PRE-SCHUR CONVEX FUNCTIONS AND SOME INTEGRAL INEQUALITIES ON DOMAINS FROM PLANE

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we introduce the concept of pre-Schur convex functions defined on general domains from plane. Then, by making use of Green's identity for double integrals, we establish some integral inequalities for this class of functions that naturally generalize the case of Schur convex functions. Some exmples for rectangles and disks are also provided.

1. Introduction

For any $x = (x_1, ..., x_n) \in \mathbb{R}^n$, let $x_{[1]} \geq ... \geq x_{[n]}$ denote the components of x in decreasing order, and let $x_{\downarrow} = (x_{[1]}, ..., x_{[n]})$ denote the decreasing rearrangement of x. For x, $y \in \mathbb{R}^n$, $x \prec y$ if, by definition,

$$\begin{cases} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \ k = 1, ..., n-1; \\ \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}. \end{cases}$$

When $x \prec y$, x is said to be *majorized* by y (y *majorizes* x). This notation and terminology was introduced by Hardy, Littlewood and Pólya in 1934.

Functions that preserve the ordering of majorization are said to be Schur-convex. Perhaps "Schur-increasing" would be more appropriate, but the term "Schur-convex" is by now well entrenched in the literature, [4, p.80].

A real-valued function ϕ defined on a set $\mathcal{A} \subset \mathbb{R}^n$ is said to be Schur-convex on $\mathcal{A}$ if

$$x \prec y \text{ on } \mathcal{A} \Rightarrow \phi(x) \leq \phi(y). \tag{1.1}$$

If, in addition, $\phi(x) < \phi(y)$ whenever $x \prec y$ but x is not a permutation of y, then ϕ is said to be strictly Schur-convex on A. If $A = \mathbb{R}^n$, then ϕ is simply said to be Schur-convex or strictly Schur-convex.

For fundamental properties of Schur convexity see the monograph [4] and the references therein. For some recent results, see [1]-[3] and [5]-[7].

The following result is known in the literature as Schur-Ostrowski theorem [4, p. 84]:

Theorem 1. *Let $I \subset \mathbb{R}$ be an open interval and let $\phi : I^n \to \mathbb{R}$ be continuously differentiable. Necessary and sufficient conditions for ϕ to be Schur-convex on I^n are*

$$\phi \text{ is symmetric on } I^n, \tag{1.2}$$

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and for all $i \neq j$, *with* $i, j \in \{1, ..., n\}$,

$$(z_i - z_j)\left[\frac{\partial\phi(z)}{\partial x_i} - \frac{\partial\phi(z)}{\partial x_j}\right] \geq 0 \text{ for all } z \in I^n, \tag{1.3}$$

where $\frac{\partial\phi}{\partial x_k}$ *denotes the partial derivative of* ϕ *with respect to its k-th argument.*

With the aid of (1.2), condition (1.3) can be replaced by the condition

$$(z_1 - z_2)\left[\frac{\partial\phi(z)}{\partial x_1} - \frac{\partial\phi(z)}{\partial x_2}\right] \geq 0 \text{ for all } z \in I^n. \tag{1.4}$$

This simplified condition is sometimes more convenient to verify.

The above condition is not sufficiently general for all applications because the domain of ϕ may not be a Cartesian product.

Let $\mathcal{A} \subset \mathbb{R}^n$ be a set with the following properties:

(i) $\mathcal{A}$ is *symmetric* in the sense that $x \in \mathcal{A} \Rightarrow x\Pi \in \mathcal{A}$ for all permutations Π;

(ii) $\mathcal{A}$ is convex and has a nonempty interior.

We have the following result, [4, p. 85].

Theorem 2. *If* ϕ *is continuously differentiable on the interior of* $\mathcal{A}$ *and continuous on* $\mathcal{A}$*, then necessary and sufficient conditions for* ϕ *to be Schur-convex on* $\mathcal{A}$ *are*

$$\phi \text{ is symmetric on } \mathcal{A} \tag{1.5}$$

and

$$(z_1 - z_2)\left[\frac{\partial\phi(z)}{\partial x_1} - \frac{\partial\phi(z)}{\partial x_2}\right] \geq 0 \text{ for all } z \in \mathcal{A}. \tag{1.6}$$

It is well known that any symmetric convex function defined on a symmetric convex set $\mathcal{A}$ is Schur convex, [4, p. 97]. If the function $\phi : \mathcal{A} \to \mathbb{R}$ is symmetric and quasi-convex, namely

$$\phi(\alpha u + (1-\alpha)v) \leq \max\{\phi(u), \phi(v)\}$$

for all $\alpha \in [0,1]$ and $u, v \in \mathcal{A}$, a symmetric convex set, then ϕ is Schur convex on $\mathcal{A}$ [4, p. 98].

In the recent paper [2] we obtained the following result for Schur convex functions defined on symmetric convex domains of $\mathbb{R}^2$.

Theorem 3. *Let* $D \subset \mathbb{R}^2$ *be symmetric, convex and has a nonempty interior. If* ϕ *is continuously differentiable on the interior of* D*, continuous and Schur convex on* D *and* ∂D *is a simple, closed counterclockwise curve in the xy-plane bounding* D*, then*

$$\int\int_D \phi(x,y)\,dxdy \leq \frac{1}{2}\oint_{\partial D} [(x-y)\,\phi(x,y)\,dx + (x-y)\,\phi(x,y)\,dy]. \tag{1.7}$$

If ϕ *is Schur concave on* D, *then the sign of inequality reverses in (1.7).*

In this paper we introduce the concept of *pre-Schur convex functions* defined on general domains from plane. Then, by making use of *Green's identity* for double integrals, we establish some integral inequalities for this class of functions that naturally generalize the case of Schur convex functions. Some examples for rectangles and disks are also provided.

2. Pre-Schur Convexity

For a function $f : D \to \mathbb{C}$ having continuous partial derivatives on the domain $D \subset \mathbb{R}^2$ we define $\Lambda_{\partial f,D} : D \to \mathbb{C}$ as

$$\Lambda_{\partial f,D}(x,y) := (x-y)\left(\frac{\partial f(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y}\right).$$

We can introduce the following concept.

Definition 1. *Let D be a measurable subset of $\mathbb{R}^2$. A function $f : D \to \mathbb{R}$ having continuous partial derivatives on $D \subset \mathbb{R}^2$ is called pre-Schur convex on D if*

$$\Lambda_{\partial f,D}(x,y) \geq 0 \text{ for all } (x,y) \in D. \tag{2.1}$$

If the sign of inequality is reversed in (2.1) then we call it pre-Schur concave. This is equivalent to the fact that $-f$ is pre-Schur convex on D.

Obviously, Schur convex functions are pre-Schur convex as pointed out below.

Lemma 1. *Let $D \subset \mathbb{R}^2$ be symmetric, convex and has a nonempty interior. If ϕ is continuously differentiable on the interior of D, continuous on D and Schur convex, then ϕ is pre-Schur convex on D.*

The proof is obvious by Schur-Ostrowski theorem applied for $D \subset \mathbb{R}^2$.

Now consider the function $f(x,y) = x^2 - y^2$, $x, y > 0$. Then

$$\Lambda_{\partial f,D}(x,y) = (x-y)\left(\frac{\partial f(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y}\right) = 2\left(x^2 - y^2\right).$$

Now, we observe that if we restrict this function to a domain

$$D \subset \left\{(x,y) \in \mathbb{R}^2 \mid x^2 - y^2 \geq 0,\ x,y > 0\right\},$$

then f is pre-Schur convex on such D but not Schur convex.

Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a region D. Let L and M be scalar functions defined at least on an open set containing D. Assume L and M have continuous first partial derivatives. Then the following equality is well known as the Green theorem (see https://en.wikipedia.org/wiki/Green%27s_theorem)

$$\int\int_D \left(\frac{\partial M(x,y)}{\partial x} - \frac{\partial L(x,y)}{\partial y}\right) dxdy = \oint_{\partial D} (L(x,y)\,dx + M(x,y)\,dy). \tag{G}$$

By applying this equality for real and imaginary parts, we can also state it for complex valued functions P and Q.

Moreover, if the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t \in [a,b]$, with x, y differentiable on (a,b) then we can calculate the path integral as

$$\oint_{\partial D} (L(x,y)\,dx + M(x,y)\,dy) = \int_a^b [L(x(t),y(t))\,x'(t) + M(x(t),y(t))\,y'(t)]\,dt.$$

We need the following identity that was obtained in [2].

Lemma 2. *Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a region D. Assume that the function $f : D \to \mathbb{C}$ has continuous partial derivatives on the domain D. Then*

$$\frac{1}{2}\oint_{\partial D}\left[(x-y)\,f(x,y)\,dx+(x-y)\,f(x,y)\,dy\right]-\int\int_D f(x,y)\,dxdy = \frac{1}{2}\int\int_D \Lambda_{\partial f,D}(x,y)\,dxdy. \tag{2.2}$$

Proof. Consider the functions

$$M(x,y) := (x-y)\,f(x,y) \text{ and } L(x,y) := (x-y)\,f(x,y)$$

for $(x,y)\in D$.

We have

$$\frac{\partial}{\partial x}\left[(x-y)\,f(x,y)\right] = f(x,y)+(x-y)\,\frac{\partial f(x,y)}{\partial x}$$

and

$$\frac{\partial}{\partial y}\left[(y-x)\,f(x,y)\right] = f(x,y)+(y-x)\,\frac{\partial f(x,y)}{\partial y}$$

for $(x,y)\in D$.

If we add these two equalities, then we get

$$\frac{\partial M(x,y)}{\partial x}-\frac{\partial L(x,y)}{\partial y} = 2f(x,y)+\Lambda_{\partial f,D}(x,y) \tag{2.3}$$

for $(x,y)\in D$.

If we integrate this equality on D, then we obtain

$$\int\int_D\left(\frac{\partial M(x,y)}{\partial x}-\frac{\partial L(x,y)}{\partial y}\right)dxdy = 2\int\int_D f(x,y)\,dxdy+\int\int_D \Lambda_{\partial f,D}(x,y)\,dxdy. \tag{2.4}$$

From Green's identity we also have

$$\begin{aligned}\int\int_D\left(\frac{\partial M(x,y)}{\partial x}-\frac{\partial L(x,y)}{\partial y}\right)dxdy &= \oint_{\partial D}\left(L(x,y)\,dx+M(x,y)\,dy\right)\\ &= \oint_{\partial D}\left[(x-y)\,f(x,y)\,dx+(x-y)\,f(x,y)\,dy\right].\end{aligned} \tag{2.5}$$

By employing (2.4) and (2.5) we deduce the desired equality (2.2). □

Corollary 1. *With the assumptions of Lemma 2 and if the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t\in[a,b]$, with x, y differentiable on (a,b), then*

$$\frac{1}{2}\int_a^b (x(t)-y(t))\,f(x(t),y(t))\,(x'(t)+y'(t))\,dt-\int\int_D f(x,y)\,dxdy = \frac{1}{2}\int\int_D \Lambda_{\partial f,D}(x,y)\,dxdy. \tag{2.6}$$

The following generalization of Theorem 3 holds:

Theorem 4. *Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a region D. Assume that the function $f : D \to \mathbb{R}$ is pre-Schur convex on D, then*

$$\int\int_D f(x,y)\,dxdy \le \frac{1}{2}\oint_{\partial D} [(x-y) f(x,y)\,dx + (x-y) f(x,y)\,dy]. \tag{2.7}$$

It follows by the identity (2.2) and the definition of pre-Schur convex functions (2.1).

Remark 1. *With the assumptions of Theorem 4 and if the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t \in [a,b]$, with x, y differentiable on (a,b), then*

$$\int\int_D f(x,y)\,dxdy \le \frac{1}{2}\int_a^b (x(t) - y(t)) f(x(t), y(t)) (x'(t) + y'(t))\,dt. \tag{2.8}$$

Corollary 2. *Let $D \subset \mathbb{R}^2$ be symmetric, convex and has a nonempty interior. If ϕ is continuously differentiable on the interior of D, continuous and Schur convex on D, then the inequality (1.7) holds true. If ϕ is Schur concave on D, then the sign of inequality reverses in (1.7).*

Let $a < b$ and $c < d$. Put $A = (a,c)$, $B = (b,c)$, $C = (b,d)$, $D = (a,d) \in \mathbb{R}^2$ the vertices of the rectangle $ABCD = [a,b] \times [c,d]$. Consider the counterclockwise segments

$$AB : \begin{cases} x = (1-t)a + tb \\ \\ y = c \end{cases}, \ t \in [0,1]$$

$$BC : \begin{cases} x = b \\ \\ y = (1-t)c + td \end{cases}, \ t \in [0,1]$$

$$CD : \begin{cases} x = (1-t)b + ta \\ \\ y = d \end{cases}, \ t \in [0,1]$$

and

$$DA : \begin{cases} x = a \\ \\ y = (1-t)d + tc \end{cases}, \ t \in [0,1].$$

Therefore $\partial(ABCD) = AB \cup BC \cup CD \cup DA$.

We have

$$\begin{aligned} &\oint_{AB} [(x-y) f(x,y)\,dx + (x-y) f(x,y)\,dy] \\ &= (b-a)\int_0^1 ((1-t)a + tb - c) f((1-t)a + tb, c)\,dt \\ &= (b-a)\int_0^1 (t(b-a) + a - c) f((1-t)a + tb, c)\,dt, \end{aligned}$$

$$\oint_{BC} [(x-y) f(x,y) dx + (x-y) f(x,y) dy]$$
$$= (d-c) \int_0^1 (b-(1-t)c-td) f(b,(1-t)c+td) dt$$
$$= (d-c) \int_0^1 (b-c-t(d-c)) f(b,(1-t)c+td) dt,$$

$$\oint_{CD} [(x-y) f(x,y) dx + (x-y) f(x,y) dy]$$
$$= (a-b) \int_0^1 ((1-t)b+ta-d) f((1-t)b+ta,d) dt$$
$$= (a-b) \int_0^1 (t(a-b)+b-d) f((1-t)b+ta,d) dt$$
$$= (a-b) \int_0^1 ((1-t)(a-b)+b-d) f((1-t)a+tb,d) dt$$
$$= (b-a) \int_0^1 (d-a-t(b-a)) f((1-t)a+tb,d) dt$$

and

$$\oint_{DA} [(x-y) f(x,y) dx + (x-y) f(x,y) dy]$$
$$= (c-d) \int_0^1 (a-(1-t)d-tc) f(a,(1-t)d+tc) dt$$
$$= (c-d) \int_0^1 (a-td-(1-t)c) f(a,(1-t)c+td) dt$$
$$= (d-c) \int_0^1 (t(d-c)+c-a) f(a,(1-t)c+td) dt.$$

Therefore

$$\oint_{\partial(ABCD)} [(x-y) f(x,y) dx + (x-y) f(x,y) dy]$$
$$= (b-a) \int_0^1 (t(b-a)+a-c) f((1-t)a+tb,c) dt$$
$$+ (b-a) \int_0^1 (d-a-t(b-a)) f((1-t)a+tb,d) dt$$
$$+ (d-c) \int_0^1 (b-c-t(d-c)) f(b,(1-t)c+td) dt$$
$$+ (d-c) \int_0^1 (t(d-c)+c-a) f(a,(1-t)c+td) dt$$

and from (2.7) we get

$$\begin{aligned}(2.9)\quad \int_a^b \int_c^d f(x,y)\,dxdy &\le \frac{1}{2}(b-a)\int_0^1 (t(b-a)+a-c)\,f((1-t)a+tb,c)\,dt \\ &+\frac{1}{2}(b-a)\int_0^1 (d-a-t(b-a))\,f((1-t)a+tb,d)\,dt \\ &+\frac{1}{2}(d-c)\int_0^1 (b-c-t(d-c))\,f(b,(1-t)c+td)\,dt \\ &+\frac{1}{2}(d-c)\int_0^1 (t(d-c)+c-a)\,f(a,(1-t)c+td)\,dt,\end{aligned}$$

provided that the function $f:[a,b]\times[c,d]\to\mathbb{R}$ is pre-Schur convex on $D=[a,b]\times[c,d]$.

If $D=[a,b]\times[a,b]=[a,b]^2$, then from (2.9) we get

$$\begin{aligned}(2.10)\quad \int_a^b \int_a^b f(x,y)\,dxdy &\le \frac{1}{2}(b-a)^2\int_0^1 tf((1-t)a+tb,a)\,dt \\ &+\frac{1}{2}(b-a)^2\int_0^1 (1-t)\,f((1-t)a+tb,b)\,dt \\ &+\frac{1}{2}(b-a)^2\int_0^1 (1-t)\,f(b,(1-t)a+tb)\,dt \\ &+\frac{1}{2}(b-a)^2\int_0^1 tf(a,(1-t)a+tb)\,dt,\end{aligned}$$

provided that the function $f:[a,b]^2\to\mathbb{R}$ is pre-Schur convex on $D=[a,b]^2$.

If we make the change of variable $(1-t)a+tb=x$, then $dx=(b-a)\,dt$, $t=\frac{x-a}{b-a}$. Also for the change of variable $(1-t)c+td=y$, we have $dy=(d-c)\,dt$ and $t=\frac{y-c}{d-c}$. From (2.9) we get

$$\begin{aligned}(2.11)\quad \int_a^b \int_c^d f(x,y)\,dxdy &\le \frac{1}{2}\int_a^b [(x-c)\,f(x,c)+(d-x)\,f(x,d)]\,dx \\ &+\frac{1}{2}\int_c^d [(b-y)\,f(b,y)+(y-a)\,f(a,y)]\,dy,\end{aligned}$$

provided that $f:[a,b]\times[c,d]\to\mathbb{R}$ is pre-Schur convex on $D=[a,b]\times[c,d]$.

For $c=a$ and $d=b$ we get

$$\begin{aligned}(2.12)\quad \int_a^b \int_a^b f(x,y)\,dxdy &\le \frac{1}{2}\int_a^b [(x-a)\,f(x,a)+(b-x)\,f(x,b)]\,dx \\ &+\frac{1}{2}\int_a^b [(b-y)\,f(b,y)+(y-a)\,f(a,y)]\,dy,\end{aligned}$$

provided that the function $f:[a,b]^2\to\mathbb{R}$ is pre-Schur convex on $D=[a,b]^2$.

Since the vast majority of examples of Schur convex functions are defined on the Cartesian product of intervals, we can state the following result of interest:

Corollary 3. *If ϕ is continuously differentiable on the interior of $D=[a,b]^2$, continuous on D and Schur convex, then*

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b \phi(x,y)\,dxdy \le \int_0^1 t\phi((1-t)a+tb,a)\,dt + \int_0^1 (1-t)\phi((1-t)a+tb,b)\,dt \tag{2.13}$$

or, equivalently,

$$\int_a^b\int_a^b \phi(x,y)\,dxdy \le \int_a^b (x-a)\phi(x,a)\,dx + \int_a^b (b-x)\phi(x,b)\,dx. \tag{2.14}$$

Proof. From (2.10) we get

$$\begin{aligned}\frac{1}{(b-a)^2}\int_a^b\int_a^b \phi(x,y)\,dxdy &\le \int_0^1 t\left[\frac{\phi((1-t)a+tb,a)+\phi(a,(1-t)a+tb)}{2}\right]dt\\ &\quad+\int_0^1 (1-t)\left[\frac{\phi((1-t)a+tb,b)+\phi(b,(1-t)a+tb)}{2}\right]dt.\end{aligned} \tag{2.15}$$

Since ϕ is symmetric on $D=[a,b]^2$, hence

$$\phi((1-t)a+tb,a)=\phi(a,(1-t)a+tb)$$

and

$$\phi((1-t)a+tb,b)=\phi(b,(1-t)a+tb)$$

for all $t\in[0,1]$ and by (2.15) we get (2.13). □

3. Lower and Upper Pre-Schur Convexity

Start with the following extensions of pre-Schur convex functions:

Definition 2. *Let D be a measurable subset of $\mathbb{R}^2$ and a function $f:D\to\mathbb{R}$ having continuous partial derivatives on $D\subset\mathbb{R}^2$.*

(i) *For $m\in\mathbb{R}$, f is called m-lower pre-Schur convex on D if*

$$m(x-y)^2 \le \Lambda_{\partial f,D}(x,y) \text{ for all } (x,y)\in D. \tag{3.1}$$

(ii) *For $M\in\mathbb{R}$, f is called M-upper pre-Schur convex on D if*

$$\Lambda_{\partial f,D}(x,y) \le M(x-y)^2 \text{ for all } (x,y)\in D. \tag{3.2}$$

(iii) *For m, $M\in\mathbb{R}$ with $m<M$, f is called (m,M)-pre-Schur convex on D if*

$$m(x-y)^2 \le \Lambda_{\partial f,D}(x,y) \le M(x-y)^2 \text{ for all } (x,y)\in D. \tag{3.3}$$

We have the following simple result:

Proposition 1. *Let D be a measurable subset of $\mathbb{R}^2$ and a function $f:D\to\mathbb{R}$ having continuous partial derivatives on $D\subset\mathbb{R}^2$.*

(i) *For $m\in\mathbb{R}$, f is m-lower pre-Schur convex on D iff $f_m:D\to\mathbb{R}$, $f_m(x,y):=f(x,y)-\frac{1}{2}m(x^2+y^2)$ is pre-Schur convex on D.*

(ii) *For $M\in\mathbb{R}$, f is M-upper pre-Schur convex on D iff $f_M:D\to\mathbb{R}$, $f_M(x,y):=\frac{1}{2}M(x^2+y^2)-f(x,y)$ is pre-Schur convex on D.*

(iii) *For $m,\ M \in \mathbb{R}$ with $m < M$, f is (m, M)-pre-Schur convex on D iff f_m and f_M are pre-Schur convex on D.*

Proof. (i). Observe that

$$\begin{aligned}\Lambda_{\partial f_m, D}(x, y) &= (x-y)\left(\frac{\partial f_m(x,y)}{\partial x} - \frac{\partial f_m(x,y)}{\partial y}\right) \\ &= (x-y)\left(\frac{\partial f(x,y)}{\partial x} - mx - \frac{\partial f(x,y)}{\partial y} + my\right) \\ &= (x-y)\left(\frac{\partial f(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y} - m(x-y)\right) \\ &= \Lambda_{\partial f, D}(x,y) - m(x-y)^2,\end{aligned}$$

for all $(x, y) \in D$, which proves the statement.

The statements (ii) and (iii) follow in a similar way. □

We have:

Theorem 5. *Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a region D.*

(i) *Assume that the function $f : D \to \mathbb{R}$ is m-lower pre-Schur convex, then*

$$\begin{aligned}&\frac{1}{2}m\int\int_D (x-y)^2\, dxdy \\ &\le \frac{1}{2}\oint_{\partial D}\left[(x-y)\, f(x,y)\, dx + (x-y)\, f(x,y)\, dy\right] - \int\int_D f(x,y)\, dxdy.\end{aligned} \tag{3.4}$$

(ii) *Assume that the function $f : D \to \mathbb{R}$ is M-upper pre-Schur convex, then*

$$\begin{aligned}&\frac{1}{2}\oint_{\partial D}\left[(x-y)\, f(x,y)\, dx + (x-y)\, f(x,y)\, dy\right] - \int\int_D f(x,y)\, dxdy \\ &\le \frac{1}{2}M\int\int_D (x-y)^2\, dxdy.\end{aligned} \tag{3.5}$$

(iii) *Assume that the function $f : D \to \mathbb{R}$ is (m, M)-pre-Schur convex, then*

$$\begin{aligned}&\frac{1}{2}m\int\int_D (x-y)^2\, dxdy \\ &\le \frac{1}{2}\oint_{\partial D}\left[(x-y)\, f(x,y)\, dx + (x-y)\, f(x,y)\, dy\right] - \int\int_D f(x,y)\, dxdy \\ &\le \frac{1}{2}M\int\int_D (x-y)^2\, dxdy.\end{aligned} \tag{3.6}$$

Proof. (i) Since $f_m(x,y) := f(x,y) - \frac{1}{2}m\left(x^2+y^2\right)$ is pre-Schur convex on D, then by (2.7) we get

$$\int\int_D f_m(x,y)\, dxdy \le \frac{1}{2}\oint_{\partial D}\left[(x-y)\, f_m(x,y)\, dx + (x-y)\, f_m(x,y)\, dy\right],$$

namely

$$\int\int_D \left[f(x,y) - \frac{1}{2}m\left(x^2+y^2\right)\right] dxdy \tag{3.7}$$
$$\le \frac{1}{2}\oint_{\partial D} \left\{(x-y)\left[f(x,y) - \frac{1}{2}m\left(x^2+y^2\right)\right] dx \right.$$
$$\left. + (x-y)\left[f(x,y) - \frac{1}{2}m\left(x^2+y^2\right)\right] dy\right\}.$$

Since

$$\int\int_D \left[f(x,y) - \frac{1}{2}m\left(x^2+y^2\right)\right] dxdy = \int\int_D f(x,y)\,dxdy - \frac{1}{2}m\int\int_D \left(x^2+y^2\right) dxdy$$

and

$$\frac{1}{2}\oint_{\partial D} \left\{(x-y)\left[f(x,y) - \frac{1}{2}m\left(x^2+y^2\right)\right] dx + (x-y)\left[f(x,y) - \frac{1}{2}m\left(x^2+y^2\right)\right] dy\right\}$$
$$= \frac{1}{2}\oint_{\partial D} \left[(x-y)f(x,y)\,dx + (x-y)f(x,y)\,dy\right] - \frac{1}{4}m\oint_{\partial D} \left[\left(x^2+y^2\right)dx + \left(x^2+y^2\right)dy\right].$$

hence (3.7) we then get

$$\frac{1}{2}m\left\{\frac{1}{2}\oint_{\partial D} \left[(x-y)\left(x^2+y^2\right)dx + (x-y)\left(x^2+y^2\right)dy\right] - \int\int_D \left(x^2+y^2\right) dxdy\right\} \tag{3.8}$$
$$\le \frac{1}{2}\oint_{\partial D} \left[(x-y)f(x,y)\,dx + (x-y)f(x,y)\,dy\right] - \int\int_D f(x,y)\,dxdy.$$

Further, if we use the identity (2.2) for the function $g(x,y) = x^2 + y^2$ we get

$$\frac{1}{2}\oint_{\partial D} \left[(x-y)\left(x^2+y^2\right)dx + (x-y)\left(x^2+y^2\right)dy\right] - \int\int_D \left(x^2+y^2\right) dxdy$$
$$= \frac{1}{2}\int\int_D 2(x-y)^2\,dxdy = \int\int_D (x-y)^2\,dxdy,$$

which together with (3.8) gives the desired result (3.4).

The statements (ii) and (iii) follow in a similar way and we omit the details. $\square$

Corollary 4. *Assume that the function $f:[a,b]\times[c,d]\to\mathbb{R}$ is (m,M)-pre-Schur convex, then*

$$\begin{aligned}(3.9)\quad & \frac{(b-c)^4-(a-c)^4-(d-b)^4+(d-a)^4}{24}m \\ &\le \frac{1}{2}\int_a^b \left[(x-c)\,f(x,c)+(d-x)\,f(x,d)\right]dx \\ &+\frac{1}{2}\int_c^d \left[(b-y)\,f(b,y)+(y-a)\,f(a,y)\right]dy \\ &-\int_a^b\int_c^d f(x,y)\,dxdy \le \frac{(b-c)^4-(a-c)^4-(d-b)^4+(d-a)^4}{24}M.\end{aligned}$$

In particular, if $[c,d]=[a,b]$, then

$$\begin{aligned}(3.10)\quad & \frac{1}{12}m\,(b-a)^4 \\ &\le \frac{1}{2}\int_a^b \left[(x-a)\,f(x,a)+(b-x)\,f(x,b)\right]dx \\ &+\frac{1}{2}\int_a^b \left[(b-y)\,f(b,y)+(y-a)\,f(a,y)\right]dy \\ &-\int_a^b\int_a^b f(x,y)\,dxdy \le \frac{1}{12}M\,(b-a)^4.\end{aligned}$$

Proof. From (3.6) we have

$$\begin{aligned}(3.11)\quad & \frac{1}{2}m\int_a^b\int_c^d (x-y)^2\,dxdy \\ &\le \frac{1}{2}\oint_{\partial(ABCD)} \left[(x-y)\,f(x,y)\,dx+(x-y)\,f(x,y)\,dy\right]-\int_a^b\int_c^d f(x,y)\,dxdy \\ &\le \frac{1}{2}M\int_a^b\int_c^d (x-y)^2\,dxdy.\end{aligned}$$

Since

$$\begin{aligned}\int_a^b\int_c^d (x-y)^2\,dxdy &= \int_a^b \left[\frac{(d-x)^3+(x-c)^3}{3}\right]dx \\ &= \frac{(b-c)^4-(a-c)^4-(d-b)^4+(d-a)^4}{12},\end{aligned}$$

hence by (3.11) we get (3.9). ☐

4. Related Results on Symmetric Domains

We have:

Lemma 3. *If $f : D \to \mathbb{C}$ is differentiable on the convex domain D, then for all $(x,y), (u,v) \in D$ we have the equality*

$$f(u,v) = f(x,y) + (u-x)\frac{\partial f}{\partial x}(x,y) + (v-y)\frac{\partial f}{\partial y}(x,y) + (u-x)\int_0^1 \left(\frac{\partial f}{\partial x}[t(u,v)+(1-t)(x,y)] - \frac{\partial f}{\partial x}(x,y)\right)dt + (v-y)\int_0^1 \left(\frac{\partial f}{\partial y}[t(u,v)+(1-t)(x,y)] - \frac{\partial f}{\partial y}(x,y)\right)dt. \tag{4.1}$$

Proof. By Taylor's multivariate theorem with integral remainder, we have

$$f(u,v) = f(x,y) + (u-x)\int_0^1 \frac{\partial f}{\partial x}[t(u,v)+(1-t)(x,y)]\,dt + (v-y)\int_0^1 \frac{\partial f}{\partial y}[t(u,v)+(1-t)(x,y)]\,dt \tag{4.2}$$

for all $(x,y), (u,v) \in D$.

Since

$$(u-x)\int_0^1 \left(\frac{\partial f}{\partial x}[t(u,v)+(1-t)(x,y)] - \frac{\partial f}{\partial x}(x,y)\right)dt = (u-x)\int_0^1 \frac{\partial f}{\partial x}[t(u,v)+(1-t)(x,y)]\,dt - (u-x)\frac{\partial f}{\partial x}(x,y)$$

and

$$(v-y)\int_0^1 \left(\frac{\partial f}{\partial y}[t(u,v)+(1-t)(x,y)] - \frac{\partial f}{\partial y}(x,y)\right)dt = (v-y)\int_0^1 \frac{\partial f}{\partial y}[t(u,v)+(1-t)(x,y)]\,dt - (v-y)\frac{\partial f}{\partial y}(x,y),$$

hence by (4.2) we get the desired result (4.1). □

Corollary 5. *With the assumptions of Lemma 3 and if D is symmetric, then for all $(x,y) \in D$ we have*

$$f(y,x) = f(x,y) + (y-x)\left(\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial y}(x,y)\right) + (y-x)\int_0^1 \left(\frac{\partial f}{\partial x}[t(y,x)+(1-t)(x,y)] - \frac{\partial f}{\partial x}(x,y)\right)dt - (y-x)\int_0^1 \left(\frac{\partial f}{\partial y}[t(y,x)+(1-t)(x,y)] - \frac{\partial f}{\partial y}(x,y)\right)dt \tag{4.3}$$

or, equivalently,

$$\Lambda_{\partial f,D}(x,y) = f(x,y) - f(y,x) + (y-x)\int_0^1 \left(\frac{\partial f}{\partial x}[t(y,x)+(1-t)(x,y)] - \frac{\partial f}{\partial x}(x,y)\right)dt - (y-x)\int_0^1 \left(\frac{\partial f}{\partial y}[t(y,x)+(1-t)(x,y)] - \frac{\partial f}{\partial y}(x,y)\right)dt. \tag{4.4}$$

We also have:

Corollary 6. *With the assumptions of Lemma 3 and if D is symmetric, then*

$$\int\int_D \Lambda_{\partial f,D}(x,y)\,dxdy \tag{4.5}$$
$$= \int\int_D (y-x)\left(\int_0^1\left(\frac{\partial f}{\partial x}(ty+(1-t)x, tx+(1-t)y) - \frac{\partial f}{\partial x}(x,y)\right)dt\right)dxdy$$
$$- \int\int_D (y-x)\left(\int_0^1\left(\frac{\partial f}{\partial y}(ty+(1-t)x, tx+(1-t)y) - \frac{\partial f}{\partial y}(x,y)\right)dt\right)dxdy.$$

The identity (4.5) follows by integrating (4.4) on D and observing that

$$\int\int_D f(x,y)\,dxdy = \int\int_D f(y,x)\,dxdy$$

since D is symmetric.

We assume that the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ satisfy the Lipschitz type conditions

$$\left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(u,v)\right| \le L_1|x-u| + K_1|y-v| \tag{4.6}$$

and

$$\left|\frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}(u,v)\right| \le L_2|x-u| + K_2|y-v| \tag{4.7}$$

for any (x,y), $(u,v) \in D$, where L_1, K_1, L_2 and K_2 are given positive constants.

Theorem 6. *If $f : D \to \mathbb{C}$ is differentiable on the convex symmetric domain D and the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ satisfy the Lipschitz type conditions (4.6) and (4.7), then*

$$\left|\int\int_D \Lambda_{\partial f,D}(x,y)\,dxdy\right| \le \frac{1}{2}(L_1+K_1+L_2+K_2)\int\int_D (y-x)^2\,dxdy. \tag{4.8}$$

Proof. From the representation (4.5) we get

$$\left|\int\int_D \Lambda_{\partial f,D}(x,y)\,dxdy\right| \tag{4.9}$$
$$\le \left|\int\int_D (y-x)\left(\int_0^1\left(\frac{\partial f}{\partial x}(ty+(1-t)x, tx+(1-t)y) - \frac{\partial f}{\partial x}(x,y)\right)dt\right)dxdy\right|$$
$$+ \left|\int\int_D (y-x)\left(\int_0^1\left(\frac{\partial f}{\partial y}(ty+(1-t)x, tx+(1-t)y) - \frac{\partial f}{\partial y}(x,y)\right)dt\right)dxdy\right|$$
$$\le \int\int_D \left|(y-x)\left(\int_0^1\left(\frac{\partial f}{\partial x}(ty+(1-t)x, tx+(1-t)y) - \frac{\partial f}{\partial x}(x,y)\right)dt\right)\right|dxdy$$
$$+ \int\int_D \left|(y-x)\left(\int_0^1\left(\frac{\partial f}{\partial y}(ty+(1-t)x, tx+(1-t)y) - \frac{\partial f}{\partial y}(x,y)\right)dt\right)\right|dxdy$$
$$\le \int\int_D |y-x|\left(\int_0^1\left|\frac{\partial f}{\partial x}(ty+(1-t)x, tx+(1-t)y) - \frac{\partial f}{\partial x}(x,y)\right|dt\right)dxdy$$
$$+ \int\int_D |y-x|\left(\int_0^1\left|\frac{\partial f}{\partial y}(ty+(1-t)x, tx+(1-t)y) - \frac{\partial f}{\partial y}(x,y)\right|dt\right)dxdy =: M.$$

Since the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ satisfy the Lipschitz type conditions (4.6) and (4.7), hence

$$\begin{aligned}&\int_0^1 \left|\frac{\partial f}{\partial x}(ty+(1-t)x, tx+(1-t)y) - \frac{\partial f}{\partial x}(x,y)\right| dt\\ &\leq \int_0^1 (L_1 |ty+(1-t)x - x| + K_1 |tx+(1-t)y-y|)\, dt\\ &= L_1 |y-x| \int_0^1 tdt + K_1 |y-x| \int_0^1 tdt = \frac{1}{2}(L_1+K_1)|y-x|\end{aligned}$$

and, similarly,

$$\begin{aligned}&\int_0^1 \left|\frac{\partial f}{\partial y}(ty+(1-t)x, tx+(1-t)y) - \frac{\partial f}{\partial y}(x,y)\right| dt\\ &\leq \frac{1}{2}(L_2+K_2)|y-x|.\end{aligned}$$

Therefore

$$\begin{aligned}M &\leq \frac{1}{2}(L_1+K_1)\int\int_D (y-x)^2\, dxdy + \frac{1}{2}(L_2+K_2)\int\int_D (y-x)^2\, dxdy\\ &= \frac{1}{2}(L_1+K_1+L_2+K_2)\int\int_D (y-x)^2\, dxdy\end{aligned}$$

and by (4.9) we get the desired result (4.8). □

Corollary 7. *Assume that $f : D \to \mathbb{R}$ is twice differentiable on the convex symmetric domain D and the second partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are bounded on D. Put*

$$\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty} := \sup_{(x,y)\in D}\left|\frac{\partial^2 f}{\partial x^2}(x,y)\right|, \quad \left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty} := \sup_{(x,y)\in D}\left|\frac{\partial^2 f}{\partial y^2}(x,y)\right|$$

and

$$\left\|\frac{\partial^2 f}{\partial x \partial y}\right\|_{D,\infty} := \sup_{(x,y)\in D}\left|\frac{\partial^2 f}{\partial x \partial y}(x,y)\right|,$$

then

$$\begin{aligned}&\left|\int\int_D \Lambda_{\partial f, D}(x,y)\, dxdy\right|\\ &\leq \frac{1}{2}\left(\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty} + 2\left\|\frac{\partial^2 f}{\partial x \partial y}\right\|_{D,\infty} + \left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty}\right)\int\int_D (y-x)^2\, dxdy.\end{aligned} \tag{4.10}$$

We have the following reverse inequality for pre-Schur convex functions:

Corollary 8. *Assume that $f : D \to \mathbb{R}$ is twice differentiable on the convex symmetric domain D and the second partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are bounded on D. If f is also pre-Schur convex on D then*

$$\begin{aligned}0 &\leq \frac{1}{2}\oint_{\partial D} [(x-y) f(x,y)\, dx + (x-y) f(x,y)\, dy] - \int\int_D f(x,y)\, dxdy\\ &\leq \frac{1}{4}\left(\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{D,\infty} + 2\left\|\frac{\partial^2 f}{\partial x \partial y}\right\|_{D,\infty} + \left\|\frac{\partial^2 f}{\partial y^2}\right\|_{D,\infty}\right)\int\int_D (y-x)^2\, dxdy.\end{aligned} \tag{4.11}$$

The proof follows by the identity (2.2) and the inequality (4.10) applied for the pre-Schur convex function f.

Remark 2. *Assume that $f:[a,b]^2\to\mathbb{R}$ is twice differentiable and the second partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x\partial y}$ are bounded on $[a,b]^2$. If f is also pre-Schur convex on $[a,b]^2$ then*

$$\begin{aligned}(4.12)\quad 0 &\le \frac{1}{2}\int_a^b \left[(x-a)f(x,a)+(b-x)f(x,b)\right]dx \\ &+\frac{1}{2}\int_a^b\left[(b-y)f(b,y)+(y-a)f(a,y)\right]dy-\int_a^b\int_a^b f(x,y)\,dxdy \\ &\le \frac{1}{24}\left(\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{[a,b]^2,\infty}+2\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{[a,b]^2,\infty}+\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{[a,b]^2,\infty}\right)(b-a)^4.\end{aligned}$$

5. Examples for Disks

We consider the closed disk $D(O,R)$ centered in $O(0,0)$ and of radius $R>0$. This is parametrized by

$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta\end{cases},\ r\in[0,R],\ \theta\in[0,2\pi]$$

and the circle $\mathcal{C}(O,R)$ is parametrized by

$$\begin{cases} x=R\cos\theta \\ y=R\sin\theta\end{cases},\ \theta\in[0,2\pi].$$

Observe that, if $\phi:D(O,R)\to\mathbb{R}$, then

$$\begin{aligned}&\oint_{\mathcal{C}(O,R)}\left[(x-y)\phi(x,y)\,dx+(x-y)\phi(x,y)\,dy\right] \\ &=-\int_0^{2\pi}R(R\cos\theta-R\sin\theta)\sin\theta\phi(R\cos\theta+a,R\sin\theta+b)\,d\theta \\ &+\int_0^{2\pi}R(R\cos\theta-R\sin\theta)\cos\theta\phi(R\cos\theta+a,R\sin\theta+b)\,d\theta \\ &=R^2\int_0^{2\pi}\phi(R\cos\theta+a,R\sin\theta+b)(\cos\theta-\sin\theta)^2\,d\theta.\end{aligned}$$

Also, we have

$$\int\int_{D(O,R)}\phi(x,y)\,dxdy=\int_0^R\int_0^{2\pi}\phi(r\cos\theta,r\sin\theta)\,rdrd\theta$$

and

$$\int\int_{D(O,R)} (x-y)^2 \, dxdy = \int_0^R \int_0^{2\pi} (R\cos\theta - R\sin\theta)^2 \, rdrd\theta$$
$$= \frac{1}{2}R^4 \int_0^{2\pi} (\cos\theta - \sin\theta)^2 \, d\theta$$
$$= \frac{1}{2}R^4 \int_0^{2\pi} (1 - 2\sin\theta\cos\theta) \, d\theta = \pi R^4.$$

Using Theorem 8 we can state the following result:

Proposition 2. *Assume that $f : D(O,R) \to \mathbb{R}$ is twice differentiable on the convex symmetric domain $D(O,R)$ and the second partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are bounded on $D(O,R)$. If f is also pre-Schur convex on $D(O,R)$, then*

$$0 \le \frac{1}{2}R^2 \int_0^{2\pi} \phi(R\cos\theta, R\sin\theta)(\cos\theta - \sin\theta)^2 \, d\theta - \int_0^R \int_0^{2\pi} \phi(r\cos\theta, r\sin\theta) \, rdrd\theta \le \frac{1}{4}\pi R^4 \left(\left\| \frac{\partial^2 f}{\partial x^2} \right\|_{D(O,R),\infty} + 2 \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_{D(O,R),\infty} + \left\| \frac{\partial^2 f}{\partial y^2} \right\|_{D(O,R),\infty} \right). \tag{5.1}$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa