INEQUALITIES FOR DOUBLE INTEGRALS OF SCHUR CONVEX FUNCTIONS ON SYMMETRIC AND CONVEX DOMAINS

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Abstract. In this paper, by making use of Green's identity for double integrals, we establish some integral inequalities for Schur convex functions defined on domains $D \subset \mathbb{R}^2$ that are symmetric, convex and have nonempty interiors. Examples for squares and disks are also provided.

1. Introduction

For any $x = (x_1, ..., x_n) \in \mathbb{R}^n$, let $x_{[1]} \geq ... \geq x_{[n]}$ denote the components of x in decreasing order, and let $x_{\downarrow} = \left(x_{[1]}, ..., x_{[n]}\right)$ denote the decreasing rearrangement of x. For $x, y \in \mathbb{R}^n$, $x \prec y$ if, by definition,

$$\begin{cases} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \; k = 1, ..., n-1; \\ \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}. \end{cases}$$

When $x \prec y$, x is said to be *majorized* by y (y *majorizes* x). This notation and terminology was introduced by Hardy, Littlewood and Pólya in 1934.

Functions that preserve the ordering of majorization are said to be Schur-convex. Perhaps "Schur-increasing" would be more appropriate, but the term "Schur-convex" is by now well entrenched in the literature, [3, p.80].

A real-valued function ϕ defined on a set $\mathcal{A} \subset \mathbb{R}^n$ is said to be *Schur-convex* on $\mathcal{A}$ if

$$x \prec y \text{ on } \mathcal{A} \Rightarrow \phi(x) \leq \phi(y). \tag{1.1}$$

If, in addition, $\phi(x) < \phi(y)$ whenever $x \prec y$ but x is not a permutation of y, then ϕ is said to be *strictly Schur-convex* on $\mathcal{A}$. If $\mathcal{A} = \mathbb{R}^n$, then ϕ is simply said to be Schur-convex or strictly Schur-convex.

For fundamental properties of Schur convexity see the monograph [3] and the references therein. For some recent results, see [1]-[2] and [4]-[6].

The following result is known in the literature as *Schur-Ostrowski theorem* [3, p. 84]:

Theorem 1. *Let $I \subset \mathbb{R}$ be an open interval and let $\phi : I^n \to \mathbb{R}$ be continuously differentiable. Necessary and sufficient conditions for ϕ to be Schur-convex on I^n are*

$$\phi \textit{ is symmetric on } I^n, \tag{1.2}$$

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and for all $i \neq j$, *with* $i, j \in \{1, ..., n\}$,

$$(z_i - z_j)\left[\frac{\partial\phi(z)}{\partial x_i} - \frac{\partial\phi(z)}{\partial x_j}\right] \geq 0 \text{ for all } z \in I^n, \tag{1.3}$$

where $\frac{\partial\phi}{\partial x_k}$ *denotes the partial derivative of* ϕ *with respect to its* k*-th argument.*

With the aid of (1.2), condition (1.3) can be replaced by the condition

$$(z_1 - z_2)\left[\frac{\partial\phi(z)}{\partial x_1} - \frac{\partial\phi(z)}{\partial x_2}\right] \geq 0 \text{ for all } z \in I^n. \tag{1.4}$$

This simplified condition is sometimes more convenient to verify.

The above condition is not sufficiently general for all applications because the domain of ϕ may not be a Cartesian product.

Let $\mathcal{A} \subset \mathbb{R}^n$ be a set with the following properties:

(i) $\mathcal{A}$ is *symmetric* in the sense that $x \in \mathcal{A} \Rightarrow x\Pi \in \mathcal{A}$ for all permutations Π of the coordinates.

(ii) $\mathcal{A}$ is convex and has a nonempty interior.

We have the following result, [3, p. 85].

Theorem 2. *If* ϕ *is continuously differentiable on the interior of* $\mathcal{A}$ *and continuous on* $\mathcal{A}$*, then necessary and sufficient conditions for* ϕ *to be Schur-convex on* $\mathcal{A}$ *are*

$$\phi \text{ is symmetric on } \mathcal{A} \tag{1.5}$$

and

$$(z_1 - z_2)\left[\frac{\partial\phi(z)}{\partial x_1} - \frac{\partial\phi(z)}{\partial x_2}\right] \geq 0 \text{ for all } z \in \mathcal{A}. \tag{1.6}$$

It is well known that any symmetric convex function defined on a symmetric convex set $\mathcal{A}$ is Schur convex, [3, p. 97]. If the function $\phi : \mathcal{A} \to \mathbb{R}$ is symmetric and quasi-convex, namely

$$\phi(\alpha u + (1-\alpha)v) \leq \max\{\phi(u), \phi(v)\}$$

for all $\alpha \in [0,1]$ and $u, v \in \mathcal{A}$, a symmetric convex set, then ϕ is Schur convex on $\mathcal{A}$ [3, p. 98].

In this paper we establish some integral inequalities for Schur convex functions defined on domains $D \subset \mathbb{R}^2$ that are symmetric, convex and have nonempty interiors. Examples for squares and disks are also provided.

2. Main Results

For a function $f : D \to \mathbb{C}$ having partial derivatives on the domain $D \subset \mathbb{R}^2$ we define $\Lambda_{\partial f, D} : D \to \mathbb{C}$ as

$$\Lambda_{\partial f, D}(x, y) := (x - y)\left(\frac{\partial f(x, y)}{\partial x} - \frac{\partial f(x, y)}{\partial y}\right).$$

Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a region D. Let L and M be scalar functions defined at least on an open set containing D. Assume L and M have continuous first partial derivatives. Then the following equality is well known as the Green theorem (see https://en.wikipedia.org/wiki/Green%27s_theorem)

$$\int\int_D \left(\frac{\partial M(x,y)}{\partial x} - \frac{\partial L(x,y)}{\partial y}\right) dxdy = \oint_{\partial D} (L(x,y)\,dx + M(x,y)\,dy)\,. \tag{G}$$

By applying this equality for real and imaginary parts, we can also state it for complex valued functions P and Q.

Moreover, if the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t \in [a,b]$, with x, y differentiable on (a,b) then we can calculate the path integral as

$$\oint_{\partial D} (L(x,y)\,dx + M(x,y)\,dy) = \int_a^b [L(x(t), y(t))\,x'(t) + M(x(t), y(t))\,y'(t)]\,dt.$$

We have the following identity of interest:

Lemma 1. *Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a region D. Assume that the function $f : D \to \mathbb{C}$ has continuous partial derivatives on the domain D. Then*

$$\frac{1}{2}\oint_{\partial D} [(x-y)\,f(x,y)\,dx + (x-y)\,f(x,y)\,dy] - \int\int_D f(x,y)\,dxdy = \frac{1}{2}\int\int_D \Lambda_{\partial f, D}(x,y)\,dxdy. \tag{2.1}$$

Proof. Consider the functions

$$M(x,y) := (x-y)\,f(x,y) \text{ and } L(x,y) := (x-y)\,f(x,y)$$

for $(x,y) \in D$.

We have

$$\frac{\partial}{\partial x}[(x-y)\,f(x,y)] = f(x,y) + (x-y)\frac{\partial f(x,y)}{\partial x}$$

and

$$\frac{\partial}{\partial y}[(y-x)\,f(x,y)] = f(x,y) + (y-x)\frac{\partial f(x,y)}{\partial y}$$

for $(x,y) \in D$.

If we add these two equalities, then we get

$$\frac{\partial M(x,y)}{\partial x} - \frac{\partial L(x,y)}{\partial y} = 2f(x,y) + \Lambda_{\partial f, D}(x,y) \tag{2.2}$$

for $(x,y) \in D$.

If we integrate this equality on D, then we obtain

$$\int\int_D \left(\frac{\partial M(x,y)}{\partial x} - \frac{\partial L(x,y)}{\partial y}\right) dxdy = 2\int\int_D f(x,y)\,dxdy + \int\int_D \Lambda_{\partial f, D}(x,y)\,dxdy. \tag{2.3}$$

From Green's identity we also have

$$\int\int_D \left(\frac{\partial M(x,y)}{\partial x} - \frac{\partial L(x,y)}{\partial y} \right) dxdy = \oint_{\partial D} (L(x,y)\,dx + M(x,y)\,dy) \tag{2.4}$$
$$= \oint_{\partial D} [(x-y) f(x,y)\,dx + (x-y) f(x,y)\,dy].$$

By employing (2.3) and (2.4) we deduce the desired equality (2.1). □

Corollary 1. *With the assumptions of Lemma 1 and if the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t \in [a,b]$, with x, y differentiable on (a,b), then*

$$\frac{1}{2}\int_a^b (x(t) - y(t)) f(x(t), y(t)) (x'(t) + y'(t))\,dt - \int\int_D f(x,y)\,dxdy \tag{2.5}$$
$$= \frac{1}{2}\int\int_D \Lambda_{\partial f, D}(x,y)\,dxdy.$$

We have the following result for Schur convex functions defined on symmetric convex domains of $\mathbb{R}^2$.

Theorem 3. *Let $D \subset \mathbb{R}^2$ be symmetric, convex and has a nonempty interior. If ϕ is continuously differentiable on the interior of D, continuous and Schur convex on D and ∂D is a simple, closed counterclockwise curve in the xy-plane bounding D, then*

$$\int\int_D \phi(x,y)\,dxdy \le \frac{1}{2}\oint_{\partial D} [(x-y)\phi(x,y)\,dx + (x-y)\phi(x,y)\,dy]. \tag{2.6}$$

If ϕ is Schur concave on D, then the sign of inequality reverses in (2.6).

The proof follows by Lemma 1 and Theorem 1.

Corollary 2. *Let $D \subset \mathbb{R}^2$ be symmetric, convex and has a nonempty interior. If ϕ is continuously differentiable on the interior of D, continuous and convex or quasi-convex on D and ∂D is a simple, closed counterclockwise curve in the xy-plane bounding D, then the inequality (2.6) is valid.*

Remark 1. *With the assumptions of Theorem 3 and if the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t \in [a,b]$, with x, y differentiable on (a,b), then*

$$\int\int_D \phi(x,y)\,dxdy \le \frac{1}{2}\int_a^b (x(t) - y(t))\phi(x(t), y(t))(x'(t) + y'(t))\,dt. \tag{2.7}$$

Let $a<b$. Put $A=(a,a)$, $B=(b,a)$, $C=(b,b)$, $D=(a,b)\in\mathbb{R}^2$ the vertices of the square $ABCD=[a,b]^2$. Consider the counterclockwise segments

$$AB:\begin{cases} x=(1-t)a+tb \\ \\ y=a \end{cases},\ t\in[0,1]$$

$$BC:\begin{cases} x=b \\ \\ y=(1-t)a+tb \end{cases},\ t\in[0,1]$$

$$CD:\begin{cases} x=(1-t)b+ta \\ \\ y=b \end{cases},\ t\in[0,1]$$

and

$$DA:\begin{cases} x=a \\ \\ y=(1-t)b+ta \end{cases},\ t\in[0,1]\ .$$

Therefore $\partial(ABCD)=AB\cup BC\cup CD\cup DA$.

For any function f defined on $ABCD$, we have

$$\begin{aligned}
&\oint_{AB}[(x-y)f(x,y)\,dx+(x-y)f(x,y)\,dy] \\
&=(b-a)\int_0^1((1-t)a+tb-a)f((1-t)a+tb,a)\,dt \\
&=(b-a)^2\int_0^1 tf((1-t)a+tb,a)\,dt,
\end{aligned}$$

$$\begin{aligned}
&\oint_{BC}[(x-y)f(x,y)\,dx+(x-y)f(x,y)\,dy] \\
&=(b-a)\int_0^1(b-(1-t)a-tb)f(b,(1-t)a+tb)\,dt \\
&=(b-a)^2\int_0^1(1-t)f(b,(1-t)a+tb)\,dt,
\end{aligned}$$

$$\begin{aligned}
&\oint_{CD}[(x-y)f(x,y)\,dx+(x-y)f(x,y)\,dy] \\
&=(a-b)\int_0^1((1-t)b+ta-b)f((1-t)b+ta,b)\,dt \\
&=(a-b)^2\int_0^1 tf((1-t)b+ta,b)\,dt \\
&=(a-b)^2\int_0^1(1-t)f((1-t)a+tb,b)\,dt \text{ (by change of variable).}
\end{aligned}$$

and

$$\oint_{DA} [(x-y) f(x,y) dx + (x-y) f(x,y) dy]$$
$$= (a-b) \int_0^1 (a - (1-t) b - ta) f(a, (1-t) b + ta) dt$$
$$= (a-b)^2 \int_0^1 (1-t) f(a, (1-t) b + ta) dt$$
$$= (a-b)^2 \int_0^1 t f(a, (1-t) a + tb) dt \text{ (by change of variable).}$$

Therefore

$$\oint_{\partial(ABCD)} [(x-y) f(x,y) dx + (x-y) f(x,y) dy] \tag{2.8}$$

$$= (b-a)^2 \int_0^1 t f((1-t) a + tb, a) dt + (b-a)^2 \int_0^1 (1-t) f(b, (1-t) a + tb) dt$$
$$+ (b-a)^2 \int_0^1 (1-t) f((1-t) a + tb, b) dt + (b-a)^2 \int_0^1 t f(a, (1-t) a + tb) dt$$
$$= (b-a)^2 \int_0^1 t [f((1-t) a + tb, a) + f(a, (1-t) a + tb)] dt$$
$$+ (b-a)^2 \int_0^1 (1-t) [f(b, (1-t) a + tb) + f((1-t) a + tb, b)] dt.$$

Since the vast majority of examples of Schur convex functions are defined on the Cartesian product of intervals, we can state the following result of interest:

Corollary 3. *If ϕ is continuously differentiable on the interior of $D = [a,b]^2$, continuous on D and Schur convex, then*

$$\frac{1}{(b-a)^2} \int_a^b \int_a^b \phi(x,y) dxdy \le \int_0^1 t\phi((1-t) a + tb, a) dt + \int_0^1 (1-t) \phi((1-t) a + tb, b) dt. \tag{2.9}$$

Proof. From (2.6) we get

$$\frac{1}{(b-a)^2} \int_a^b \int_a^b \phi(x,y) dxdy \le \int_0^1 t \left[\frac{\phi((1-t) a + tb, a) + \phi(a, (1-t) a + tb)}{2}\right] dt + \int_0^1 (1-t) \left[\frac{\phi((1-t) a + tb, b) + \phi(b, (1-t) a + tb)}{2}\right] dt. \tag{2.10}$$

Since ϕ is symmetric on $D = [a,b]^2$, hence

$$\phi((1-t) a + tb, a) = \phi(a, (1-t) a + tb)$$

and

$$\phi((1-t)a+tb,b)=\phi(b,(1-t)a+tb)$$

for all $t\in[0,1]$ and by (2.10) we get (2.9). $\square$

Remark 2. *By making the change of variable* $x=(1-t)a+tb,\ t\in[0,1]$, *then* $dx=(b-a)\,dt,\ t=\frac{x-a}{b-a}$ *and by (2.9) we get*

$$\begin{aligned}\frac{1}{(b-a)^2}\int_a^b\int_a^b\phi(x,y)\,dxdy \\ \leq \frac{1}{b-a}\int_a^b\frac{x-a}{b-a}\phi(x,a)\,dx+\frac{1}{b-a}\int_a^b\frac{b-x}{b-a}\phi(x,b)\,dx,\end{aligned}\tag{2.11}$$

or, equivalently,

$$\int_a^b\int_a^b\phi(x,y)\,dxdy\leq\int_a^b(x-a)\,\phi(x,a)\,dx+\int_a^b(b-x)\,\phi(x,b)\,dx.\tag{2.12}$$

3. Lower and Upper Schur Convexity

Start with the following extensions of Schur convex functions:

Definition 1. *Let D be symmetric, convex and has a nonempty interior in $\mathbb{R}^2$ and a symmetric function $f:D\to\mathbb{R}$ having continuous partial derivatives on $D\subset\mathbb{R}^2$.*

(i) *For $m\in\mathbb{R}$, f is called m-lower Schur convex on D if*

$$m(x-y)^2\leq\Lambda_{\partial f,D}(x,y)\ \text{ for all }\ (x,y)\in D.\tag{3.1}$$

(ii) *For $M\in\mathbb{R}$, f is called M-upper Schur convex on D if*

$$\Lambda_{\partial f,D}(x,y)\leq M(x-y)^2\ \text{ for all }\ (x,y)\in D.\tag{3.2}$$

(iii) *For $m,\ M\in\mathbb{R}$ with $m<M$, f is called (m,M)-Schur convex on D if*

$$m(x-y)^2\leq\Lambda_{\partial f,D}(x,y)\leq M(x-y)^2\ \text{ for all }\ (x,y)\in D.\tag{3.3}$$

We have the following simple but useful result :

Proposition 1. *Let D be symmetric, convex and has a nonempty interior in $\mathbb{R}^2$ and a symmetric function $f:D\to\mathbb{R}$ having continuous partial derivatives on $D\subset\mathbb{R}^2$.*

(i) *For $m\in\mathbb{R}$, f is m-lower Schur convex on D iff $f_m:D\to\mathbb{R}$,*

$$f_m(x,y):=f(x,y)-\frac{1}{2}m\left(x^2+y^2\right)$$

is Schur convex on D.

(ii) *For $M\in\mathbb{R}$, f is M-upper Schur convex on D iff $f_M:D\to\mathbb{R}$,*

$$f_M(x,y):=\frac{1}{2}M\left(x^2+y^2\right)-f(x,y)$$

is Schur convex on D.

(iii) *For $m,\ M\in\mathbb{R}$ with $m<M$, f is (m,M)-Schur convex on D iff f_m and f_M are Schur convex on D.*

Proof. (i). Observe that

$$\begin{aligned}\Lambda_{\partial f_m,D}(x,y) &= (x-y)\left(\frac{\partial f_m(x,y)}{\partial x}-\frac{\partial f_m(x,y)}{\partial y}\right)\\ &= (x-y)\left(\frac{\partial f(x,y)}{\partial x}-mx-\frac{\partial f(x,y)}{\partial y}+my\right)\\ &= (x-y)\left(\frac{\partial f(x,y)}{\partial x}-\frac{\partial f(x,y)}{\partial y}-m(x-y)\right)\\ &= \Lambda_{\partial f,D}(x,y)-m(x-y)^2,\end{aligned}$$

for all $(x,y)\in D$, which proves the statement.

The statements (ii) and (iii) follow in a similar way. □

We have:

Theorem 4. *Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a domain $D\subset\mathbb{R}^2$ that is symmetric, convex and has a nonempty interior.*

(i) *Assume that the function $f:D\to\mathbb{R}$ is m-lower Schur convex, then*

$$\begin{aligned}&\frac{1}{2}m\int\int_D (x-y)^2\,dxdy \\ &\leq \frac{1}{2}\oint_{\partial D}\left[(x-y)f(x,y)\,dx+(x-y)f(x,y)\,dy\right]-\int\int_D f(x,y)\,dxdy.\end{aligned}\tag{3.4}$$

(ii) *Assume that the function $f:D\to\mathbb{R}$ is M-upper Schur convex, then*

$$\begin{aligned}&\frac{1}{2}\oint_{\partial D}\left[(x-y)f(x,y)\,dx+(x-y)f(x,y)\,dy\right]-\int\int_D f(x,y)\,dxdy \\ &\leq \frac{1}{2}M\int\int_D (x-y)^2\,dxdy.\end{aligned}\tag{3.5}$$

(iii) *Assume that the function $f:D\to\mathbb{R}$ is (m,M)-Schur convex, then*

$$\begin{aligned}&\frac{1}{2}m\int\int_D (x-y)^2\,dxdy \\ &\leq \frac{1}{2}\oint_{\partial D}\left[(x-y)f(x,y)\,dx+(x-y)f(x,y)\,dy\right]-\int\int_D f(x,y)\,dxdy \\ &\leq \frac{1}{2}M\int\int_D (x-y)^2\,dxdy.\end{aligned}\tag{3.6}$$

Proof. (i) Since $f_m(x,y):=f(x,y)-\frac{1}{2}m\left(x^2+y^2\right)$ is Schur convex on D, then by (2.6) we get

$$\int\int_D f_m(x,y)\,dxdy\leq\frac{1}{2}\oint_{\partial D}\left[(x-y)f_m(x,y)\,dx+(x-y)f_m(x,y)\,dy\right],$$

namely

$$\begin{aligned}\int\int_D \left[f(x,y) - \frac{1}{2}m\left(x^2+y^2\right)\right] dxdy \\ \leq \frac{1}{2}\oint_{\partial D} \left\{(x-y)\left[f(x,y) - \frac{1}{2}m\left(x^2+y^2\right)\right] dx \right. \\ \left. + (x-y)\left[f(x,y) - \frac{1}{2}m\left(x^2+y^2\right)\right] dy\right\}.\end{aligned} \tag{3.7}$$

Since

$$\begin{aligned}\int\int_D \left[f(x,y) - \frac{1}{2}m\left(x^2+y^2\right)\right] dxdy = \int\int_D f(x,y)\, dxdy \\ - \frac{1}{2}m\int\int_D \left(x^2+y^2\right) dxdy\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\oint_{\partial D} \left\{(x-y)\left[f(x,y) - \frac{1}{2}m\left(x^2+y^2\right)\right] dx \right. \\ \left. + (x-y)\left[f(x,y) - \frac{1}{2}m\left(x^2+y^2\right)\right] dy\right\} \\ = \frac{1}{2}\oint_{\partial D} \left[(x-y) f(x,y)\, dx + (x-y) f(x,y)\, dy\right] \\ - \frac{1}{4}m\oint_{\partial D} \left[\left(x^2+y^2\right) dx + \left(x^2+y^2\right) dy\right],\end{aligned}$$

hence, by (3.7), we get

$$\begin{aligned}\frac{1}{2}m\left\{\frac{1}{2}\oint_{\partial D} \left[(x-y)\left(x^2+y^2\right) dx + (x-y)\left(x^2+y^2\right) dy\right] \right. \\ \left. - \int\int_D \left(x^2+y^2\right) dxdy\right\} \\ \leq \frac{1}{2}\oint_{\partial D} \left[(x-y) f(x,y)\, dx + (x-y) f(x,y)\, dy\right] - \int\int_D f(x,y)\, dxdy.\end{aligned} \tag{3.8}$$

Further, if we use the identity (2.1) for the function $g(x,y) = x^2+y^2$ we get

$$\begin{aligned}\frac{1}{2}\oint_{\partial D} \left[(x-y)\left(x^2+y^2\right) dx + (x-y)\left(x^2+y^2\right) dy\right] - \int\int_D \left(x^2+y^2\right) dxdy \\ = \frac{1}{2}\int\int_D 2(x-y)^2\, dxdy = \int\int_D (x-y)^2\, dxdy,\end{aligned}$$

which together with (3.8) gives the desired result (3.4).

The statements (ii) and (iii) follow in a similar way and we omit the details. □

If f is symmetric on D we have

$$\begin{aligned}\Lambda_{\partial f,D}(x,y) &= (x-y)\left(\frac{\partial f(x,y)}{\partial x}-\frac{\partial f(x,y)}{\partial y}\right)\\ &= (x-y)\left(\frac{\partial f(x,y)}{\partial x}-\frac{\partial f(y,x)}{\partial x}\right)\end{aligned}$$

for all $(x,y)\in D$.

If

$$0<k\le\left|\frac{\frac{\partial f(x,y)}{\partial x}-\frac{\partial f(y,x)}{\partial x}}{x-y}\right|\le K<\infty \text{ for all } (x,y)\in D \text{ with } x\neq y, \tag{3.9}$$

then

$$0\le k(x-y)^2\le\Lambda_{\partial f,D}(x,y)\le K(x-y)^2 \text{ for all } (x,y)\in D.$$

By making use of Theorem 4 we can state the following result:

Corollary 4. *Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a domain $D\subset\mathbb{R}^2$ that is symmetric, convex and has a nonempty interior. If f is continuously differentiable on the interior of D, continuous and symmetric on D and the partial derivative $\frac{\partial f}{\partial x}$ satisfies the condition (3.9), then we have the inequalities*

$$\begin{aligned}0 &\le \frac{1}{2}k\int\int_D(x-y)^2\,dxdy \\ &\le \frac{1}{2}\oint_{\partial D}[(x-y)f(x,y)\,dx+(x-y)f(x,y)\,dy]-\int\int_D f(x,y)\,dxdy \\ &\le \frac{1}{2}K\int\int_D(x-y)^2\,dxdy.\end{aligned}\tag{3.10}$$

Remark 3. *If $D=[a,b]^2$ and since*

$$\int_a^b\int_a^b(x-y)^2\,dxdy=\int_a^b\frac{(b-x)^3+(x-a)^3}{3}dx=\frac{1}{6}(b-a)^4$$

hence by (3.10) we get

$$\begin{aligned}0 &\le \frac{1}{12}k(b-a)^4 \\ &\le \int_a^b(x-a)f(x,a)\,dx+\int_a^b(b-x)f(x,b)\,dx-\int_a^b\int_a^b f(x,y)\,dxdy \\ &\le \frac{1}{12}K(b-a)^4,\end{aligned}\tag{3.11}$$

provided that f is continuously differentiable on the interior of $[a,b]^2$, continuous and symmetric on $[a,b]^2$ and the partial derivative $\frac{\partial f}{\partial x}$ satisfies the condition (3.9).

4. Examples for Disks

We consider the closed disk $D(O,R)$ centered in $O(0,0)$ and of radius $R>0$. This is parametrized by

$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta\end{cases},\ r\in[0,R],\ \theta\in[0,2\pi]$$

and the circle $\mathcal{C}(O,R)$ is parametrized by

$$\begin{cases} x = R\cos\theta \\ y = R\sin\theta \end{cases}, \ \theta \in [0, 2\pi].$$

Observe that, if $\phi : D(O,R) \to \mathbb{R}$, then

$$\begin{aligned} &\oint_{\mathcal{C}(O,R)} [(x-y)\phi(x,y)\,dx + (x-y)\phi(x,y)\,dy] \\ &= -\int_0^{2\pi} R(R\cos\theta - R\sin\theta)\sin\theta\phi(R\cos\theta, R\sin\theta)\,d\theta \\ &+ \int_0^{2\pi} R(R\cos\theta - R\sin\theta)\cos\theta\phi(R\cos\theta, R\sin\theta)\,d\theta \\ &= R^2\int_0^{2\pi} \phi(R\cos\theta, R\sin\theta)(\cos\theta - \sin\theta)^2\,d\theta. \end{aligned}$$

Also, we have

$$\int\int_{D(O,R)} \phi(x,y)\,dxdy = \int_0^R \int_0^{2\pi} \phi(r\cos\theta, r\sin\theta)\,rdrd\theta.$$

Using Theorem 3 we can state the following result:

Proposition 2. *If ϕ is continuously differentiable on the interior of $D(O,R)$, continuous and Schur convex on $D(O,R)$, then*

$$\begin{aligned} &\int_0^R \int_0^{2\pi} \phi(r\cos\theta, r\sin\theta)\,rdrd\theta \\ &\le \frac{1}{2}R^2\int_0^{2\pi} \phi(R\cos\theta, R\sin\theta)(\cos\theta - \sin\theta)^2\,d\theta. \end{aligned} \tag{4.1}$$

Now, observe that

$$\begin{aligned} \int\int_{D(O,R)} (x-y)^2\,dxdy &= \int_0^R \int_0^{2\pi} (R\cos\theta - R\sin\theta)^2\,rdrd\theta \\ &= \frac{1}{2}R^4\int_0^{2\pi} (\cos\theta - \sin\theta)^2\,d\theta \\ &= \frac{1}{2}R^4\int_0^{2\pi} (1 - 2\sin\theta\cos\theta)\,d\theta = \pi R^4. \end{aligned}$$

We also have, by Corollary 4, that:

Proposition 3. *If ϕ is continuously differentiable on the interior of $D(O,R)$, continuous and Schur convex on $D(O,R)$ and the derivative $\frac{\partial f}{\partial x}$ satisfies the condition (3.9) on $D(O,R)$, then*

$$\begin{aligned} \frac{1}{2}\pi kR^4 \le \frac{1}{2}R^2 &\int_0^{2\pi} \phi(R\cos\theta, R\sin\theta)(\cos\theta - \sin\theta)^2\,d\theta \\ &- \int_0^R \int_0^{2\pi} \phi(r\cos\theta, r\sin\theta)\,rdrd\theta \le \frac{1}{2}\pi KR^4. \end{aligned} \tag{4.2}$$

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