1}{V(B)} \int_B f(x) \, dx - \frac{1}{n+1} f(u) \right| \leq \frac{1}{n+1} \sum_{k=1}^n \frac{1}{V(B)} \left| \int_{\partial B} (x_k - u_k) f(x) n_k(x) \, dA \right|
\]

\[
\leq \frac{1}{2(n+1)} \sum_{k=1}^n \sum_{j=1}^n L_{kj} \frac{1}{V(B)} \int_B |x_k - u_k| |x_j - u_j| \, dx.
\]
and

\[
(4.3) \quad \left| \sum_{k=1}^{n} \frac{1}{V(B)} \int_{B} (u_k - x_k) \frac{\partial f(x)}{\partial x_k} \, dx - \frac{n}{n+1} f(u) \right|
\]

\[
- \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (u_k - x_k) f(x) n_k(x) \, dA
\]

\[
\leq \frac{n}{2(n+1)} \sum_{k=1}^{n} \sum_{j=1}^{n} L_{kj} \frac{1}{V(B)} \int_{B} |x_k - u_k| |x_j - u_j| \, dx.
\]

**Proof.** Using the identity (3.22) we have

\[
\left| \frac{1}{V(B)} \int_{B} f(x) \, dx - \frac{1}{n+1} f(u) \right|
\]

\[
- \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - u_k) f(x) n_k(x) \, dA
\]

\[
\leq \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)}
\]

\[
\times \left| \int_{B} [(x_k - u_k) \int_{0}^{1} \left( \frac{\partial f}{\partial x_k} [tu + (1-t)x] - \frac{\partial f(x)}{\partial x_k} \right) dt] \, dx \right|
\]

\[
\leq \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)}
\]

\[
\times \int_{B} [(x_k - u_k) \int_{0}^{1} \left( \frac{\partial f}{\partial x_k} [tu + (1-t)x] - \frac{\partial f(x)}{\partial x_k} \right) dt] \, dx
\]

\[
\leq \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)}
\]

\[
\times \int_{B} |x_k - u_k| \left( \int_{0}^{1} \left| \frac{\partial f}{\partial x_k} [tu + (1-t)x] - \frac{\partial f(x)}{\partial x_k} \right| dt \right) \, dx
\]

\[=: B
\]

Since \( \frac{\partial f}{\partial x_k} \), \( k \in \{1, \ldots, n\} \) satisfy the Lipschitz type conditions (4.1), hence

\[
\int_{0}^{1} \left| \frac{\partial f}{\partial x_k} [tu + (1-t)x] - \frac{\partial f(x)}{\partial x_k} \right| dt \leq \sum_{j=1}^{n} L_{kj} \int_{0}^{1} |tu_j + (1-t)x_j - x_j| \, dt
\]

\[
= \frac{1}{2} \sum_{j=1}^{n} L_{kj} |x_j - u_j|
\]

for all \( x = (x_1, \ldots, x_n) \), \( u = (u_1, \ldots, u_n) \) \( \in B \).
Therefore
\[
B \leq \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{B} |x_k - u_k| \left( \frac{1}{2} \sum_{j=1}^{n} L_{kj} |x_j - u_j| \right) dx
\]
\[
= \frac{1}{2(n+1)} \sum_{k=1}^{n} \sum_{j=1}^{n} L_{kj} \frac{1}{V(B)} \int_{B} |x_k - u_k| |x_j - u_j| dx,
\]
which proves the desired result (4.2).

The inequality (4.3) follows in a similar way by (3.23). \(\square\)

If \(f : B \to \mathbb{C}\) is twice differentiable on the convex set \(B\) and the second partial derivatives \(\frac{\partial^2 f}{\partial x_k \partial x_j}\) for \(k, j \in \{1, \ldots, n\}\) are bounded on \(B\) (for \(j = k\) we denote \(\frac{\partial^2 f}{\partial x_k \partial x_j} = \frac{\partial^2 f}{\partial x_k^2}\)), namely
\[
\left\| \frac{\partial^2 f}{\partial x_k \partial x_j} \right\|_{B, \infty} := \sup_{x \in B} \left| \frac{\partial^2 f(x)}{\partial x_k \partial x_j} \right| < \infty,
\]
then \(\frac{\partial f}{\partial x_k}, k \in \{1, \ldots, n\}\) satisfy the Lipschitz type conditions (4.1) for \(L_{kj} = \left\| \frac{\partial^2 f}{\partial x_k \partial x_j} \right\|_{B, \infty}\), where \(k, j \in \{1, \ldots, n\}\). We observe that for \(k \neq j\), \(L_{kj} = L_{jk}\).

Corollary 6. Let \(B\) be a bounded closed convex subset of \(\mathbb{R}^n\) \((n \geq 2)\) with smooth (or piecewise smooth) boundary \(\partial B\). Let \(f\) be a twice differentiable function defined in \(\mathbb{R}^n\), or at least in an open neighborhood of \(B\) and with complex values and such that the second partial derivatives \(\frac{\partial^2 f}{\partial x_k \partial x_j}\) for \(k, j \in \{1, \ldots, n\}\) are bounded on \(B\). Then for all \(u = (u_1, \ldots, u_n) \in B\) we have the inequalities

\[
\frac{1}{V(B)} \int_{B} f(x) dx - \frac{1}{n+1} f(u)
\]
\[
= \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - u_k) f(x) n_k(x) dA
\]
\[
\leq \frac{1}{2(n+1)} \sum_{k=1}^{n} \left\| \frac{\partial^2 f}{\partial x_k^2} \right\|_{B, \infty} \frac{1}{V(B)} \int_{B} (x_k - u_k)^2 dx
\]
\[
+ \frac{1}{n+1} \sum_{1 \leq k < j \leq n} \left\| \frac{\partial^2 f}{\partial x_k \partial x_j} \right\|_{B, \infty} \frac{1}{V(B)} \int_{B} |x_k - u_k| |x_j - u_j| dx.
\]

and

\[
\frac{n}{V(B)} \int_{B} (u_k - x_k) \frac{\partial f(x)}{\partial x_k} dx - \frac{n}{n+1} f(u)
\]
\[
= \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (u_k - x_k) f(x) n_k(x) dA
\]
\[
\leq \frac{n}{2(n+1)} \sum_{k=1}^{n} \left\| \frac{\partial^2 f}{\partial x_k^2} \right\|_{B, \infty} \frac{1}{V(B)} \int_{B} (x_k - u_k)^2 dx
\]
\[
+ \frac{n}{n+1} \sum_{1 \leq k < j \leq n} \left\| \frac{\partial^2 f}{\partial x_k \partial x_j} \right\|_{B, \infty} \frac{1}{V(B)} \int_{B} |x_k - u_k| |x_j - u_j| dx.
\]
Remark 1. If we take in Corollary 6 \( u \) to be the centre of gravity \( G = (\overline{x}_{B,1}, \ldots, \overline{x}_{B,n}) \), then we get the inequalities

\[
(4.6) \quad \left| \frac{1}{V(B)} \int_B f(x) \, dx - \frac{1}{n+1} f(G) \right|
- \frac{1}{n+1} \sum_{k=1}^n \frac{1}{V(B)} \int_{\partial B} (x_k - \overline{x}_{B,k}) f(x) n_k(x) \, dA
\leq \frac{1}{2(n+1)} \sum_{k=1}^n \left\| \frac{\partial^2 f}{\partial x_k^2} \right\|_{B,\infty} \frac{1}{V(B)} \int_B (x_k - \overline{x}_{B,k})^2 \, dx
+ \frac{1}{(n+1)} \sum_{1 \leq k < j \leq n} \left\| \frac{\partial^2 f}{\partial x_k \partial x_j} \right\|_{B,\infty} \frac{1}{V(B)} \int_B |x_k - \overline{x}_{B,k}| |x_j - \overline{x}_{B,j}| \, dx
\]

and

\[
(4.7) \quad \left| \frac{1}{V(B)} \int_B (\overline{x}_{B,k} - x_k) \frac{\partial f(x)}{\partial x_k} \, dx - \frac{n}{n+1} f(G) \right|
- \frac{1}{n+1} \sum_{k=1}^n \frac{1}{V(B)} \int_{\partial B} (\overline{x}_{B,k} - x_k) f(x) n_k(x) \, dA
\leq \frac{n}{2(n+1)} \sum_{k=1}^n \left\| \frac{\partial^2 f}{\partial x_k^2} \right\|_{B,\infty} \frac{1}{V(B)} \int_B (\overline{x}_{B,k} - x_k)^2 \, dx
+ \frac{n}{(n+1)} \sum_{1 \leq k < j \leq n} \left\| \frac{\partial^2 f}{\partial x_k \partial x_j} \right\|_{B,\infty} \frac{1}{V(B)} \int_B |\overline{x}_{B,k} - x_k| |x_j - \overline{x}_{B,j}| \, dx.
\]

Also, if we take in Corollary 6 \( u \) to be \( S = (\overline{x}_{A,f,1}, \ldots, \overline{x}_{A,f,n}) \), then we get

\[
(4.8) \quad \left| \frac{1}{V(B)} \int_B f(x) \, dx - \frac{1}{n+1} f(S) \right|
\leq \frac{1}{2(n+1)} \sum_{k=1}^n \left\| \frac{\partial^2 f}{\partial x_k^2} \right\|_{B,\infty} \frac{1}{V(B)} \int_B (x_k - \overline{x}_{A,f,k})^2 \, dx
+ \frac{1}{(n+1)} \sum_{1 \leq k < j \leq n} \left\| \frac{\partial^2 f}{\partial x_k \partial x_j} \right\|_{B,\infty} \int_B |x_k - \overline{x}_{A,f,k}| |x_j - \overline{x}_{A,f,j}| \, dx
\]

and

\[
(4.9) \quad \left| \sum_{k=1}^n \frac{1}{V(B)} \int_B (\overline{x}_{A,f,k} - x_k) \frac{\partial f(x)}{\partial x_k} \, dx - \frac{n}{n+1} f(S) \right|
\leq \frac{n}{2(n+1)} \sum_{k=1}^n \left\| \frac{\partial^2 f}{\partial x_k^2} \right\|_{B,\infty} \frac{1}{V(B)} \int_B (\overline{x}_{A,f,k} - x_k)^2 \, dx
+ \frac{n}{(n+1)} \sum_{1 \leq k < j \leq n} \left\| \frac{\partial^2 f}{\partial x_k \partial x_j} \right\|_{B,\infty} \int_B |\overline{x}_{A,f,k} - x_k| |x_j - \overline{x}_{A,f,j}| \, dx.
\]
5. Example for 3-Dimensional Spaces

Let $B$ be a bounded closed convex subset of $\mathbb{R}^3$ with smooth (or piecewise smooth) boundary $\partial B$. Let $f$ be a twice differentiable function defined in $\mathbb{R}^3$, or at least in on open neighborhood of $B$ and with complex values and assume that

$$\left\| \frac{\partial^2 f}{\partial x_k \partial x_j} \right\|_{B,\infty} := \sup_{x \in B} \left| \frac{\partial^2 f(x)}{\partial x_k \partial x_j} \right| < \infty$$

for all $k, j \in \{1, ..., 3\}$.

Consider a surface described by the vector equation

$$r(u, v) = x_1(u, v) \hat{i} + x_2(u, v) \hat{j} + x_3(u, v) \hat{k}$$

where $(u, v) \in [a, b] \times [c, d]$. Then, by using the notations from the second section, we have from (4.4) that

\begin{align*}
(5.1) & \quad \left| \frac{1}{V(B)} \int_B f(x) \, dx - \frac{1}{4} f(y) \right| \\
& \leq \frac{1}{8} \sum_{k=1}^{3} \left\| \frac{\partial^2 f}{\partial x_k^2} \right\|_{B,\infty} \frac{1}{V(B)} \int_B (x_k - y_k)^2 \, dx \\
& \quad + \frac{1}{4} \sum_{1 \leq k < j \leq 3} \left\| \frac{\partial^2 f}{\partial x_k \partial x_j} \right\|_{B,\infty} \frac{1}{V(B)} \int_B |x_k - y_k| |x_j - y_j| \, dx
\end{align*}

for all $y = (y_1, y_2, y_3) \in B$.

If we take $y = (\bar{x}_{B,1}, \bar{x}_{B,2}, \bar{x}_{B,3})$ in (5.1), then we get

\begin{align*}
(5.2) & \quad \left| \frac{1}{V(B)} \int_B f(x) \, dx - \frac{1}{4} f(\bar{x}_{B,1}, \bar{x}_{B,2}, \bar{x}_{B,3}) \right| \\
& \leq \frac{1}{8} \sum_{k=1}^{3} \left\| \frac{\partial^2 f}{\partial x_k^2} \right\|_{B,\infty} \frac{1}{V(B)} \int_B (x_k - \bar{x}_{B,k})^2 \, dx \\
& \quad + \frac{1}{4} \sum_{1 \leq k < j \leq 3} \left\| \frac{\partial^2 f}{\partial x_k \partial x_j} \right\|_{B,\infty} \frac{1}{V(B)} \int_B |x_k - \bar{x}_{B,k}| |x_j - \bar{x}_{B,j}| \, dx.
\end{align*}
Consider the 3-dimensional ball centered in \( C = (a, b, c) \) and having the radius \( R > 0 \),
\[
B(C, R) := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1 - a)^2 + (x_2 - b)^2 + (x_3 - c)^2 \leq R^2 \right\}
\]
and the sphere
\[
S(C, R) := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1 - a)^2 + (x_2 - b)^2 + (x_3 - c)^2 = R^2 \right\}.
\]
Consider the parametrization of \( B(C, R) \) and \( S(C, R) \) given by:
\[
B(C, R) : \begin{cases} 
  x_1 = r \cos \psi \cos \varphi + a \\
  x_2 = r \cos \psi \sin \varphi + b \\
  x_3 = r \sin \psi + c
\end{cases} \quad ; \quad (r, \psi, \varphi) \in [0, R] \times \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \times [0, 2\pi]
\]
and
\[
S(C, R) : \begin{cases} 
  x_1 = R \cos \psi \cos \varphi + a \\
  x_2 = R \cos \psi \sin \varphi + b \\
  x_3 = R \sin \psi + c
\end{cases} \quad ; \quad (\psi, \varphi) \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \times [0, 2\pi].
\]
Observe that
\[
\begin{vmatrix}
  \frac{\partial x_2}{\partial \psi} & \frac{\partial x_2}{\partial \varphi} \\
  \frac{\partial x_3}{\partial \psi} & \frac{\partial x_3}{\partial \varphi}
\end{vmatrix} = -R^2 \cos^2 \psi \cos \varphi,
\]
\[
\begin{vmatrix}
  \frac{\partial x_1}{\partial \psi} & \frac{\partial x_1}{\partial \varphi} \\
  \frac{\partial x_3}{\partial \psi} & \frac{\partial x_3}{\partial \varphi}
\end{vmatrix} = R^2 \cos^2 \psi \sin \varphi,
\]
and
\[
\begin{vmatrix}
  \frac{\partial x_1}{\partial \psi} & \frac{\partial x_2}{\partial \varphi} \\
  \frac{\partial x_2}{\partial \psi} & \frac{\partial x_2}{\partial \varphi}
\end{vmatrix} = -R^2 \sin \psi \cos \psi.
\]
Let us consider the transformation \( T_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) given by:
\[
T_2 (r, \psi, \varphi) := (r \cos \psi \cos \varphi + a, r \cos \psi \sin \varphi + b, r \sin \psi + c).
\]
It is well known that the Jacobian of \( T_2 \) is
\[
J (T_2) = r^2 \cos \psi
\]
and \( T_2 \) is a one-to-one mapping defined on the interval of \( \mathbb{R}^3, [0, R] \times \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \times [0, 2\pi] \), with values in the ball \( B(C, R) \) from \( \mathbb{R}^3 \). Thus we have the change of variable:
\[
(6.1) \quad \iiint_{B(C, R)} f (x_1, x_2, x_3) \, dx_1 \, dx_2 \, dx_3
= \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} f (r \cos \psi \cos \varphi + a, r \cos \psi \sin \varphi + b, r \sin \psi + c) \, r^2 \cos \psi \, dr \, d\psi \, d\varphi.
\]
Moreover

\[
\frac{1}{4V(B)} \int_a^b \int_c^d (x_1(u,v) - \frac{\mathbf{x}_B,1}{R}) f(x_1(u,v), x_2(u,v), x_3(u,v)) \frac{\partial (x_2, x_3)}{\partial (u,v)} dudv
\]

\[
= - \frac{3}{16\pi} \int_a^b \int_c^d f(R \cos \psi \cos \varphi + a, R \cos \psi \sin \varphi + b, R \sin \psi + c)
\]

\[
\times \cos^3 \psi \cos^2 \varphi d\psi d\varphi,
\]

\[
\frac{1}{4V(B)} \int_a^b \int_c^d (x_2(u,v) - \frac{\mathbf{x}_B,2}{R}) f(x_1(u,v), x_2(u,v), x_3(u,v)) \frac{\partial (x_3, x_1)}{\partial (u,v)} dudv
\]

\[
= \frac{3}{16\pi} \int_a^b \int_c^d f(R \cos \psi \cos \varphi + a, R \cos \psi \sin \varphi + b, R \sin \psi + c)
\]

\[
\times \cos^3 \psi \sin^2 \varphi d\psi d\varphi
\]

and

\[
\frac{1}{4V(B)} \int_a^b \int_c^d (x_3(u,v) - \frac{\mathbf{x}_B,3}{R}) f(x_1(u,v), x_2(u,v), x_3(u,v)) \frac{\partial (x_1, x_2)}{\partial (u,v)} dudv
\]

\[
= - \frac{3}{16\pi} \int_a^b \int_c^d f(R \cos \psi \cos \varphi + a, R \cos \psi \sin \varphi + b, R \sin \psi + c)
\]

\[
\times \sin^2 \psi \cos \psi d\psi d\varphi
\]

We also have

\[
\iiint_{B(C,R)} |x_3 - \frac{\mathbf{x}_B(C,R)}{R}|^2 dx_1 dx_2 dx_3
\]

\[
= \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} r^2 \sin^2 \psi \cos^2 \varphi dr d\psi d\varphi
\]

\[
= \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} r^4 \sin^2 \psi \cos \varphi dr d\psi d\varphi = \frac{4}{15} \pi R^5
\]

and, similarly

\[
\iiint_{B(C,R)} |x_1 - \frac{\mathbf{x}_B(C,R)}{R}|^2 dx_1 dx_2 dx_3 = \iiint_{B(C,R)} |x_2 - \frac{\mathbf{x}_B(C,R)}{R}|^2 dx_1 dx_2 dx_3
\]

\[
= \frac{4}{15} \pi R^5.
\]

Also

\[
\iiint_{B(C,R)} |x_1 - \frac{\mathbf{x}_B(C,R)}{R}| |x_2 - \frac{\mathbf{x}_B(C,R)}{R}| dx_1 dx_2 dx_3
\]

\[
= \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} |r \cos \psi \cos \varphi| |r \cos \psi \sin \varphi| r^2 \cos \psi dr d\psi d\varphi
\]

\[
= \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} r^4 \cos^3 \psi |\sin \varphi \cos \varphi| dr d\psi d\varphi = \frac{8}{15} R^5
\]
and, similarly

\[
\iiint_{B(C, R)} |x_1 - \frac{x_1}{2B(C, R)}| |x_3 - \frac{x_3}{2B(C, R)}| dx_1 dx_2 dx_3 = \frac{8}{15} R^5.
\]

Then by (5.2) we get

\[
(6.2) \quad \frac{3}{4\pi R^3} \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} f (r \cos \psi \cos \phi + a, r \cos \psi \sin \phi + b, r \sin \psi + c) \times r^2 \cos \psi dr d\psi d\phi - \frac{1}{4} \int_{a, b, c} f (a, b, c)
\]

\[+ \frac{3}{16\pi} \int_a^b \int_c^d \int_0^{2\pi} f (R \cos \psi \cos \phi + a, R \cos \psi \sin \phi + b, R \sin \psi + c) \times \cos^3 \psi \cos^2 \phi d\psi d\phi
\]

\[\quad - \frac{3}{16\pi} \int_a^b \int_c^d \int_0^{2\pi} f (R \cos \psi \cos \phi + a, R \cos \psi \sin \phi + b, R \sin \psi + c) \times \cos^3 \psi \sin^2 \phi d\psi d\phi
\]

\[\quad + \frac{3}{16\pi} \int_a^b \int_c^d \int_0^{2\pi} f (R \cos \psi \cos \phi + a, R \cos \psi \sin \phi + b, R \sin \psi + c) \times \sin^2 \psi \cos \psi d\psi d\phi
\]

\[\leq \frac{1}{40} R^2 \left[ \left\| \frac{\partial^2 f}{\partial x^2} \right\|_{B(C, R), \infty} + \left\| \frac{\partial^2 f}{\partial y^2} \right\|_{B(C, R), \infty} + \left\| \frac{\partial^2 f}{\partial z^2} \right\|_{B(C, R), \infty} \right]
\]

\[+ \frac{1}{10\pi} R^2 \left[ \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_{B(C, R), \infty} + \left\| \frac{\partial^2 f}{\partial y \partial z} \right\|_{B(C, R), \infty} + \left\| \frac{\partial^2 f}{\partial z \partial x} \right\|_{B(C, R), \infty} \right].
\]

### References


