PERTURBED OSTROWSKI TYPE INEQUALITIES FOR MULTIPLE INTEGRAL ON GENERAL DOMAINS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper, by making use of Divergence Theorem for multiple integral, we establish some perturbed Ostrowski type inequalities for functions of independent variables defined on closed and bounded convex subsets of the Euclidian space $\mathbb{R}^n$. Some examples for 3-dimensional space are also provided.

1. Introduction

Let f be a complex valued function defined on an open set containing $D \subset \mathbb{R}^2$ and having continuous partial derivatives on D. We assume that the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ satisfy the *Lipschitz type conditions*

$$\left| \frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(u,v) \right| \le L_1 |x-u| + K_1 |y-v| \tag{1.1}$$

and

$$\left| \frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}(u,v) \right| \le L_2 |x-u| + K_2 |y-v| \tag{1.2}$$

for any $(x,y), (u,v) \in D$, where L_1, K_1, L_2 and K_2 are given positive constants.

In the recent paper [8] we obtained among others the following result:

Theorem 1. *Let ∂D be a simple, closed counterclockwise curve bounding a convex region D and f defined on an open set containing D and having continuous partial derivatives on D. If the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ satisfy the Lipschitz type conditions (1.1) and (1.2), then*

$$\begin{aligned} &\left| \frac{1}{A_D} \int\int_D f(x,y)\,dxdy \right. \\ &\left. -\frac{1}{3} f(u,v) - \frac{2}{3}\frac{1}{A_D} \oint_{\partial D} \left[\frac{(v-y) f(x,y)\,dx + (x-u) f(x,y)\,dy}{2} \right] \right| \\ &\le \frac{1}{6}\left[L_1 \frac{1}{A_D} \iint_D (x-u)^2\,dxdy + K_2 \frac{1}{A_D} \iint_D (y-v)^2\,dxdy \right] \\ &+ \frac{K_1+L_2}{6} \frac{1}{A_D} \iint_D |x-u|\,|y-v|\,dxdy \end{aligned} \tag{1.3}$$

1991 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Ostrowski inequality, Hermite-Hadamard inequality, Multiple integral inequalities, Divergence theorem.

RGMIA Res. Rep. Coll. 22 (2019), Art. 68, 20 pp. Received 26/06/19

and

$$\left| \frac{1}{A_D} \int \int_D \left[(u-x) \frac{\partial f}{\partial x}(x,y) + (v-y) \frac{\partial f}{\partial y}(x,y) \right] dxdy \right.$$
$$\left. -\frac{2}{3} f(u,v) - \frac{1}{3} \frac{1}{A_D} \oint_{\partial D} \left[(y-v) f(x,y) \, dx + (u-x) f(x,y) \, dy \right] \right|$$
$$\leq \frac{1}{3} \left[L_1 \frac{1}{A_D} \iint_D (x-u)^2 \, dxdy + K_2 \frac{1}{A_D} \iint_D (y-v)^2 \, dxdy \right]$$
$$+ \frac{K_1 + L_2}{3} \frac{1}{A_D} \iint_D |x-u| \, |y-v| \, dxdy \tag{1.4}$$

for all $(u,v) \in D$.

For other integral inequalities for multiple integrals see [3]-[16].

In this paper, by making use of Divergence Theorem for multiple integral, we establish some perturbed Ostrowski type inequalities for functions of independent variables defined on closed and bounded convex subsets of the Euclidian space $\mathbb{R}^n$. Some examples for 3-dimensional space are also provided.

2. Some Preliminary Facts

Let B be a bounded open subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary ∂B. Let $F = (F_1, ..., F_n)$ be a smooth vector field defined in $\mathbb{R}^n$, or at least in $B \cup \partial B$. Let $\mathbf{n}$ be the unit outward-pointing normal of ∂B. Then the *Divergence Theorem* states, see for instance [17]:

$$\int_B \operatorname{div} F dV = \int_{\partial B} F \cdot n dA, \tag{2.1}$$

where

$$\operatorname{div} F = \nabla \cdot F = \sum_{k=1}^{n} \frac{\partial F_k}{\partial x_k},$$

dV is the element of volume in $\mathbb{R}^n$ and dA is the element of surface area on ∂B.

If $\mathbf{n} = (\mathbf{n}_1, ..., \mathbf{n}_n)$, $x = (x_1, ..., x_n) \in B$ and use the notation dx for dV we can write (2.1) more explicitly as

$$\sum_{k=1}^{n} \int_B \frac{\partial F_k(x)}{\partial x_k} dx = \sum_{k=1}^{n} \int_{\partial B} F_k(x) \, n_k(x) \, dA. \tag{2.2}$$

By taking the real and imaginary part, we can extend the above equality for complex valued functions F_k, $k \in \{1, ..., n\}$ defined on B.

If $n = 2$, the normal is obtained by rotating the tangent vector through $90°$ (in the correct direction so that it points out). The quantity tds can be written (dx_1, dx_2) along the surface, so that

$$ndA := nds = (dx_2, -dx_1).$$

Here t is the tangent vector along the boundary curve and ds is the element of arc-length.

From (2.2) we get for $B \subset \mathbb{R}^2$ that

$$\int_B \frac{\partial F_1(x_1,x_2)}{\partial x_1} dx_1 dx_2 + \int_B \frac{\partial F_2(x_1,x_2)}{\partial x_2} dx_1 dx_2 = \int_{\partial B} F_1(x_1,x_2)\, dx_2 - \int_{\partial B} F_2(x_1,x_2)\, dx_1, \tag{2.3}$$

which is *Green's theorem* in plane.

If $n = 3$ and if ∂B is described as a level-set of a function of 3 variables i.e. $\partial B = \{x_1, x_2, x_3 \in \mathbb{R}^3 \mid G(x_1,x_2,x_3) = 0\}$, then a vector pointing in the direction of $\mathbf{n}$ is $\operatorname{grad} G$. We shall use the case where $G(x_1,x_2,x_3) = x_3 - g(x_1,x_2)$, $(x_1,x_2) \in D$, a domain in $\mathbb{R}^2$ for some differentiable function g on D and B corresponds to the inequality $x_3 < g(x_1,x_2)$, namely

$$B = \left\{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_3 < g(x_1,x_2)\right\}.$$

Then

$$\mathbf{n} = \frac{(-g_{x_1}, -g_{x_2}, 1)}{\left(1 + g_{x_1}^2 + g_{x_2}^2\right)^{1/2}},\ dA = \left(1 + g_{x_1}^2 + g_{x_2}^2\right)^{1/2} dx_1 dx_2$$

and

$$\mathbf{n} dA = (-g_{x_1}, -g_{x_2}, 1)\, dx_1 dx_2.$$

From (2.2) we get

$$\begin{aligned} &\int_B \left(\frac{\partial F_1(x_1,x_2,x_3)}{\partial x_1} + \frac{\partial F_2(x_1,x_2,x_3)}{\partial x_2} + \frac{\partial F_3(x_1,x_2,x_3)}{\partial x_3} \right) dx_1 dx_2 dx_3 \\ &= -\int_D F_1(x_1,x_2,g(x_1,x_2))\, g_{x_1}(x_1,x_2)\, dx_1 dx_2 \\ &- \int_D F_1(x_1,x_2,g(x_1,x_2))\, g_{x_2}(x_1,x_2) dx_1 dx_2 \\ &+ \int_D F_3(x_1,x_2,g(x_1,x_2))\, dx_1 dx_2, \end{aligned} \tag{2.4}$$

which is the *Gauss-Ostrogradsky theorem* in space.

Following Apostol [1], consider a surface described by the vector equation

$$r(u,v) = x_1(u,v)\, \overrightarrow{i} + x_2(u,v)\, \overrightarrow{j} + x_3(u,v)\, \overrightarrow{k}$$

where $(u,v) \in [a,b] \times [c,d]$.

If x_1, x_2, x_3 are differentiable on $[a,b] \times [c,d]$ we consider the two vectors

$$\frac{\partial r}{\partial u} = \frac{\partial x_1}{\partial u} \overrightarrow{i} + \frac{\partial x_2}{\partial u} \overrightarrow{j} + \frac{\partial x_3}{\partial u} \overrightarrow{k}$$

and

$$\frac{\partial r}{\partial v} = \frac{\partial x_1}{\partial v} \overrightarrow{i} + \frac{\partial x_2}{\partial v} \overrightarrow{j} + \frac{\partial x_3}{\partial v} \overrightarrow{k}.$$

The *cross product* of these two vectors $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ will be referred to as the fundamental vector product of the representation r. Its components can be expressed as *Jacobian* *determinants*. In fact, we have [1, p. 420]

$$\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}=\begin{vmatrix}\frac{\partial x_2}{\partial u} & \frac{\partial x_3}{\partial u}\\ \frac{\partial x_2}{\partial v} & \frac{\partial x_3}{\partial v}\end{vmatrix}\overrightarrow{i}+\begin{vmatrix}\frac{\partial x_3}{\partial u} & \frac{\partial x_1}{\partial u}\\ \frac{\partial x_3}{\partial v} & \frac{\partial x_1}{\partial v}\end{vmatrix}\overrightarrow{j}+\begin{vmatrix}\frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u}\\ \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v}\end{vmatrix}\overrightarrow{k}$$

$$=\frac{\partial(x_2,x_3)}{\partial(u,v)}\overrightarrow{i}+\frac{\partial(x_3,x_1)}{\partial(u,v)}\overrightarrow{j}+\frac{\partial(x_1,x_2)}{\partial(u,v)}\overrightarrow{k}.$$

Let $\partial B=r(T)$ be a parametric surface described by a vector-valued function r defined on the box $T=[a,b]\times[c,d]$. The area of ∂B denoted $A_{\partial B}$ is defined by the double integral [1, p. 424-425]

$$\begin{aligned}A_{\partial B}&=\int_a^b\int_c^d\left\|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right\|dudv\\&=\int_a^b\int_c^d\sqrt{\left(\frac{\partial(x_2,x_3)}{\partial(u,v)}\right)^2+\left(\frac{\partial(x_3,x_1)}{\partial(u,v)}\right)^2+\left(\frac{\partial(x_1,x_2)}{\partial(u,v)}\right)^2}dudv.\end{aligned}$$

We define surface integrals in terms of a parametric representation for the surface. One can prove that under certain general conditions the value of the integral is independent of the representation.

Let $\partial B=r(T)$ be a parametric surface described by a vector-valued differentiable function r defined on the box $T=[a,b]\times[c,d]$ and let $f:\partial B\to\mathbb{C}$ defined and bounded on ∂B. The surface integral of f over ∂B is defined by [1, p. 430]

$$\begin{aligned}\int\int_{\partial B}fdA&=\int_a^b\int_c^d f(x_1,x_2,x_3)\left\|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right\|dudv\\&=\int_a^b\int_c^d f(x_1(u,v),x_2(u,v),x_3(u,v))\\&\quad\times\sqrt{\left(\frac{\partial(x_2,x_3)}{\partial(u,v)}\right)^2+\left(\frac{\partial(x_3,x_1)}{\partial(u,v)}\right)^2+\left(\frac{\partial(x_1,x_2)}{\partial(u,v)}\right)^2}dudv.\end{aligned}$$

If $\partial B=r(T)$ is a parametric surface, the fundamental vector product $N=\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}$ is normal to ∂B at each regular point of the surface. At each such point there are two unit normals, a unit normal $\mathbf{n}_1$, which has the same direction as N, and a unit normal $\mathbf{n}_2$ which has the opposite direction. Thus

$$\mathbf{n}_1=\frac{N}{\|N\|}\text{ and }\mathbf{n}_2=-\mathbf{n}_1.$$

Let $\mathbf{n}$ be one of the two normals $\mathbf{n}_1$ or $\mathbf{n}_2$. Let also F be a vector field defined on ∂B and assume that the surface integral,

$$\int\int_{\partial B}(F\cdot\mathbf{n})\,dA,$$

called the flux surface integral, exists. Here $F\cdot\mathbf{n}$ is the dot or inner product.

We can write [1, p. 434]

$$\int\int_{\partial B}(F\cdot\mathbf{n})\,dA=\pm\int_a^b\int_c^d F(r(u,v))\cdot\left(\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right)dudv$$

where the sign " $+$ " is used if $\mathbf{n}=\mathbf{n}_1$ and the " $-$ " sign is used if $\mathbf{n}=\mathbf{n}_2$.

If

$$F(x_1,x_2,x_3)=F_1(x_1,x_2,x_3)\overrightarrow{i}+F_2(x_1,x_2,x_3)\overrightarrow{j}+F_3(x_1,x_2,x_3)\overrightarrow{k}$$

and

$$r(u,v)=x_1(u,v)\overrightarrow{i}+x_2(u,v)\overrightarrow{j}+x_3(u,v)\overrightarrow{k}\text{ where }(u,v)\in[a,b]\times[c,d]$$

then the flux surface integral for $\mathbf{n}=\mathbf{n}_1$ can be explicitly calculated as [1, p. 435]

$$\begin{aligned}\int\int_{\partial B}(F\cdot\mathbf{n})\,dA&=\int_a^b\int_c^d F_1(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_2,x_3)}{\partial(u,v)}dudv\\&+\int_a^b\int_c^d F_2(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_3,x_1)}{\partial(u,v)}dudv\\&+\int_a^b\int_c^d F_3(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_1,x_2)}{\partial(u,v)}dudv.\end{aligned}\tag{2.5}$$

The sum of the double integrals on the right is often written more briefly as [1, p. 435]

$$\begin{aligned}&\int\int_{\partial B}F_1(x_1,x_2,x_3)\,dx_2\wedge dx_3+\int\int_{\partial B}F_2(x_1,x_2,x_3)\,dx_3\wedge dx_1\\&+\int\int_{\partial B}F_3(x_1,x_2,x_3)\,dx_1\wedge dx_2\end{aligned}$$

Let $B\subset\mathbb{R}^3$ be a solid in 3-space bounded by an orientable closed surface ∂B, and let $\mathbf{n}$ be the unit outer normal to ∂B. If F is a continuously differentiable vector field defined on B, we have the *Gauss-Ostrogradsky identity*

$$\iiint_B(\operatorname{div}F)\,dV=\int\int_{\partial B}(F\cdot\mathbf{n})\,dA.\tag{GO}$$

If we express

$$F(x_1,x_2,x_3)=F_1(x_1,x_2,x_3)\overrightarrow{i}+F_2(x_1,x_2,x_3)\overrightarrow{j}+F_3(x_1,x_2,x_3)\overrightarrow{k},$$

then (2.4) can be written as

$$\begin{aligned}&\iiint_B\left(\frac{\partial F_1(x_1,x_2,x_3)}{\partial x_1}+\frac{\partial F_2(x_1,x_2,x_3)}{\partial x_2}+\frac{\partial F_3(x_1,x_2,x_3)}{\partial x_3}\right)dx_1dx_2dx_3\\&=\int\int_{\partial B}F_1(x_1,x_2,x_3)\,dx_2\wedge dx_3+\int\int_{\partial B}F_2(x_1,x_2,x_3)\,dx_3\wedge dx_1\\&+\int\int_{\partial B}F_3(x_1,x_2,x_3)\,dx_1\wedge dx_2.\end{aligned}\tag{2.6}$$

3. Identities of Interest

We have the following identity of interest:

Lemma 1. *Let B be a bounded closed subset of $\mathbb{R}^n$ $(n\geq 2)$ with smooth (or piecewise smooth) boundary ∂B. Let f be a continuously differentiable function* *defined in $\mathbb{R}^n$, or at least in an open neighborhood of B and with complex values. If $\alpha_k,\ \gamma_k \in \mathbb{C}$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} \alpha_k = 1$, then*

$$\int_B f(x)\,dx = \sum_{k=1}^{n} \alpha_k \int_B (\gamma_k - x_k) \frac{\partial f(x)}{\partial x_k} dx \tag{3.1}$$

$$+ \sum_{k=1}^{n} \alpha_k \int_{\partial B} (x_k - \gamma_k) f(x) n_k(x)\,dA. \tag{3.2}$$

In particular,

$$\int_B f(x)\,dx = \frac{1}{n} \sum_{k=1}^{n} \int_B (\gamma_k - x_k) \frac{\partial f(x)}{\partial x_k} dx \tag{3.3}$$

$$+ \frac{1}{n} \sum_{k=1}^{n} \int_{\partial B} (x_k - \gamma_k) f(x) n_k(x)\,dA. \tag{3.4}$$

If $u = (u_1, ..., u_n) \in B$, then

$$\frac{1}{V(B)} \int_B f(x)\,dx = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{V(B)} \int_B (u_k - x_k) \frac{\partial f(x)}{\partial x_k} dx \tag{3.5}$$

$$+ \frac{1}{n} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - u_k) f(x) n_k(x)\,dA. \tag{3.6}$$

Proof. Let $x = (x_1, ..., x_n) \in B$. We consider

$$F_k(x) = \alpha_k (x_k - \gamma_k) f(x), \ \ k \in \{1, ..., n\}$$

and take the partial derivatives $\frac{\partial F_k(x)}{\partial x_k}$ to get

$$\frac{\partial F_k(x)}{\partial x_k} = \alpha_k f(x) + \alpha_k (x_k - \gamma_k) \frac{\partial f(x)}{\partial x_k}, \ \ k \in \{1, ..., n\}.$$

If we sum this equality over k from 1 to n we get

$$\begin{aligned} \sum_{k=1}^{n} \frac{\partial F_k(x)}{\partial x_k} &= \sum_{k=1}^{n} \alpha_k f(x) + \sum_{k=1}^{n} \alpha_k (x_k - \gamma_k) \frac{\partial f(x)}{\partial x_k} \\ &= f(x) + \sum_{k=1}^{n} \alpha_k (x_k - \gamma_k) \frac{\partial f(x)}{\partial x_k} \end{aligned} \tag{3.7}$$

for all $x = (x_1, ..., x_n) \in B$.

Now, if we take the integral in the equality (3.7) over $(x_1, ..., x_n) \in B$ we get

$$\int_B \left(\sum_{k=1}^{n} \frac{\partial F_k(x)}{\partial x_k} \right) dx = \int_B f(x)\,dx + \sum_{k=1}^{n} \int_B \alpha_k (x_k - \gamma_k) \frac{\partial f(x)}{\partial x_k} dx. \tag{3.8}$$

By the Divergence Theorem (2.2) we also have

$$\int_B \left(\sum_{k=1}^{n} \frac{\partial F_k(x)}{\partial x_k} \right) dx = \sum_{k=1}^{n} \int_{\partial B} \alpha_k (x_k - \gamma_k) f(x) n_k(x)\,dA \tag{3.9}$$

and by making use of (3.8) and (3.9) we get

$$\int_B f(x)\,dx+\sum_{k=1}^{n}\alpha_k\int_B (x_k-\gamma_k)\frac{\partial f(x)}{\partial x_k}dx$$
$$=\sum_{k=1}^{n}\alpha_k\int_{\partial B}(x_k-\gamma_k)\,f(x)\,n_k(x)\,dA,$$

which gives the desired representation (3.1).

The identity (3.3) follows by (3.1) for $\alpha_k=\frac{1}{n}$, $k\in\{1,...,n\}$. The identity (3.5) follows by (3.3) for $\gamma_k=u_k$, $k\in\{1,...,n\}$. □

For the body B we consider the coordinates for the *centre of gravity*

$$G=\left(\overline{x_{B,1}},...,\overline{x_{B,n}}\right)$$

defined by

$$\overline{x_{B,k}}:=\frac{1}{V(B)}\int_B x_k dx,\ \ k\in\{1,...,n\},$$

where

$$V(B):=\int_B x dx$$

is the volume of B.

Corollary 1. *With the assumptions of Theorem 2 we have*

$$\int_B f(x)\,dx=\sum_{k=1}^{n}\int_B\alpha_k\left(\overline{x_{B,k}}-x_k\right)\frac{\partial f(x)}{\partial x_k}dx \tag{3.10}$$

$$+\sum_{k=1}^{n}\int_{\partial B}\alpha_k\left(x_k-\overline{x_{B,k}}\right)f(x)\,n_k(x)\,dA \tag{3.11}$$

and, in particular,

$$\int_B f(x)\,dx=\frac{1}{n}\sum_{k=1}^{n}\int_B\left(\overline{x_{B,k}}-x_k\right)\frac{\partial f(x)}{\partial x_k}dx \tag{3.12}$$

$$+\frac{1}{n}\sum_{k=1}^{n}\int_{\partial B}\left(x_k-\overline{x_{B,k}}\right)f(x)\,n_k(x)\,dA. \tag{3.13}$$

The proof follows by (3.1) on taking $\gamma_k=\overline{x_{B,k}}$, $k\in\{1,...,n\}$.

For a function f as in Theorem 2 above, we define the points

$$x_{B,\partial f,k}:=\frac{\int_B x_k\frac{\partial f(x)}{\partial x_k}dx}{\int_B\frac{\partial f(x)}{\partial x_k}dx},\ \ k\in\{1,...,n\},$$

provided that all denominators are not zero.

Corollary 2. *With the assumptions of Theorem 2 we have*

$$\int_B f(x)\,dx=\sum_{k=1}^{n}\int_{\partial B}\alpha_k\left(x_k-x_{B,\partial f,k}\right)f(x)\,n_k(x)\,dA \tag{3.14}$$

and, in particular,

$$\int_B f(x)\,dx=\frac{1}{n}\sum_{k=1}^{n}\int_{\partial B}\left(x_k-x_{B,\partial f,k}\right)f(x)\,n_k(x)\,dA. \tag{3.15}$$

The proof follows by (3.1) on taking $\gamma_k = x_{B,\partial f,k}$, $k \in \{1,...,n\}$ and observing that
$$\sum_{k=1}^{n} \alpha_k \int_B (x_{B,\partial f,k} - x_k) \frac{\partial f(x)}{\partial x_k} dx = 0.$$

For a function f as in Theorem 2 above, we define the points
$$x_{\partial B,f,k} := \frac{\int_{\partial B} x_k f(x) n_k(x) dA}{\int_{\partial B} f(x) n_k(x) dA},\ k \in \{1,...,n\}$$
provided that all denominators are not zero.

Corollary 3. *With the assumptions of Theorem 2 we have*
$$\int_B f(x) dx = \sum_{k=1}^{n} \int_B \alpha_k (x_{\partial B,f,k} - x_k) \frac{\partial f(x)}{\partial x_k} dx \tag{3.16}$$
and, in particular,
$$\int_B f(x) dx = \frac{1}{n} \sum_{k=1}^{n} \int_B (x_{\partial B,f,k} - x_k) \frac{\partial f(x)}{\partial x_k} dx. \tag{3.17}$$

The proof follows by (3.1) on taking $\gamma_k = x_{\partial B,f,k}$, $k \in \{1,...,n\}$ and observing that
$$\sum_{k=1}^{n} \int_{\partial B} \alpha_k (x_k - x_{\partial B,f,k}) f(x) n_k(x) dA = 0.$$

We have:

Lemma 2. *If $f : B \to \mathbb{C}$ is differentiable on B, a convex subset of $\mathbb{R}^n$ $(n \geq 2)$, then for all $x = (x_1,...,x_n)$, $u = (u_1,...,u_n) \in B$ and $\lambda_k \in \mathbb{C}$, $k \in \{1,...,n\}$ we have the equality*
$$\begin{aligned} f(u) = f(x) &+ \sum_{k=1}^{n} (u_k - x_k) \lambda_k \\ &+ \sum_{k=1}^{n} (u_k - x_k) \int_0^1 \left(\frac{\partial f}{\partial x_k} [tu + (1-t)x] - \lambda_k \right) dt. \end{aligned} \tag{3.18}$$
In particular,
$$\begin{aligned} f(u) = f(x) &+ \sum_{k=1}^{n} (u_k - x_k) \frac{\partial f(x)}{\partial x_k} \\ &+ \sum_{k=1}^{n} (u_k - x_k) \int_0^1 \left(\frac{\partial f}{\partial x_k} [tu + (1-t)x] - \frac{\partial f(x)}{\partial x_k} \right) dt \end{aligned} \tag{3.19}$$
and
$$\begin{aligned} f(u) = {}& \frac{1}{V(B)} \int_B f(x) dx + \sum_{k=1}^{n} \frac{1}{V(B)} \int_B (u_k - x_k) \frac{\partial f(x)}{\partial x_k} dx \\ &+ \sum_{k=1}^{n} \frac{1}{V(B)} \int_B \left[(u_k - x_k) \int_0^1 \left(\frac{\partial f}{\partial x_k} [tu + (1-t)x] - \frac{\partial f(x)}{\partial x_k} \right) dt \right] dx. \end{aligned} \tag{3.20}$$

Proof. By Taylor's multivariate theorem with integral remainder, we have

$$f(u) = f(x) + \sum_{k=1}^{n} (u_k - x_k) \int_0^1 \frac{\partial f}{\partial x_k} [tu + (1-t)x] \, dt \tag{3.21}$$

for all $x = (x_1, ..., x_n)$, $u = (u_1, ..., u_n) \in B$.

If $\lambda_k \in \mathbb{C}$, $k \in \{1, ..., n\}$, then

$$\begin{aligned} &(u_k - x_k) \int_0^1 \left(\frac{\partial f}{\partial x_k} [tu + (1-t)x] - \lambda_k \right) dt \\ &= (u_k - x_k) \int_0^1 \frac{\partial f}{\partial x_k} [tu + (1-t)x] \, dt - (u_k - x_k) \lambda_k \end{aligned}$$

and by (3.21) we get the desired result (3.18).

The identity (3.19) follows by (3.18) on taking $\lambda_k = \frac{\partial f(x)}{\partial x_k}$, $k \in \{1, ..., n\}$, while the identity (3.20) follows by (3.19) on taking the integral mean $\frac{1}{V(B)} \int_B$ over the variable x. □

We have the following equalities that will be used in the sequel:

Theorem 2. *Let B be a bounded closed convex subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary ∂B. Let f be a continuously differentiable function defined in $\mathbb{R}^n$, or at least in an open neighborhood of B and with complex values. Then for all $u = (u_1, ..., u_n) \in B$ we have*

$$\begin{aligned} &\frac{1}{V(B)} \int_B f(x) \, dx \\ &= \frac{1}{n+1} f(u) + \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - u_k) f(x) n_k(x) \, dA \\ &+ \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \\ &\times \int_B \left[(x_k - u_k) \int_0^1 \left(\frac{\partial f}{\partial x_k} [tu + (1-t)x] - \frac{\partial f(x)}{\partial x_k} \right) dt \right] dx \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} &\sum_{k=1}^{n} \frac{1}{V(B)} \int_B (u_k - x_k) \frac{\partial f(x)}{\partial x_k} dx \\ &= \frac{n}{n+1} f(u) + \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (u_k - x_k) f(x) n_k(x) \, dA \\ &+ \frac{n}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \\ &\times \int_B \left[(x_k - u_k) \int_0^1 \left(\frac{\partial f}{\partial x_k} [tu + (1-t)x] - \frac{\partial f(x)}{\partial x_k} \right) dt \right] dx. \end{aligned} \tag{3.23}$$

Proof. From the equality (3.5) we have

$$\sum_{k=1}^{n}\frac{1}{V(B)}\int_{B}(u_k-x_k)\frac{\partial f(x)}{\partial x_k}dx$$
$$=\frac{n}{V(B)}\int_{B}f(x)\,dx-\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k-u_k)f(x)\,n_k(x)\,dA.$$

By replacing this into (3.20) we get

$$\begin{aligned}f(u)&=\frac{1}{V(B)}\int_{B}f(x)\,dx+\frac{n}{V(B)}\int_{B}f(x)\,dx\\&-\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k-u_k)f(x)\,n_k(x)\,dA\\&+\sum_{k=1}^{n}\frac{1}{V(B)}\int_{B}\left[(u_k-x_k)\int_0^1\left(\frac{\partial f}{\partial x_k}[tu+(1-t)x]-\frac{\partial f(x)}{\partial x_k}\right)dt\right]dx\\&=\frac{n+1}{V(B)}\int_{B}f(x)\,dx-\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k-u_k)f(x)\,n_k(x)\,dA\\&+\sum_{k=1}^{n}\frac{1}{V(B)}\int_{B}\left[(u_k-x_k)\int_0^1\left(\frac{\partial f}{\partial x_k}[tu+(1-t)x]-\frac{\partial f(x)}{\partial x_k}\right)dt\right]dx,\end{aligned}$$

namely

$$\begin{aligned}&\frac{n+1}{V(B)}\int_{B}f(x)\,dx\\&=f(u)+\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k-u_k)f(x)\,n_k(x)\,dA\\&+\sum_{k=1}^{n}\frac{1}{V(B)}\int_{B}\left[(x_k-u_k)\int_0^1\left(\frac{\partial f}{\partial x_k}[tu+(1-t)x]-\frac{\partial f(x)}{\partial x_k}\right)dt\right]dx\end{aligned}$$

which is equivalent to (3.22).

Further, if we replace $\frac{1}{V(B)}\int_B f(x)\,dx$ from (3.5) into (3.20) we get

$$\begin{aligned}f(u)&=\frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{B}(u_k-x_k)\frac{\partial f(x)}{\partial x_k}dx\\&+\sum_{k=1}^{n}\frac{1}{V(B)}\int_{B}(u_k-x_k)\frac{\partial f(x)}{\partial x_k}dx\\&+\frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k-u_k)f(x)\,n_k(x)\,dA\\&+\sum_{k=1}^{n}\frac{1}{V(B)}\int_{B}\left[(u_k-x_k)\int_0^1\left(\frac{\partial f}{\partial x_k}[tu+(1-t)x]-\frac{\partial f(x)}{\partial x_k}\right)dt\right]dx\end{aligned}$$

$$
\begin{aligned}
&= \frac{n+1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_B (u_k - x_k)\frac{\partial f(x)}{\partial x_k}dx \\
&+ \frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k-u_k) f(x) n_k(x)\,dA \\
&+ \sum_{k=1}^{n}\frac{1}{V(B)}\int_B\left[(u_k-x_k)\int_0^1\left(\frac{\partial f}{\partial x_k}[tu+(1-t)x]-\frac{\partial f(x)}{\partial x_k}\right)dt\right]dx,
\end{aligned}
$$

which gives

$$
\begin{aligned}
&\frac{n+1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_B (u_k - x_k)\frac{\partial f(x)}{\partial x_k}dx \\
&= f(u) + \frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(u_k-x_k) f(x) n_k(x)\,dA \\
&+ \sum_{k=1}^{n}\frac{1}{V(B)}\int_B\left[(x_k-u_k)\int_0^1\left(\frac{\partial f}{\partial x_k}[tu+(1-t)x]-\frac{\partial f(x)}{\partial x_k}\right)dt\right]dx
\end{aligned}
$$

namely, the desired identity (3.23). □

Corollary 4. *With the assumptions of Theorem 2 we have the following inequalities concerning the centre of gravity* $G=(\overline{x_{B,1}},...,\overline{x_{B,n}})$

$$
\begin{aligned}
&\frac{1}{V(B)}\int_B f(x)\,dx \\
&= \frac{1}{n+1}f(G) + \frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k-\overline{x_{B,k}}) f(x) n_k(x)\,dA \\
&+ \frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{V(B)} \\
&\times\int_B\left[(x_k-\overline{x_{B,k}})\int_0^1\left(\frac{\partial f}{\partial x_k}[tG+(1-t)x]-\frac{\partial f(x)}{\partial x_k}\right)dt\right]dx
\end{aligned}
\tag{3.24}
$$

and

$$
\begin{aligned}
&\sum_{k=1}^{n}\frac{1}{V(B)}\int_B(\overline{x_{B,k}}-x_k)\frac{\partial f(x)}{\partial x_k}dx \\
&= \frac{n}{n+1}f(G) + \frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(\overline{x_{B,k}}-x_k) f(x) n_k(x)\,dA \\
&+ \frac{n}{n+1}\sum_{k=1}^{n}\frac{1}{V(B)} \\
&\times\int_B\left[(x_k-\overline{x_{B,k}})\int_0^1\left(\frac{\partial f}{\partial x_k}[tG+(1-t)x]-\frac{\partial f(x)}{\partial x_k}\right)dt\right]dx.
\end{aligned}
\tag{3.25}
$$

We consider the point

$$S=(x_{\partial B,f,1},...,x_{\partial B,f,n})$$

where

$$x_{\partial B,f,k} := \frac{\int_{\partial B} x_k f(x) n_k(x) dA}{\int_{\partial B} f(x) n_k(x) dA}, \ k \in \{1, ..., n\}.$$

Corollary 5. *With the assumptions of Theorem 2 and if $S \in B$, then we have the equalities*

$$\begin{aligned} (3.26) \qquad & \frac{1}{V(B)} \int_B f(x) dx \\ &= \frac{1}{n+1} f(S) + \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \\ &\times \int_B \left[(x_k - x_{\partial B,f,k}) \int_0^1 \left(\frac{\partial f}{\partial x_k} [tS + (1-t)x] - \frac{\partial f(x)}{\partial x_k} \right) dt \right] dx \end{aligned}$$

and

$$\begin{aligned} (3.27) \qquad & \sum_{k=1}^{n} \frac{1}{V(B)} \int_B (x_{\partial B,f,k} - x_k) \frac{\partial f(x)}{\partial x_k} dx \\ &= \frac{n}{n+1} f(S) + \frac{n}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \\ &\times \int_B \left[(x_k - x_{\partial B,f,k}) \int_0^1 \left(\frac{\partial f}{\partial x_k} [tS + (1-t)x] - \frac{\partial f(x)}{\partial x_k} \right) dt \right] dx. \end{aligned}$$

4. Some Inequalities

We assume that the partial derivatives $\frac{\partial f}{\partial x_k}$, $k \in \{1, ..., n\}$ satisfy the *Lipschitz type conditions*

$$\left| \frac{\partial f(x)}{\partial x_k} - \frac{\partial f(u)}{\partial x_k} \right| \le \sum_{j=1}^{n} L_{kj} |x_j - u_j| \tag{4.1}$$

for all $x = (x_1, ..., x_n)$, $u = (u_1, ..., u_n) \in B$ where $L_{kj} > 0$, k, $j \in \{1, ..., n\}$ are given positive constants.

We have the following generalization for n-dimensional spaces of the result from Theorem 1:

Theorem 3. *Let B be a bounded closed convex subset of $\mathbb{R}^n$ $(n \ge 2)$ with smooth (or piecewise smooth) boundary ∂B. Let f be a continuously differentiable function defined in $\mathbb{R}^n$, or at least in an open neighborhood of B and with complex values and assume that the partial derivatives $\frac{\partial f}{\partial x_k}$, $k \in \{1, ..., n\}$ satisfy the Lipschitz type conditions (4.1). Then for all $u = (u_1, ..., u_n) \in B$ we have the inequalities*

$$\begin{aligned} (4.2) \qquad & \left| \frac{1}{V(B)} \int_B f(x) dx - \frac{1}{n+1} f(u) \right. \\ & \left. - \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - u_k) f(x) n_k(x) dA \right| \\ & \le \frac{1}{2(n+1)} \sum_{k=1}^{n} \sum_{j=1}^{n} L_{kj} \frac{1}{V(B)} \int_B |x_k - u_k| |x_j - u_j| dx \end{aligned}$$

and

$$\begin{aligned}(4.3)\qquad &\left|\sum_{k=1}^{n}\frac{1}{V(B)}\int_{B}(u_k-x_k)\frac{\partial f(x)}{\partial x_k}dx-\frac{n}{n+1}f(u)\right.\\&\left.-\frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(u_k-x_k)f(x)n_k(x)\,dA\right|\\&\leq\frac{n}{2(n+1)}\sum_{k=1}^{n}\sum_{j=1}^{n}L_{kj}\frac{1}{V(B)}\int_{B}|x_k-u_k|\,|x_j-u_j|\,dx.\end{aligned}$$

Proof. Using the identity (3.22) we have

$$\begin{aligned}&\left|\frac{1}{V(B)}\int_{B}f(x)\,dx-\frac{1}{n+1}f(u)\right.\\&\left.-\frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k-u_k)f(x)n_k(x)\,dA\right|\\&\leq\frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{V(B)}\\&\times\left|\int_{B}\left[(x_k-u_k)\int_0^1\left(\frac{\partial f}{\partial x_k}[tu+(1-t)x]-\frac{\partial f(x)}{\partial x_k}\right)dt\right]dx\right|\end{aligned}$$

$$\begin{aligned}&\leq\frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{V(B)}\\&\times\int_{B}\left|(x_k-u_k)\int_0^1\left(\frac{\partial f}{\partial x_k}[tu+(1-t)x]-\frac{\partial f(x)}{\partial x_k}\right)dt\right|dx\\&\leq\frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{V(B)}\\&\times\int_{B}|x_k-u_k|\left(\int_0^1\left|\frac{\partial f}{\partial x_k}[tu+(1-t)x]-\frac{\partial f(x)}{\partial x_k}\right|dt\right)dx\\&=:B\end{aligned}$$

Since $\frac{\partial f}{\partial x_k}$, $k\in\{1,...,n\}$ satisfy the *Lipschitz type conditions (4.1),* hence

$$\begin{aligned}\int_0^1\left|\frac{\partial f}{\partial x_k}[tu+(1-t)x]-\frac{\partial f(x)}{\partial x_k}\right|dt&\leq\sum_{j=1}^{n}L_{kj}\int_0^1|tu_j+(1-t)x_j-x_j|\,dt\\&=\frac{1}{2}\sum_{j=1}^{n}L_{kj}|x_j-u_j|\end{aligned}$$

for all $x=(x_1,...,x_n)$, $u=(u_1,...,u_n)\in B$.

Therefore

$$B \leq \frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{B}|x_k-u_k|\left(\frac{1}{2}\sum_{j=1}^{n}L_{kj}|x_j-u_j|\right)dx$$
$$=\frac{1}{2(n+1)}\sum_{k=1}^{n}\sum_{j=1}^{n}L_{kj}\frac{1}{V(B)}\int_{B}|x_k-u_k||x_j-u_j|\,dx,$$

which proves the desired result (4.2).

The inequality (4.3) follows in a similar way by (3.23). □

If $f: B \to \mathbb{C}$ is twice differentiable on the convex set B and the second partial derivatives $\frac{\partial^2 f}{\partial x_k \partial x_j}$ for $k,\ j \in \{1,...,n\}$ are bounded on B (for $j = k$ we denote $\frac{\partial^2 f}{\partial x_k \partial x_j} = \frac{\partial^2 f}{\partial x_k^2}$), namely

$$\left\|\frac{\partial^2 f}{\partial x_k \partial x_j}\right\|_{B,\infty} := \sup_{x\in B}\left|\frac{\partial^2 f(x)}{\partial x_k \partial x_j}\right| < \infty,$$

then $\frac{\partial f}{\partial x_k}$, $k \in \{1,...,n\}$ satisfy the Lipschitz type conditions *(4.1)* for $L_{kj} = \left\|\frac{\partial^2 f}{\partial x_k \partial x_j}\right\|_{B,\infty}$ where $k,\ j \in \{1,...,n\}$. We observe that for $k \neq j$, $L_{kj} = L_{jk}$.

Corollary 6. *Let B be a bounded closed convex subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary ∂B. Let f be a twice differentiable function defined in $\mathbb{R}^n$, or at least in an open neighborhood of B and with complex values and such that the second partial derivatives $\frac{\partial^2 f}{\partial x_k \partial x_j}$ for $k,\ j \in \{1,...,n\}$ are bounded on B. Then for all $u = (u_1,...,u_n) \in B$ we have the inequalities*

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx - \frac{1}{n+1}f(u) - \frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k-u_k)f(x)n_k(x)\,dA\right|$$
$$\leq \frac{1}{2(n+1)}\sum_{k=1}^{n}\left\|\frac{\partial^2 f}{\partial x_k^2}\right\|_{B,\infty}\frac{1}{V(B)}\int_B (x_k-u_k)^2\,dx + \frac{1}{(n+1)}\sum_{1\leq k<j\leq n}\left\|\frac{\partial^2 f}{\partial x_k \partial x_j}\right\|_{B,\infty}\frac{1}{V(B)}\int_B|x_k-u_k||x_j-u_j|\,dx \tag{4.4}$$

and

$$\left|\sum_{k=1}^{n}\frac{1}{V(B)}\int_B (u_k-x_k)\frac{\partial f(x)}{\partial x_k}dx - \frac{n}{n+1}f(u) - \frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(u_k-x_k)f(x)n_k(x)\,dA\right|$$
$$\leq \frac{n}{2(n+1)}\sum_{k=1}^{n}\left\|\frac{\partial^2 f}{\partial x_k^2}\right\|_{B,\infty}\frac{1}{V(B)}\int_B (x_k-u_k)^2\,dx + \frac{n}{(n+1)}\sum_{1\leq k<j\leq n}\left\|\frac{\partial^2 f}{\partial x_k \partial x_j}\right\|_{B,\infty}\frac{1}{V(B)}\int_B|x_k-u_k||x_j-u_j|\,dx. \tag{4.5}$$

Remark 1. *If we take in Corollary 6 u to be the centre of gravity* $G = \left(\overline{x_{B,1}}, ..., \overline{x_{B,n}}\right)$, *then we get the inequalities*

$$
\begin{aligned}
(4.6)\qquad & \left| \frac{1}{V(B)} \int_B f(x)\,dx - \frac{1}{n+1} f(G) \right. \\
& \left. - \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} \left(x_k - \overline{x_{B,k}}\right) f(x)\, n_k(x)\, dA \right| \\
& \leq \frac{1}{2(n+1)} \sum_{k=1}^{n} \left\| \frac{\partial^2 f}{\partial x_k^2} \right\|_{B,\infty} \frac{1}{V(B)} \int_B \left(x_k - \overline{x_{B,k}}\right)^2 dx \\
& + \frac{1}{(n+1)} \sum_{1 \leq k < j \leq n} \left\| \frac{\partial^2 f}{\partial x_k \partial x_j} \right\|_{B,\infty} \frac{1}{V(B)} \int_B \left|x_k - \overline{x_{B,k}}\right| \left|x_j - \overline{x_{B,j}}\right| dx
\end{aligned}
$$

and

$$
\begin{aligned}
(4.7)\qquad & \left| \sum_{k=1}^{n} \frac{1}{V(B)} \int_B \left(\overline{x_{B,k}} - x_k\right) \frac{\partial f(x)}{\partial x_k} dx - \frac{n}{n+1} f(G) \right. \\
& \left. - \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} \left(\overline{x_{B,k}} - x_k\right) f(x)\, n_k(x)\, dA \right| \\
& \leq \frac{n}{2(n+1)} \sum_{k=1}^{n} \left\| \frac{\partial^2 f}{\partial x_k^2} \right\|_{B,\infty} \frac{1}{V(B)} \int_B \left(x_k - \overline{x_{B,k}}\right)^2 dx \\
& + \frac{n}{(n+1)} \sum_{1 \leq k < j \leq n} \left\| \frac{\partial^2 f}{\partial x_k \partial x_j} \right\|_{B,\infty} \frac{1}{V(B)} \int_B \left|x_k - \overline{x_{B,k}}\right| \left|x_j - \overline{x_{B,j}}\right| dx.
\end{aligned}
$$

Also, if we take in Corollary 6 u to be $S = \left(x_{\partial B,f,1}, ..., x_{\partial B,f,n}\right)$, *then we get*

$$
\begin{aligned}
(4.8)\qquad & \left| \frac{1}{V(B)} \int_B f(x)\,dx - \frac{1}{n+1} f(S) \right. \\
& \leq \frac{1}{2(n+1)} \sum_{k=1}^{n} \left\| \frac{\partial^2 f}{\partial x_k^2} \right\|_{B,\infty} \frac{1}{V(B)} \int_B \left(x_k - x_{\partial B,f,n}\right)^2 dx \\
& + \frac{1}{(n+1)} \sum_{1 \leq k < j \leq n} \left\| \frac{\partial^2 f}{\partial x_k \partial x_j} \right\|_{B,\infty} \\
& \times \frac{1}{V(B)} \int_B \left|x_k - x_{\partial B,f,k}\right| \left|x_j - x_{\partial B,f,j}\right| dx
\end{aligned}
$$

and

$$
\begin{aligned}
(4.9)\qquad & \left| \sum_{k=1}^{n} \frac{1}{V(B)} \int_B \left(x_{\partial B,f,k} - x_k\right) \frac{\partial f(x)}{\partial x_k} dx - \frac{n}{n+1} f(S) \right. \\
& \leq \frac{n}{2(n+1)} \sum_{k=1}^{n} \left\| \frac{\partial^2 f}{\partial x_k^2} \right\|_{B,\infty} \frac{1}{V(B)} \int_B \left(x_k - x_{\partial B,f,k}\right)^2 dx \\
& + \frac{n}{(n+1)} \sum_{1 \leq k < j \leq n} \left\| \frac{\partial^2 f}{\partial x_k \partial x_j} \right\|_{B,\infty} \\
& \times \frac{1}{V(B)} \int_B \left|x_k - x_{\partial B,f,k}\right| \left|x_j - x_{\partial B,f,j}\right| dx.
\end{aligned}
$$

5. Example for 3-Dimensional Spaces

Let B be a bounded closed convex subset of $\mathbb{R}^3$ with smooth (or piecewise smooth) boundary ∂B. Let f be a twice differentiable function defined in $\mathbb{R}^3$, or at least in on open neighborhood of B and with complex values and assume that

$$\left\|\frac{\partial^2 f}{\partial x_k \partial x_j}\right\|_{B,\infty} := \sup_{x\in B}\left|\frac{\partial^2 f(x)}{\partial x_k \partial x_j}\right| < \infty$$

for all $k,\, j \in \{1,...,3\}$.

Consider a surface described by the vector equation

$$r(u,v) = x_1(u,v)\overrightarrow{i} + x_2(u,v)\overrightarrow{j} + x_3(u,v)\overrightarrow{k}$$

where $(u,v)\in[a,b]\times[c,d]$. Then, by using the notations from the second section, we have from (4.4) that

$$\begin{aligned}(5.1)\quad &\Bigg|\frac{1}{V(B)}\int_B f(x)\,dx - \frac{1}{4}f(y)\\
&-\frac{1}{4V(B)}\int_a^b\int_c^d (x_1(u,v)-y_1)\, f(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_2,x_3)}{\partial(u,v)}dudv\\
&-\frac{1}{4V(B)}\int_a^b\int_c^d (x_2(u,v)-y_2)\, f(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_3,x_1)}{\partial(u,v)}dudv\\
&-\frac{1}{4V(B)}\int_a^b\int_c^d (x_3(u,v)-y_3)\, f(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_1,x_2)}{\partial(u,v)}dudv\\
&\qquad\le \frac{1}{8}\sum_{k=1}^{3}\left\|\frac{\partial^2 f}{\partial x_k^2}\right\|_{B,\infty}\frac{1}{V(B)}\int_B (x_k-y_k)^2\,dx\\
&\qquad\quad+\frac{1}{4}\sum_{1\le k<j\le 3}\left\|\frac{\partial^2 f}{\partial x_k\partial x_j}\right\|_{B,\infty}\frac{1}{V(B)}\int_B |x_k-y_k|\,|x_j-y_j|\,dx\end{aligned}$$

for all $y=(y_1,y_2,y_3)\in B$.

If we take $y=(\overline{x_{B,1}},\overline{x_{B,2}},\overline{x_{B,3}})$ in (5.1), then we get

$$\begin{aligned}(5.2)\quad &\Bigg|\frac{1}{V(B)}\int_B f(x)\,dx - \frac{1}{4}f(\overline{x_{B,1}},\overline{x_{B,2}},\overline{x_{B,3}})\\
&-\frac{1}{4V(B)}\int_a^b\int_c^d (x_1(u,v)-\overline{x_{B,1}})\, f(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_2,x_3)}{\partial(u,v)}dudv\\
&-\frac{1}{4V(B)}\int_a^b\int_c^d (x_2(u,v)-\overline{x_{B,2}})\, f(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_3,x_1)}{\partial(u,v)}dudv\\
&-\frac{1}{4V(B)}\int_a^b\int_c^d (x_3(u,v)-\overline{x_{B,3}})\, f(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_1,x_2)}{\partial(u,v)}dudv\\
&\qquad\le \frac{1}{8}\sum_{k=1}^{3}\left\|\frac{\partial^2 f}{\partial x_k^2}\right\|_{B,\infty}\frac{1}{V(B)}\int_B (x_k-\overline{x_{B,k}})^2\,dx\\
&\qquad\quad+\frac{1}{4}\sum_{1\le k<j\le 3}\left\|\frac{\partial^2 f}{\partial x_k\partial x_j}\right\|_{B,\infty}\frac{1}{V(B)}\int_B |x_k-\overline{x_{B,k}}|\,|x_j-\overline{x_{B,j}}|\,dx.\end{aligned}$$

6. Example for 3-Dimensional Balls

Consider the 3-dimensional ball centered in $C=(a,b,c)$ and having the radius $R>0$,

$$B(C,R):=\left\{(x_1,x_2,x_3)\in\mathbb{R}^3\big|\,(x_1-a)^2+(x_2-b)^2+(x_3-c)^2\le R^2\right\}$$

and the sphere

$$S(C,R):=\left\{(x_1,x_2,x_3)\in\mathbb{R}^3\big|\,(x_1-a)^2+(x_2-b)^2+(x_3-c)^2=R^2\right\}.$$

Consider the parametrization of $B(C,R)$ and $S(C,R)$ given by:

$$B(C,R):\begin{cases} x_1=r\cos\psi\cos\varphi+a\\ x_2=r\cos\psi\sin\varphi+b\\ x_3=r\sin\psi+c\end{cases}\ ;\ (r,\psi,\varphi)\in[0,R]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi]$$

and

$$S(C,R):\begin{cases} x_1=R\cos\psi\cos\varphi+a\\ x_2=R\cos\psi\sin\varphi+b\\ x_3=R\sin\psi+c\end{cases}\ ;\ (\psi,\varphi)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi].$$

Observe that

$$\begin{vmatrix}\frac{\partial x_2}{\partial\psi} & \frac{\partial x_3}{\partial\psi}\\ \frac{\partial x_2}{\partial\varphi} & \frac{\partial x_3}{\partial\varphi}\end{vmatrix}=-R^2\cos^2\psi\cos\varphi,$$

$$\begin{vmatrix}\frac{\partial x_1}{\partial\psi} & \frac{\partial x_3}{\partial\psi}\\ \frac{\partial x_1}{\partial\varphi} & \frac{\partial x_3}{\partial\varphi}\end{vmatrix}=R^2\cos^2\psi\sin\varphi,$$

and

$$\begin{vmatrix}\frac{\partial x_1}{\partial\psi} & \frac{\partial x_2}{\partial\psi}\\ \frac{\partial x_1}{\partial\varphi} & \frac{\partial x_2}{\partial\varphi}\end{vmatrix}=-R^2\sin\psi\cos\psi.$$

Let us consider the transformation $T_2:\mathbb{R}^3\to\mathbb{R}^3$ given by:

$$T_2(r,\psi,\varphi):=(r\cos\psi\cos\varphi+a,r\cos\psi\sin\varphi+b,r\sin\psi+c).$$

It is well known that the Jacobian of T_2 is

$$J(T_2)=r^2\cos\psi$$

and T_2 is a one-to-one mapping defined on the interval of $\mathbb{R}^3$, $[0,R]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi]$, with values in the ball $B(C,R)$ from $\mathbb{R}^3$. Thus we have the change of variable:

$$\iiint_{B(C,R)}f(x_1,x_2,x_3)\,dx_1dx_2dx_3 \tag{6.1}$$
$$=\int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}f(r\cos\psi\cos\varphi+a,r\cos\psi\sin\varphi+b,r\sin\psi+c)\,r^2\cos\psi drd\psi d\varphi.$$

Moreover

$$\frac{1}{4V(B)}\int_a^b\int_c^d \left(x_1(u,v)-\overline{x_{B,1}}\right) f\left(x_1(u,v),x_2(u,v),x_3(u,v)\right)\frac{\partial(x_2,x_3)}{\partial(u,v)}dudv$$
$$=-\frac{3}{16\pi}\int_a^b\int_c^d f\left(R\cos\psi\cos\varphi+a,R\cos\psi\sin\varphi+b,R\sin\psi+c\right)$$
$$\times\cos^3\psi\cos^2\varphi d\psi d\varphi,$$

$$\frac{1}{4V(B)}\int_a^b\int_c^d \left(x_2(u,v)-\overline{x_{B,2}}\right) f\left(x_1(u,v),x_2(u,v),x_3(u,v)\right)\frac{\partial(x_3,x_1)}{\partial(u,v)}dudv$$
$$=\frac{3}{16\pi}\int_a^b\int_c^d f\left(R\cos\psi\cos\varphi+a,R\cos\psi\sin\varphi+b,R\sin\psi+c\right)$$
$$\times\cos^3\psi\sin^2\varphi d\psi d\varphi$$

and

$$\frac{1}{4V(B)}\int_a^b\int_c^d \left(x_3(u,v)-\overline{x_{B,3}}\right) f\left(x_1(u,v),x_2(u,v),x_3(u,v)\right)\frac{\partial(x_1,x_2)}{\partial(u,v)}dudv$$
$$=-\frac{3}{16\pi}\int_a^b\int_c^d f\left(R\cos\psi\cos\varphi+a,R\cos\psi\sin\varphi+b,R\sin\psi+c\right)$$
$$\times\sin^2\psi\cos\psi d\psi d\varphi.$$

We also have

$$\iiint_{B(C,R)}\left|x_3-\overline{x_{3B(C,R)}}\right|^2dx_1dx_2dx_3$$
$$=\int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}r^2\sin^2\psi r^2\cos\psi drd\psi d\varphi$$
$$=\int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}r^4\sin^2\psi\cos\psi drd\psi d\varphi=\frac{4}{15}\pi R^5$$

and, similarly

$$\iiint_{B(C,R)}\left|x_1-\overline{x_{1B(C,R)}}\right|^2dx_1dx_2dx_3=\iiint_{B(C,R)}\left|x_2-\overline{x_{2B(C,R)}}\right|^2dx_1dx_2dx_3$$
$$=\frac{4}{15}\pi R^5.$$

Also

$$\iiint_{B(C,R)}\left|x_1-\overline{x_{1B(C,R)}}\right|\left|x_2-\overline{x_{2B(C,R)}}\right|dx_1dx_2dx_3$$
$$=\int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\left|r\cos\psi\cos\varphi\right|\left|r\cos\psi\sin\varphi\right|r^2\cos\psi drd\psi d\varphi$$
$$=\int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}r^4\cos^3\psi\left|\sin\varphi\cos\varphi\right|drd\psi d\varphi=\frac{8}{15}R^5$$

and, similarly

$$\iiint_{B(C,R)} \left| x_1 - \overline{x_{1B(C,R)}} \right| \left| x_3 - \overline{x_{3B(C,R)}} \right| dx_1 dx_2 dx_3$$
$$= \iiint_{B(C,R)} \left| x_2 - \overline{x_{2B(C,R)}} \right| \left| x_3 - \overline{x_{3B(C,R)}} \right| dx_1 dx_2 dx_3 = \frac{8}{15} R^5.$$

Then by (5.2) we get

$$\begin{aligned}
(6.2)\quad & \Bigg| \frac{3}{4\pi R^3} \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} f\left(r\cos\psi\cos\varphi + a, r\cos\psi\sin\varphi + b, r\sin\psi + c\right) \\
& \qquad \times r^2 \cos\psi dr d\psi d\varphi - \frac{1}{4} f\left(a,b,c\right) \\
& + \frac{3}{16\pi} \int_a^b \int_c^d f\left(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c\right) \\
& \qquad \times \cos^3\psi \cos^2\varphi d\psi d\varphi \\
& - \frac{3}{16\pi} \int_a^b \int_c^d f\left(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c\right) \\
& \qquad \times \cos^3\psi \sin^2\varphi d\psi d\varphi \\
& + \frac{3}{16\pi} \int_a^b \int_c^d f\left(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c\right) \\
& \qquad \times \sin^2\psi \cos\psi d\psi d\varphi \Bigg| \\
& \le \frac{1}{40} R^2 \left[\left\| \frac{\partial^2 f}{\partial^2 x} \right\|_{B(C,R),\infty} + \left\| \frac{\partial^2 f}{\partial^2 y} \right\|_{B(C,R),\infty} + \left\| \frac{\partial^2 f}{\partial^2 z} \right\|_{B(C,R),\infty} \right] \\
& \quad + \frac{1}{10\pi} R^2 \left[\left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_{B(C,R),\infty} + \left\| \frac{\partial^2 f}{\partial y \partial z} \right\|_{B(C,R),\infty} + \left\| \frac{\partial^2 f}{\partial z \partial x} \right\|_{B(C,R),\infty} \right].
\end{aligned}$$

References

[1] Apostol, T. M. *Calculus Volume II, Multi Variable Calculus and Linear Algebra, with Applications to Differential Equations and Probability*, Second Edition, John Wiley & Sons, New York London Sydney Toronto, 1969.

[2] Barnett, N. S.; Cîrstea, F. C. and Dragomir, S. S. Some inequalities for the integral mean of Hölder continuous functions defined on disks in a plane, in *Inequality Theory and Applications*, Vol. **2** (Chinju/Masan, 2001), 7–18, Nova Sci. Publ., Hauppauge, NY, 2003. Preprint *RGMIA Res. Rep. Coll.* **5** (2002), Nr. 1, Art. 7, 10 pp. [Online https://rgmia.org/papers/v5n1/BCD.pdf].

[3] Barnett, N. S.; Dragomir, S. S. An Ostrowski type inequality for double integrals and applications for cubature formulae. Soochow J. Math. 27 (2001), no. 1, 1–10.

[4] Barnett, N. S.; Dragomir, S. S.; Pearce, C. E. M. A quasi-trapezoid inequality for double integrals. ANZIAM J. 44 (2003), no. 3, 355–364.

[5] Budak, Hüseyin; Sarıkaya, Mehmet Zeki An inequality of Ostrowski-Grüss type for double integrals. Stud. Univ. Babeş-Bolyai Math. 62 (2017), no. 2, 163–173.

[6] Dragomir, S. S.; Cerone, P.; Barnett, N. S.; Roumeliotis, J. An inequality of the Ostrowski type for double integrals and applications for cubature formulae. Tamsui Oxf. J. Math. Sci. 16 (2000), no. 1, 1–16.

[7] Dragomir S. S., Ostrowski type integral inequalities for double integral on general domains, Preprint *RGMIA Res. Rep. Coll.* **22** (2019), Art. 47, 16 pp. [Online http://rgmia.org/papers/v22/v22a47.pdf].

[8] S. S. Dragomir, Perturbed Ostrowski type inequalities for double integral on general domains, Preprint *RGMIA Res. Rep. Coll.* **22** (2019), Art. 58, 16 pp. [Online http://rgmia.org/papers/v22/v22a58.pdf].

[9] Erden, Samet; Sarikaya, Mehmet Zeki On exponential Pompeiu's type inequalities for double integrals with applications to Ostrowski's inequality. New Trends Math. Sci. 4 (2016), no. 1, 256–267.

[10] Hanna, George Some results for double integrals based on an Ostrowski type inequality. Ostrowski type inequalities and applications in numerical integration, 331–371, Kluwer Acad. Publ., Dordrecht, 2002.

[11] Hanna, G.; Dragomir, S. S.; Cerone, P. A general Ostrowski type inequality for double integrals. Tamkang J. Math. 33 (2002), no. 4, 319–333.

[12] Liu, Zheng A sharp general Ostrowski type inequality for double integrals. Tamsui Oxf. J. Inf. Math. Sci. 28 (2012), no. 2, 217–226.

[13] Özdemir, M. Emin; Akdemir, Ahmet Ocak; Set, Erhan A new Ostrowski-type inequality for double integrals. J. Inequal. Spec. Funct. 2 (2011), no. 1, 27–34.

[14] Pachpatte, B. G. A new Ostrowski type inequality for double integrals. Soochow J. Math. 32 (2006), no. 2, 317–322.

[15] Sarikaya, Mehmet Zeki On the Ostrowski type integral inequality for double integrals. Demonstratio Math. 45 (2012), no. 3, 533–540.

[16] Sarikaya, Mehmet Zeki; Ogunmez, Hasan On the weighted Ostrowski-type integral inequality for double integrals. Arab. J. Sci. Eng. 36 (2011), no. 6, 1153–1160.

[17] Singer, M. The divergence theorem, [Online https://www.maths.ed.ac.uk/~jmf/Teaching/Lectures/divthm.pdf].

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa