# VOLUME AND SURFACE INTEGRALS INEQUALITIES FOR FUNCTIONS DEFINED ON BODIES FROM $n$-DIMENSIONAL SPACES

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Abstract. In this paper, by making use of Divergence theorem for multiple integrals, we establish some volume and surface integrals inequalities for functions define on bodies from $n$-nimensional spaces $\mathbb{R}^n$. Examples for three dimensional balls are also provided.

## 1. Introduction

Let $\partial D$ be a simple, closed counterclockwise curve in the $xy$-plane, bounding a region $D$. We consider the following Lebesgue norms for a measurable function $g : D \to \mathbb{C}$

$$\|g\|_{D,p} := \left( \int \int_D |g(x,y)|^p \, dxdy \right)^{1/p} < \infty \text{ for } p \geq 1$$

and

$$\|g\|_{D,\infty} := \sup_{(x,y)\in D} |g(x,y)| < \infty \text{ for } p = \infty.$$

In the recent paper [7] we obtained the following result:

**Theorem 1.** *Let $\partial D$ be a simple, closed counterclockwise curve in the xy-plane, bounding a region $D$. Assume that the function $f : D \to \mathbb{C}$ has continuous partial derivatives on the domain $D$. Then*

$$\left| \int \int_D f(x,y)\, dxdy - \frac{1}{2} \oint_{\partial D} [(x-y) f(x,y)\, dx + (x-y) f(x,y)\, dy] \right| \tag{1.1}$$
$$\leq \frac{1}{2} \int \int_D |x-y| \left| \frac{\partial f(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y} \right| dxdy$$
$$\leq \frac{1}{2} \begin{cases} \left\| \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \right\|_{D,\infty} \int \int_D |x-y|\, dxdy; \\ \\ \left\| \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \right\|_{D,p} \left( \int \int_D |x-y|^q \, dxdy \right)^{1/q} \\ \text{where } p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \left\| \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \right\|_{D,1} \sup_{(x,y)\in D} |x-y| . \end{cases}$$

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Assume that the function $f:[a,b]^2 \to \mathbb{C}$ has continuous partial derivatives on the domain $[a,b]^2$. Then

$$\begin{aligned}(1.2)\quad &\left|\frac{1}{2}\int_a^b [(x-a) f(x,a) + (b-x) f(x,b)]\, dx\right. \\ &\left.+\frac{1}{2}\int_a^b [(b-y) f(b,y) + (y-a) f(a,y)]\, dy - \int_a^b\int_a^b f(x,y)\, dxdy\right| \\ &\leq \frac{1}{2}\begin{cases} \frac{(b-a)^3}{3}\left\|\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right\|_{[a,b]^2,\infty}; \\ \frac{2^{1/q}(b-a)^{1+2/q}}{(q+1)^{1/q}(q+2)^{1/q}}\left\|\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right\|_{[a,b]^2,p} \\ \text{where } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ (b-a)\left\|\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right\|_{[a,b]^2,1}. \end{cases}\end{aligned}$$

Assume that the function $f: D(O,R) \to \mathbb{C}$ has continuous partial derivatives on the disk $D(O,R)$ centered in $O(0,0)$ and of radius $R>0$ and there exist the constant $L>0$ such that

$$\left|\frac{\partial f(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y}\right| \leq L|x-y| \text{ for } (x,y) \in D(O,R),$$

then

$$\begin{aligned}(1.3)\quad &\left|\frac{1}{2}R^2\int_0^{2\pi} f(R\cos\theta, R\sin\theta)(\cos\theta - \sin\theta)^2\, d\theta\right. \\ &\left.-\int_0^R\int_0^{2\pi} f(r\cos\theta, r\sin\theta)\, rdrd\theta\right| \\ &\leq \frac{1}{2}L\pi R^4.\end{aligned}$$

In this paper, by making use of Divergence theorem for multiple integrals, we establish some volume and surface integrals inequalities for functions defined on bodies from $n$-dimensional spaces $\mathbb{R}^n$. Examples for three dimensional balls are also provided.

## 2. Some Preliminary Facts

Let $B$ be a bounded open subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary $\partial B$. Let $F=(F_1,...,F_n)$ be a smooth vector field defined in $\mathbb{R}^n$, or at least in $B \cup \partial B$. Let $\mathbf{n}$ be the unit outward-pointing normal of $\partial B$. Then the *Divergence Theorem* states, see for instance [10]:

$$(2.1)\quad \int_B \operatorname{div} F dV = \int_{\partial B} F\cdot \mathbf{n} dA,$$

where

$$\operatorname{div} F = \nabla \cdot F = \sum_{k=1}^n \frac{\partial F_i}{\partial x_i},$$

$dV$ is the element of volume in $\mathbb{R}^n$ and $dA$ is the element of surface area on $\partial B$.

If $\mathbf{n} = (\mathbf{n}_1, ..., \mathbf{n}_n)$, $x = (x_1, ..., x_n) \in B$ and use the notation $dx$ for $dV$ we can write (2.1) more explicitly as

$$\sum_{k=1}^{n} \int_B \frac{\partial F_k(x)}{\partial x_k} dx = \sum_{k=1}^{n} \int_{\partial B} F_k(x) \mathbf{n}_k(x) dA. \tag{2.2}$$

By taking the real and imaginary part, we can extend the above equality for complex valued functions $F_k$, $k \in \{1, ..., n\}$ defined on $B$.

If $n = 2$, the normal is obtained by rotating the tangent vector through $90°$ (in the correct direction so that it points out). The quantity $tds$ can be written $(dx_1, dx_2)$ along the surface, so that

$$\mathbf{n}dA := \mathbf{n}ds = (dx_2, -dx_1)$$

Here $t$ is the tangent vector along the boundary curve and $ds$ is the element of arc-length.

From (2.2) we get for $B \subset \mathbb{R}^2$ that

$$\begin{aligned} &\int_B \frac{\partial F_1(x_1, x_2)}{\partial x_1} dx_1 dx_2 + \int_B \frac{\partial F_2(x_1, x_2)}{\partial x_2} dx_1 dx_2 \\ &= \int_{\partial B} F_1(x_1, x_2) dx_2 - \int_{\partial B} F_2(x_1, x_2) dx_1, \end{aligned} \tag{2.3}$$

which is *Green's theorem* in plane.

If $n = 3$ and if $\partial B$ is described as a level-set of a function of 3 variables i.e. $\partial B = \{x_1, x_2, x_3 \in \mathbb{R}^3 \mid G(x_1, x_2, x_3) = 0\}$, then a vector pointing in the direction of $\mathbf{n}$ is $\operatorname{grad} G$. We shall use the case where $G(x_1, x_2, x_3) = x_3 - g(x_1, x_2)$, $(x_1, x_2) \in D$, a domain in $\mathbb{R}^2$ for some differentiable function $g$ on $D$ and $B$ corresponds to the inequality $x_3 < g(x_1, x_2)$, namely

$$B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 < g(x_1, x_2)\}.$$

Then

$$\mathbf{n} = \frac{(-g_{x_1}, -g_{x_2}, 1)}{\left(1 + g_{x_1}^2 + g_{x_2}^2\right)^{1/2}},\ dA = \left(1 + g_{x_1}^2 + g_{x_2}^2\right)^{1/2} dx_1 dx_2$$

and

$$\mathbf{n}dA = (-g_{x_1}, -g_{x_2}, 1) dx_1 dx_2.$$

From (2.2) we get

$$\begin{aligned} &\int_B \left( \frac{\partial F_1(x_1, x_2, x_3)}{\partial x_1} + \frac{\partial F_2(x_1, x_2, x_3)}{\partial x_2} + \frac{\partial F_3(x_1, x_2, x_3)}{\partial x_3} \right) dx_1 dx_2 dx_3 \\ &= -\int_D F_1(x_1, x_2, g(x_1, x_2)) g_{x_1}(x_1, x_2) dx_1 dx_2 \\ &- \int_D F_1(x_1, x_2, g(x_1, x_2)) g_{x_2}(x_1, x_2) dx_1 dx_2 \\ &+ \int_D F_3(x_1, x_2, g(x_1, x_2)) dx_1 dx_2 \end{aligned} \tag{2.4}$$

which is the *Gauss-Ostrogradsky theorem* in space.

Following Apostol [1], we can also consider a surface described by the vector equation

$$r(u, v) = x_1(u, v) \overrightarrow{i} + x_2(u, v) \overrightarrow{j} + x_3(u, v) \overrightarrow{k} \tag{2.5}$$

where $(u,v)\in[a,b]\times[c,d]$.

If $x_1$, $x_2$, $x_3$ are differentiable on $[a,b]\times[c,d]$ we consider the two vectors

$$\frac{\partial r}{\partial u}=\frac{\partial x_1}{\partial u}\overrightarrow{i}+\frac{\partial x_2}{\partial u}\overrightarrow{j}+\frac{\partial x_3}{\partial u}\overrightarrow{k}$$

and

$$\frac{\partial r}{\partial v}=\frac{\partial x_1}{\partial v}\overrightarrow{i}+\frac{\partial x_2}{\partial v}\overrightarrow{j}+\frac{\partial x_3}{\partial v}\overrightarrow{k}.$$

The *cross product* of these two vectors $\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}$ will be referred to as the fundamental vector product of the representation $r$. Its components can be expressed as *Jacobian determinants*. In fact, we have [1, p. 420]

$$\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}=\begin{vmatrix}\frac{\partial x_2}{\partial u} & \frac{\partial x_3}{\partial u}\\ \frac{\partial x_2}{\partial v} & \frac{\partial x_3}{\partial v}\end{vmatrix}\overrightarrow{i}+\begin{vmatrix}\frac{\partial x_3}{\partial u} & \frac{\partial x_1}{\partial u}\\ \frac{\partial x_3}{\partial v} & \frac{\partial x_1}{\partial v}\end{vmatrix}\overrightarrow{j}+\begin{vmatrix}\frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u}\\ \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v}\end{vmatrix}\overrightarrow{k}\tag{2.6}$$

$$=\frac{\partial(x_2,x_3)}{\partial(u,v)}\overrightarrow{i}+\frac{\partial(x_3,x_1)}{\partial(u,v)}\overrightarrow{j}+\frac{\partial(x_1,x_2)}{\partial(u,v)}\overrightarrow{k}.$$

Let $\partial B=r(T)$ be a parametric surface described by a vector-valued function $r$ defined on the box $T=[a,b]\times[c,d]$. The area of $\partial B$ denoted $A_{\partial B}$ is defined by the double integral [1, p. 424-425]

$$\begin{aligned}A_{\partial B}&=\int_a^b\int_c^d\left\|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right\|dudv\\&=\int_a^b\int_c^d\sqrt{\left(\frac{\partial(x_2,x_3)}{\partial(u,v)}\right)^2+\left(\frac{\partial(x_3,x_1)}{\partial(u,v)}\right)^2+\left(\frac{\partial(x_1,x_2)}{\partial(u,v)}\right)^2}dudv.\end{aligned}\tag{2.7}$$

We define surface integrals in terms of a parametric representation for the surface. One can prove that under certain general conditions the value of the integral is independent of the representation.

Let $\partial B=r(T)$ be a parametric surface described by a vector-valued differentiable function $r$ defined on the box $T=[a,b]\times[c,d]$ and let $f:\partial B\to\mathbb{C}$ defined and bounded on $\partial B$. The surface integral of $f$ over $\partial B$ is defined by [1, p. 430]

$$\begin{aligned}\int\int_{\partial B}fdA&=\int_a^b\int_c^d f(x_1,x_2,x_3)\left\|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right\|dudv\\&=\int_a^b\int_c^d f(x_1(u,v),x_2(u,v),x_3(u,v))\\&\quad\times\sqrt{\left(\frac{\partial(x_2,x_3)}{\partial(u,v)}\right)^2+\left(\frac{\partial(x_3,x_1)}{\partial(u,v)}\right)^2+\left(\frac{\partial(x_1,x_2)}{\partial(u,v)}\right)^2}dudv.\end{aligned}\tag{2.8}$$

If $\partial B=r(T)$ is a parametric surface, the fundamental vector product $N=\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}$ is normal to $\partial B$ at each regular point of the surface. At each such point there are two unit normals, a unit normal $\mathbf{n}_1$, which has the same direction as $N$, and a unit normal $\mathbf{n}_2$ which has the opposite direction. Thus

$$\mathbf{n}_1=\frac{N}{\|N\|}\text{ and }\mathbf{n}_2=-\mathbf{n}_1.$$

Let $\mathbf{n}$ be one of the two normals $\mathbf{n}_1$ or $\mathbf{n}_2$. Let also $F$ be a vector field defined on $\partial B$ and assume that the surface integral,

$$\int\int_{\partial B} (F \cdot \mathbf{n})\, dA,$$

called the flux surface integral, exists. Here $F \cdot \mathbf{n}$ is the dot or inner product.

We can write [1, p. 434]

$$\int\int_{\partial B} (F \cdot \mathbf{n})\, dA = \pm \int_a^b \int_c^d F\left(r\left(u,v\right)\right) \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right) dudv$$

where the sign " + " is used if $\mathbf{n} = \mathbf{n}_1$ and the " − " sign is used if $\mathbf{n} = \mathbf{n}_2$.

If

$$F\left(x_1, x_2, x_3\right) = F_1\left(x_1, x_2, x_3\right) \overrightarrow{i} + F_2\left(x_1, x_2, x_3\right) \overrightarrow{j} + F_3\left(x_1, x_2, x_3\right) \overrightarrow{k}$$

and

$$r\left(u,v\right) = x_1\left(u,v\right) \overrightarrow{i} + x_2\left(u,v\right) \overrightarrow{j} + x_3\left(u,v\right) \overrightarrow{k} \text{ where } \left(u,v\right) \in [a,b] \times [c,d]$$

then the flux surface integral for $\mathbf{n} = \mathbf{n}_1$ can be explicitly calculated as [1, p. 435]

$$\begin{aligned} \text{(2.9)} \quad \int\int_{\partial B} (F \cdot \mathbf{n})\, dA &= \int_a^b \int_c^d F_1\left(x_1\left(u,v\right), x_2\left(u,v\right), x_3\left(u,v\right)\right) \frac{\partial\left(x_2, x_3\right)}{\partial\left(u,v\right)} dudv \\ &+ \int_a^b \int_c^d F_2\left(x_1\left(u,v\right), x_2\left(u,v\right), x_3\left(u,v\right)\right) \frac{\partial\left(x_3, x_1\right)}{\partial\left(u,v\right)} dudv \\ &+ \int_a^b \int_c^d F_3\left(x_1\left(u,v\right), x_2\left(u,v\right), x_3\left(u,v\right)\right) \frac{\partial\left(x_1, x_2\right)}{\partial\left(u,v\right)} dudv. \end{aligned}$$

The sum of the double integrals on the right is often written more briefly as [1, p. 435]

$$\int\int_{\partial B} F_1\left(x_1, x_2, x_3\right) dx_2 \wedge dx_3 + \int\int_{\partial B} F_2\left(x_1, x_2, x_3\right) dx_3 \wedge dx_1$$
$$+ \int\int_{\partial B} F_3\left(x_1, x_2, x_3\right) dx_1 \wedge dx_2$$

Let $B \subset \mathbb{R}^3$ be a solid in 3-space bounded by an orientable closed surface $\partial B$, and let $\mathbf{n}$ be the unit outer normal to $\partial B$. If $F$ is a continuously differentiable vector field defined on $B$, we have the *Gauss-Ostrogradsky identity*

$$\text{(GO)} \qquad \int\int\int_B (\operatorname{div} F)\, dV = \int\int_{\partial B} (F \cdot \mathbf{n})\, dA.$$

If we express

$$F\left(x_1, x_2, x_3\right) = F_1\left(x_1, x_2, x_3\right) \overrightarrow{i} + F_2\left(x_1, x_2, x_3\right) \overrightarrow{j} + F_3\left(x_1, x_2, x_3\right) \overrightarrow{k},$$

then (2.4) can be written as

$$\begin{aligned} \text{(2.10)} \quad & \int\int\int_B \left(\frac{\partial F_1\left(x_1, x_2, x_3\right)}{\partial x_1} + \frac{\partial F_2\left(x_1, x_2, x_3\right)}{\partial x_2} + \frac{\partial F_3\left(x_1, x_2, x_3\right)}{\partial x_3}\right) dx_1 dx_2 dx_3 \\ &= \int\int_{\partial B} F_1\left(x_1, x_2, x_3\right) dx_2 \wedge dx_3 + \int\int_{\partial B} F_2\left(x_1, x_2, x_3\right) dx_3 \wedge dx_1 \\ &+ \int\int_{\partial B} F_3\left(x_1, x_2, x_3\right) dx_1 \wedge dx_2. \end{aligned}$$

## 3. Some Identities

We start with the following identity that is of interest in itself:

**Lemma 1.** *Assume that $f : B \to \mathbb{C}$ has partial derivatives on the domain $B \subset \mathbb{R}^n$, $n \geq 2$. Define for $j \neq i$*

$$\Lambda_{\partial f,B}\left(x_i, x_j\right) := \left(x_i - x_j\right)\left(\frac{\partial f\left(x_1, ..., x_n\right)}{\partial x_i} - \frac{\partial f\left(x_1, ..., x_n\right)}{\partial x_j}\right),$$

*where $(x_1, ..., x_n) \in B$. Then we have*

$$\begin{aligned}(3.1)\qquad &\frac{1}{n-1}\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\left(x_k - \frac{1}{n}\sum_{j=1}^{n}x_j\right)f\left(x_1, ..., x_n\right)\right) \\ &= f\left(x_1, ..., x_n\right) + \frac{1}{n\left(n-1\right)}\sum_{1\leq i<j\leq n}\Lambda_{\partial f,B}\left(x_i, x_j\right).\end{aligned}$$

*Proof.* For $j \neq i$ we have

$$\frac{\partial}{\partial x_i}\left(\left(x_i - x_j\right)f\left(x_1, ..., x_n\right)\right) = f\left(x_1, ..., x_n\right) + \left(x_i - x_j\right)\frac{\partial f\left(x_1, ..., x_n\right)}{\partial x_i}$$

and

$$\frac{\partial}{\partial x_j}\left(\left(x_i - x_j\right)f\left(x_1, ..., x_n\right)\right) = -f\left(x_1, ..., x_n\right) + \left(x_i - x_j\right)\frac{\partial f\left(x_1, ..., x_n\right)}{\partial x_j},$$

which gives

$$\begin{aligned}&\frac{\partial}{\partial x_i}\left(\left(x_i - x_j\right)f\left(x_1, ..., x_n\right)\right) - \frac{\partial}{\partial x_j}\left(\left(x_i - x_j\right)f\left(x_1, ..., x_n\right)\right) \\ &= 2f\left(x_1, ..., x_n\right) + \left(x_i - x_j\right)\left(\frac{\partial f\left(x_1, ..., x_n\right)}{\partial x_i} - \frac{\partial f\left(x_1, ..., x_n\right)}{\partial x_j}\right)\end{aligned}$$

for $j \neq i$.

If we take the sum over $i, j \in \{1, ..., n\}$ with $j \neq i$ we get

$$\begin{aligned}(3.2)\qquad &\sum_{i,j=1,j\neq i}^{n}\left[\frac{\partial}{\partial x_i}\left(\left(x_i - x_j\right)f\left(x_1, ..., x_n\right)\right) - \frac{\partial}{\partial x_j}\left(\left(x_i - x_j\right)f\left(x_1, ..., x_n\right)\right)\right] \\ &= 2\sum_{i,j=1,j\neq i}^{n}f\left(x_1, ..., x_n\right) \\ &+ \sum_{i,j=1,j\neq i}^{n}\left(x_i - x_j\right)\left(\frac{\partial f\left(x_1, ..., x_n\right)}{\partial x_i} - \frac{\partial f\left(x_1, ..., x_n\right)}{\partial x_j}\right).\end{aligned}$$

We have

$$\sum_{i,j=1,j\neq i}^{n}f\left(x_1, ..., x_n\right) = n\left(n-1\right)f\left(x_1, ..., x_n\right)$$

and

$$\sum_{i,j=1,j\neq i}^{n}(x_i-x_j)\left(\frac{\partial f(x_1,...,x_n)}{\partial x_i}-\frac{\partial f(x_1,...,x_n)}{\partial x_j}\right)$$
$$=2\sum_{1\leq i<j\leq n}^{n}(x_i-x_j)\left(\frac{\partial f(x_1,...,x_n)}{\partial x_i}-\frac{\partial f(x_1,...,x_n)}{\partial x_j}\right).$$

Also

$$\sum_{i,j=1,j\neq i}^{n}\left[\frac{\partial}{\partial x_i}((x_i-x_j)f(x_1,...,x_n))-\frac{\partial}{\partial x_j}((x_i-x_j)f(x_1,...,x_n))\right]$$
$$=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\sum_{j=1,j\neq i}^{n}(x_i-x_j)f(x_1,...,x_n)\right)$$
$$-\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\sum_{i=1,j\neq i}^{n}(x_i-x_j)f(x_1,...,x_n)\right)$$
$$=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left((n-1)x_i-\sum_{j=1,j\neq i}^{n}x_j\right)f(x_1,...,x_n)\right)$$
$$-\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\left(\sum_{i=1,j\neq i}^{n}x_i-(n-1)x_j\right)f(x_1,...,x_n)\right)$$
$$=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left((n-1)x_i-\sum_{j=1,j\neq i}^{n}x_j\right)f(x_1,...,x_n)\right)$$
$$+\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\left((n-1)x_j-\sum_{i=1,j\neq i}^{n}x_i\right)f(x_1,...,x_n)\right)$$
$$=2\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\left((n-1)x_k-\sum_{j=1,j\neq k}^{n}x_j\right)f(x_1,...,x_n)\right)$$
$$=2\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\left(nx_k-\sum_{j=1}^{n}x_j\right)f(x_1,...,x_n)\right).$$

By (3.2) we get

$$2\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\left(nx_k-\sum_{j=1}^{n}x_j\right)f(x_1,...,x_n)\right)$$
$$=2n(n-1)f(x_1,...,x_n)$$
$$+2\sum_{1\leq i<j\leq n}(x_i-x_j)\left(\frac{\partial f(x_1,...,x_n)}{\partial x_i}-\frac{\partial f(x_1,...,x_n)}{\partial x_j}\right),$$

which is equivalent to the desired result. □

**Remark 1.** *For $n=2$ we get*

$$\frac{1}{2}\left[\frac{\partial}{\partial x_1}\left[\left(x_1-x_2\right)f\left(x_1,x_2\right)\right]+\frac{\partial}{\partial x_1}\left[\left(x_2-x_1\right)f\left(x_1,x_2\right)\right]\right] = f\left(x_1,x_2\right)+\frac{1}{2}\Lambda_{\partial f,B}\left(x_1,x_2\right), \tag{3.3}$$

*for $(x_1,x_2)\in B$.*

*For $n=3$ we get*

$$\begin{aligned}&\frac{1}{3}\left[\frac{\partial}{\partial x_1}\left(\left(x_1-\frac{x_2+x_3}{2}\right)f\left(x_1,x_2,x_3\right)\right)\right.\\&+\frac{\partial}{\partial x_2}\left(\left(x_2-\frac{x_1+x_3}{2}\right)f\left(x_1,x_2,x_3\right)\right)\\&\left.+\frac{\partial}{\partial x_2}\left(\left(x_3-\frac{x_1+x_2}{2}\right)f\left(x_1,x_2,x_3\right)\right)\right]\\&=f\left(x_1,x_2,x_3\right)+\frac{1}{6}\left[\Lambda_{\partial f,B}\left(x_1,x_2\right)+\Lambda_{\partial f,B}\left(x_2,x_3\right)+\Lambda_{\partial f,B}\left(x_1,x_3\right)\right]\end{aligned} \tag{3.4}$$

*for $(x_1,x_2,x_3)\in B$.*

We have the following identity of interest:

**Lemma 2.** *Let $B$ be a bounded closed subset of $\mathbb{R}^n$ $(n\geq 2)$ with smooth (or piecewise smooth) boundary $\partial B$ and $\mathbf{n}=(\mathbf{n}_1,...,\mathbf{n}_n)$ be the unit outward-pointing normal of $\partial B$. If $f$ is a continuously differentiable function on an open neighborhood of $B$, then we have the representation*

$$\begin{aligned}&\frac{1}{n-1}\sum_{k=1}^{n}\int_{\partial B}f\left(x\right)\left(x_k-\frac{1}{n}\sum_{j=1}^{n}x_j\right)\mathbf{n}_k\left(x\right)dA-\int_B f\left(x\right)dx\\&=\frac{1}{n\left(n-1\right)}\sum_{1\leq i<j\leq n}\int_B\Lambda_{\partial f,B}\left(x_i,x_j\right)dx.\end{aligned} \tag{3.5}$$

*Proof.* We use the identity (3.1) on $B$ for $x=(x_1,...,x_n)$ and take the volume integral to get

$$\begin{aligned}&\frac{1}{n-1}\int_B\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\left(x_k-\frac{1}{n}\sum_{j=1}^{n}x_j\right)f\left(x\right)\right)dx\\&=\int_B f\left(x\right)dx+\frac{1}{n\left(n-1\right)}\sum_{1\leq i<j\leq n}\int_B\Lambda_{\partial f,B}\left(x_i,x_j\right)dx.\end{aligned} \tag{3.6}$$

Define

$$F_k\left(x\right)=\left(x_k-\frac{1}{n}\sum_{j=1}^{n}x_j\right)f\left(x\right),\ k\in\{1,...,n\},\ x\in B$$

and use the Divergence theorem (2.2) to get

$$\int_B \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left( \left( x_k - \frac{1}{n} \sum_{j=1}^{n} x_j \right) f(x) \right) dx \tag{3.7}$$
$$= \sum_{k=1}^{n} \int_{\partial B} \left( x_k - \frac{1}{n} \sum_{j=1}^{n} x_j \right) f(x) \mathbf{n}_k(x) \, dA.$$

On utilising (3.6) and (3.7) we obtain

$$\int_B f(x) \, dx + \frac{1}{n(n-1)} \sum_{1 \le i < j \le n} \int_B \Lambda_{\partial f, B}(x_i, x_j) \, dx$$
$$= \frac{1}{n-1} \sum_{k=1}^{n} \int_{\partial B} \left( x_k - \frac{1}{n} \sum_{j=1}^{n} x_j \right) f(x) \mathbf{n}_k(x) \, dA$$

that is equivalent to (3.5). □

**Remark 2.** *For $n = 2$ we obtain the identity*

$$\frac{1}{2} \int_{\partial B} \left[ (x_1 - x_2) f(x_1, x_2) \, dx_1 + (x_1 - x_2) f(x_1, x_2) \, dx_2 \right] \tag{3.8}$$
$$- \int_B f(x_1, x_2) \, dx_1 dx_2$$
$$= \frac{1}{2} \int_B \Lambda_{\partial f, B}(x_1, x_2) \, dx_1 dx_2,$$

*where $B$ is a bounded closed subset of $\mathbb{R}^2$ with smooth (or piecewise smooth) boundary $\partial B$ and $f$ is a continuously differentiable function on an open neighborhood of $B$.*

*For $n = 3$ we obtain the identity*

$$\frac{1}{3} \left[ \int_{\partial B} \left( x_1 - \frac{x_2 + x_3}{2} \right) f(x_1, x_2, x_3) \, dx_2 \wedge dx_3 \right. \tag{3.9}$$
$$+ \int_{\partial B} \left( x_2 - \frac{x_1 + x_3}{2} \right) f(x_1, x_2, x_3) \, dx_3 \wedge dx_1$$
$$\left. + \int_{\partial B} \left( x_3 - \frac{x_1 + x_2}{2} \right) f(x_1, x_2, x_3) \, dx_1 \wedge dx_2 \right]$$
$$- \int_B f(x_1, x_2, x_3) \, dx_1 dx_2 dx_3$$
$$= \frac{1}{6} \int_B \left[ \Lambda_{\partial f, B}(x_1, x_2) + \Lambda_{\partial f, B}(x_2, x_3) + \Lambda_{\partial f, B}(x_1, x_3) \right] dx_1 dx_2 dx_3,$$

*where $B$ is a bounded closed subset of $\mathbb{R}^3$ with smooth (or piecewise smooth) boundary $\partial B$ and $f$ is a continuously differentiable function on an open neighborhood of $B$.*

If the surface described by the vector equation

$$r(u, v) = x_1(u, v) \overrightarrow{i} + x_2(u, v) \overrightarrow{j} + x_3(u, v) \overrightarrow{k}$$

where $(u,v)\in[a,b]\times[c,d]$, then the parametric version of (3.9) is

$$\begin{aligned}
(3.10)\qquad & \frac{1}{3}\left[\int_a^b\int_c^d\left(x_1(u,v)-\frac{x_2(u,v)+x_3(u,v)}{2}\right)\right.\\
&\times f(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_2,x_3)}{\partial(u,v)}dudv\\
&+\int_a^b\int_c^d\left(x_2(u,v)-\frac{x_1(u,v)+x_3(u,v)}{2}\right)\\
&\times f(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_3,x_1)}{\partial(u,v)}dudv\\
&+\int_a^b\int_c^d\left(x_3(u,v)-\frac{x_1(u,v)+x_2(u,v)}{2}\right)\\
&\left.\times f(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_1,x_2)}{\partial(u,v)}dudv\right]\\
&-\int_B f(x_1,x_2,x_3)\,dx_1dx_2dx_3\\
&=\frac{1}{6}\int_B\left[\Lambda_{\partial f,B}(x_1,x_2)+\Lambda_{\partial f,B}(x_2,x_3)+\Lambda_{\partial f,B}(x_1,x_3)\right]dx_1dx_2dx_3.
\end{aligned}$$

## 4. Some Inequalities

We consider the following Lebesgue norms for a measurable function $g:B\to\mathbb{C}$

$$\|g\|_{B,p}:=\left(\int\int_B|g(x,y)|^p\,dxdy\right)^{1/p}<\infty \text{ for } p\geq 1$$

and

$$\|g\|_{B,\infty}:=\sup_{(x,y)\in B}|g(x,y)|<\infty \text{ for } p=\infty.$$

We have the following result:

**Theorem 2.** *Let $B$ be a bounded closed subset of $\mathbb{R}^n$ $(n\geq 2)$ with smooth (or piecewise smooth) boundary $\partial B$ and $\mathbf{n}=(\mathbf{n}_1,...,\mathbf{n}_n)$ be the unit outward-pointing normal of $\partial B$. If $f$ is a continuously differentiable function on an open neighborhood of $B$, then we have the inequalities*

$$\begin{aligned}
(4.1)\qquad & \left|\frac{1}{n-1}\sum_{k=1}^n\int_{\partial B}f(x)\left(x_k-\frac{1}{n}\sum_{j=1}^n x_j\right)\mathbf{n}_k(x)\,dA-\int_B f(x)\,dx\right|\\
&\leq\frac{1}{n(n-1)}\sum_{1\leq i<j\leq n}\int_B|x_i-x_j|\left|\frac{\partial f(x)}{\partial x_i}-\frac{\partial f(x)}{\partial x_j}\right|dx\\
&\leq\frac{1}{n(n-1)}\sum_{1\leq i<j\leq n}\begin{cases}\left\|\frac{\partial f}{\partial x_i}-\frac{\partial f}{\partial x_j}\right\|_{B,\infty}\int_B|x_i-x_j|\,dx;\\[2ex]
\left\|\frac{\partial f}{\partial x_i}-\frac{\partial f}{\partial x_j}\right\|_{B,p}\left(\int_B|x_i-x_j|^q\,dx\right)^{1/q}\\
\text{where } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1;\\[2ex]
\left\|\frac{\partial f}{\partial x_i}-\frac{\partial f}{\partial x_j}\right\|_{B,1}\sup_{x\in B}|x_i-x_j|.\end{cases}
\end{aligned}$$

*Proof.* By the identity (3.5) we have

$$\begin{aligned}(4.2)\quad &\left|\frac{1}{n-1}\sum_{k=1}^{n}\int_{\partial B}\left(x_k-\frac{1}{n}\sum_{j=1}^{n}x_j\right)f(x)\,\mathbf{n}_k(x)\,dA-\int_B f(x)\,dx\right|\\ &\leq\frac{1}{n(n-1)}\sum_{1\leq i<j\leq n}\left|\int_B\Lambda_{\partial f,B}(x_i,x_j)\,dx\right|\\ &\leq\frac{1}{n(n-1)}\sum_{1\leq i<j\leq n}\int_B\left|\Lambda_{\partial f,B}(x_i,x_j)\right|dx\\ &=\frac{1}{n(n-1)}\sum_{1\leq i<j\leq n}\int_B|x_i-x_j|\left|\frac{\partial f(x_1,...,x_n)}{\partial x_i}-\frac{\partial f(x_1,...,x_n)}{\partial x_j}\right|dx.\end{aligned}$$

Using Hölder's integral inequality we have for $1\leq i<j\leq n$ that

$$\begin{aligned}&\int_B|x_i-x_j|\left|\frac{\partial f(x_1,...,x_n)}{\partial x_i}-\frac{\partial f(x_1,...,x_n)}{\partial x_j}\right|dx\\ &\leq\begin{cases}\sup_{x\in B}\left|\frac{\partial f(x_1,...,x_n)}{\partial x_i}-\frac{\partial f(x_1,...,x_n)}{\partial x_j}\right|\int_B|x_i-x_j|\,dx;\\ \\ \left(\int_B\left|\frac{\partial f(x_1,...,x_n)}{\partial x_i}-\frac{\partial f(x_1,...,x_n)}{\partial x_j}\right|^p dx\right)^{1/p}\left(\int_B|x_i-x_j|^q\,dx\right)^{1/q}\\ \text{where } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1;\\ \\ \sup_{x\in B}|x_i-x_j|\int_B\left|\frac{\partial f(x_1,...,x_n)}{\partial x_i}-\frac{\partial f(x_1,...,x_n)}{\partial x_j}\right|dx\end{cases}\\ &=\begin{cases}\left\|\frac{\partial f}{\partial x_i}-\frac{\partial f}{\partial x_j}\right\|_{B,\infty}\int_B|x_i-x_j|\,dx;\\ \\ \left\|\frac{\partial f}{\partial x_i}-\frac{\partial f}{\partial x_j}\right\|_{B,p}\left(\int_B|x_i-x_j|^q\,dx\right)^{1/q}\\ \text{where } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1;\\ \\ \left\|\frac{\partial f}{\partial x_i}-\frac{\partial f}{\partial x_j}\right\|_{B,1}\sup_{x\in B}|x_i-x_j|.\end{cases}\end{aligned}$$

By making use of (4.2) we get the desired result (4.1). □

**Corollary 1.** *With the assumptions of Theorem 2 and if there exists the constants* $L_{ij}>0$, $1\leq i<j\leq n$ *such that*

$$(4.3)\quad \left|\frac{\partial f(x_1,...,x_n)}{\partial x_i}-\frac{\partial f(x_1,...,x_n)}{\partial x_j}\right|\leq L_{ij}|x_i-x_j|$$

*for all* $x=(x_1,...,x_n)\in B$. *Then*

$$\begin{aligned}(4.4)\quad &\left|\frac{1}{n-1}\sum_{k=1}^{n}\int_{\partial B}f(x)\left(x_k-\frac{1}{n}\sum_{j=1}^{n}x_j\right)\mathbf{n}_k(x)\,dA-\int_B f(x)\,dx\right|\\ &\leq\frac{1}{n(n-1)}\sum_{1\leq i<j\leq n}L_{ij}\int_B(x_i-x_j)^2\,dx.\end{aligned}$$

The proof follows by the first inequality in (4.1) and by condition (4.3).

We can define the function $\Phi$ defined on the surface $\partial B$ and given by

$$\Phi(x_1,...,x_n) := \frac{1}{n-1}\sum_{k=1}^{n}\left(x_k - \frac{1}{n}\sum_{j=1}^{n}x_j\right)\mathbf{n}_k(x), \quad (x_1,...,x_n) \in B. \tag{4.5}$$

As above, we define

$$\|f\|_{\partial B,\infty} := \sup_{x\in\partial B}|f(x)| < \infty.$$

**Theorem 3.** *Let $B$ be a bounded closed subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary $\partial B$ and $\mathbf{n} = (\mathbf{n}_1,...,\mathbf{n}_n)$ be the unit outward-pointing normal of $\partial B$. If $f$ is a continuously differentiable function on an open neighborhood of $B$, then we have the inequalities*

$$\begin{aligned}
&\left|\int_B f(x)\,dx + \frac{1}{n(n-1)}\sum_{1\leq i<j\leq n}\int_B \Lambda_{\partial f,B}(x_i,x_j)\,dx\right| \\
&\leq \int_{\partial B}|f(x)|\,|\Phi(x_1,...,x_n)|\,dA \\
&\leq \frac{1}{n-1}\begin{cases}
\int_{\partial B}|f(x)|\max_{k\in\{1,...,n\}}\left|x_k - \frac{1}{n}\sum_{j=1}^{n}x_j\right|\sum_{k=1}^{n}|\mathbf{n}_k(x)|\,dA; \\
\int_{\partial B}|f(x)|\left(\sum_{k=1}^{n}\left|x_k - \frac{1}{n}\sum_{j=1}^{n}x_j\right|^p\right)^{1/p} \\
\left(\sum_{k=1}^{n}|\mathbf{n}_k(x)|^q\right)^{1/q}dA,\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\
\int_{\partial B}|f(x)|\max_{k\in\{1,...,n\}}|\mathbf{n}_k(x)|\sum_{k=1}^{n}\left|x_k - \frac{1}{n}\sum_{j=1}^{n}x_j\right|dA,
\end{cases} \\
&\leq \frac{1}{n-1}\|f\|_{\partial B,\infty} \\
&\times\begin{cases}
\int_{\partial B}\max_{k\in\{1,...,n\}}\left|x_k - \frac{1}{n}\sum_{j=1}^{n}x_j\right|\sum_{k=1}^{n}|n_k(x)|\,dA; \\
\int_{\partial B}\left(\sum_{k=1}^{n}\left|x_k - \frac{1}{n}\sum_{j=1}^{n}x_j\right|^p\right)^{1/p} \\
\left(\sum_{k=1}^{n}|n_k(x)|^q\right)^{1/q}dA,\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\
\int_{\partial B}\max_{k\in\{1,...,n\}}|n_k(x)|\sum_{k=1}^{n}\left|x_k - \frac{1}{n}\sum_{j=1}^{n}x_j\right|dA,
\end{cases}
\end{aligned} \tag{4.6}$$

*where the function $\Phi$ is defined above in (4.5).*

*In particular, we have*

$$\begin{aligned}
&\left|\int_B f(x)\,dx+\frac{1}{n(n-1)}\sum_{1\le i<j\le n}\int_B \Lambda_{\partial f,B}(x_i,x_j)\,dx\right| \\
&\le \int_{\partial B}|f(x)|\,|\Phi(x_1,...,x_n)|\,dA \\
&\le \frac{\sqrt{n}}{(n-1)\,n}\int_{\partial B}|f(x)|\left(n\sum_{k=1}^n x_k^2-\left(\sum_{j=1}^n x_j\right)^2\right)^{1/2} dA \\
&\le \frac{\sqrt{n}}{(n-1)\,n}\|f\|_{\partial B,\infty}\int_{\partial B}\left(n\sum_{k=1}^n x_k^2-\left(\sum_{j=1}^n x_j\right)^2\right)^{1/2} dA.
\end{aligned}\tag{4.7}$$

*Proof.* From (3.5) we get

$$\begin{aligned}
&\left|\int_B f(x)\,dx+\frac{1}{n(n-1)}\sum_{1\le i<j\le n}\int_B \Lambda_{\partial f,B}(x_i,x_j)\,dx\right| \\
&=\left|\int_{\partial B} f(x)\,\frac{1}{n-1}\sum_{k=1}^n\left(x_k-\frac{1}{n}\sum_{j=1}^n x_j\right)\mathbf{n}_k(x)\,dA\right| \\
&\le \int_{\partial B}|f(x)|\,|\Phi(x_1,...,x_n)|\,dA.
\end{aligned}\tag{4.8}$$

Using Hölder's type discrete inequalities we have

$$\begin{aligned}
&|\Phi(x_1,...,x_n)| \\
&\le \frac{1}{n-1}\sum_{k=1}^n\left|x_k-\frac{1}{n}\sum_{j=1}^n x_j\right||\mathbf{n}_k(x)| \\
&\le \frac{1}{n-1}\begin{cases}
\max_{k\in\{1,...,n\}}\left|x_k-\frac{1}{n}\sum_{j=1}^n x_j\right|\sum_{k=1}^n|\mathbf{n}_k(x)|; \\[2ex]
\left(\sum_{k=1}^n\left|x_k-\frac{1}{n}\sum_{j=1}^n x_j\right|^p\right)^{1/p}\left(\sum_{k=1}^n|\mathbf{n}_k(x)|^q\right)^{1/q}, \\
p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
\max_{k\in\{1,...,n\}}|\mathbf{n}_k(x)|\sum_{k=1}^n\left|x_k-\frac{1}{n}\sum_{j=1}^n x_j\right|.
\end{cases}
\end{aligned}\tag{4.9}$$

In particular, we have

$$
\begin{aligned}
(4.10)\quad |\Phi(x_1,...,x_n)| &\le \frac{1}{n-1}\left(\sum_{k=1}^{n}\left(x_k-\frac{1}{n}\sum_{j=1}^{n}x_j\right)^2\right)^{1/2}\left(\sum_{k=1}^{n}\mathbf{n}_k^2(x)\right)^{1/2} \\
&= \frac{1}{n-1}\left(\sum_{k=1}^{n}x_k^2-\frac{2}{n}\sum_{k=1}^{n}x_k\sum_{j=1}^{n}x_j+n\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)^2\right)^{1/2} \\
&= \frac{1}{n-1}\left(\frac{n\sum_{k=1}^{n}x_k^2-\left(\sum_{j=1}^{n}x_j\right)^2}{n}\right)^{1/2} \\
&= \frac{\sqrt{n}}{(n-1)\,n}\left(n\sum_{k=1}^{n}x_k^2-\left(\sum_{j=1}^{n}x_j\right)^2\right)^{1/2}
\end{aligned}
$$

since $\sum_{k=1}^{n}\mathbf{n}_k^2(x)=1$ for $(x_1,...,x_n)\in B$.

Therefore, by (4.8) we get the desired results (4.6) and (4.7). □

One can also separate the terms in the upper bounds by utilising the Lebesgue norms

$$\|g\|_{\partial B,p}:=\left(\int_{\partial B}|g(x)|^p\,dA\right)^{1/p}<\infty \text{ for } p\ge 1$$

and Hölder's integral inequality for the surface multiple integrals as follows:

**Corollary 2.** *With the assumptions of Theorem 3 we also have*

$$
\begin{aligned}
(4.11)\quad &\left|\int_B f(x)\,dx+\frac{1}{n(n-1)}\sum_{1\le i<j\le n}\int_B \Lambda_{\partial f,B}(x_i,x_j)\,dx\right| \\
&\le \int_{\partial B}|f(x)|\,|\Phi(x_1,...,x_n)|\,dA \\
&\le \begin{cases} \|f\|_{\partial B,\infty}\int_{\partial B}|\Phi(x)|\,dA; \\[2mm] \|f\|_{\partial B,p}\left(\int_{\partial B}|\Phi(x)|^q\,dA\right)^{1/q},\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\[2mm] \|f\|_{\partial B,1}\sup_{x\in\partial B}|\Phi(x)|. \end{cases}
\end{aligned}
$$

**Remark 3.** *Since upper bounds for $|\Phi(x)|$ are available, see (4.9), one can get further upper bounds. However, they are quite complicate in general and we do not state them here. We give only one example in the case of Euclidian norms, namely,* *via (4.7) we get*

$$\left|\int_B f(x)\,dx + \frac{1}{n(n-1)}\sum_{1\le i<j\le n}\int_B \Lambda_{\partial f,B}(x_i,x_j)\,dx\right| \tag{4.12}$$
$$\le \int_{\partial B} |f(x)|\,|\Phi(x_1,...,x_n)|\,dA$$
$$\le \frac{\sqrt{n}}{(n-1)\,n}\int_{\partial B}|f(x)|\left(n\sum_{k=1}^n x_k^2 - \left(\sum_{j=1}^n x_j\right)^2\right)^{1/2} dA$$
$$\le \frac{\sqrt{n}}{(n-1)\,n}\|f\|_{\partial B,2}\left(\int_{\partial B}\left[n\sum_{k=1}^n x_k^2 - \left(\sum_{j=1}^n x_j\right)^2\right]dA\right)^{1/2}.$$

## 5. An Example for Three Dimensional Balls

Consider the 3-dimensional ball centered in $O=(0,0,0)$ and having the radius $R>0$,

$$B(O,R) := \left\{ (x_1,x_2,x_3)\in\mathbb{R}^3 \middle|\, x_1^2+x_2^2+x_3^2\le R^2\right\}$$

and the sphere

$$S(O,R) := \left\{ (x_1,x_2,x_3)\in\mathbb{R}^3 \middle|\, x_1^2+x_2^2+x_3^2 = R^2\right\}.$$

Consider the parametrization of $B(O,R)$ and $S(O,R)$ given by:

$$B(O,R):\begin{cases} x_1 = r\cos\psi\cos\varphi \\ x_2 = r\cos\psi\sin\varphi \\ x_3 = r\sin\psi\end{cases};\quad (r,\psi,\varphi)\in[0,R]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi]$$

and

$$S(O,R):\begin{cases} x_1 = R\cos\psi\cos\varphi \\ x_2 = R\cos\psi\sin\varphi \\ x_3 = R\sin\psi\end{cases};\quad (\psi,\varphi)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi].$$

We have

$$\begin{vmatrix} \frac{\partial x_2}{\partial\psi} & \frac{\partial x_3}{\partial\psi} \\ \frac{\partial x_2}{\partial\varphi} & \frac{\partial x_3}{\partial\varphi}\end{vmatrix} = -R^2\cos^2\psi\cos\varphi,\quad \begin{vmatrix} \frac{\partial x_1}{\partial\psi} & \frac{\partial x_3}{\partial\psi} \\ \frac{\partial x_1}{\partial\varphi} & \frac{\partial x_3}{\partial\varphi}\end{vmatrix} = R^2\cos^2\psi\sin\varphi,$$

and

$$\begin{vmatrix} \frac{\partial x_1}{\partial\psi} & \frac{\partial x_2}{\partial\psi} \\ \frac{\partial x_1}{\partial\varphi} & \frac{\partial x_2}{\partial\varphi}\end{vmatrix} = -R^2\sin\psi\cos\psi.$$

Now, observe that

$$\int_{B(O,R)} (x_1-x_2)^2\,dx_1dx_2dx_3$$
$$= \int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}(r\cos\psi\cos\varphi - r\cos\psi\sin\varphi)^2\,r^2\cos\psi\,dr\,d\psi\,d\varphi$$
$$= \int_0^R r^4dr\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3\psi\,d\psi\int_0^{2\pi}(\cos\varphi-\sin\varphi)^2\,d\varphi = \frac{R^5}{5}\left(\frac{4}{3}\right)2\pi = \frac{8}{15}\pi R^5$$

and, similarly

$$\int_{B(O,R)} (x_2 - x_3)^2 \, dx_1 dx_2 dx_3 = \int_{B(O,R)} (x_1 - x_3)^2 \, dx_1 dx_2 dx_3 = \frac{8}{15}\pi R^5.$$

From Corollary 1 we have

$$\left| \frac{1}{2} \sum_{k=1}^{3} \int_{\partial B} f(x) \left( x_k - \frac{1}{3} \sum_{j=1}^{n} x_j \right) \mathbf{n}_k(x) \, dA - \int_B f(x) \, dx \right| \leq \frac{1}{6} \sum_{1 \leq i < j \leq 3} L_{ij} \int_B (x_i - x_j)^2 \, dx, \tag{5.1}$$

provided

$$\left| \frac{\partial f(x)}{\partial x_i} - \frac{\partial f(x)}{\partial x_j} \right| \leq L_{ij} |x_i - x_j| \tag{5.2}$$

with $L_{ij} > 0$, $1 \leq i < j \leq 3$ for all $x = (x_1, x_2, x_3) \in B$.

By writing (5.1) in polar coordinates, we get

$$\begin{aligned} &\left| \frac{1}{3} R^3 \left[ - \int_{S(O,R)} \left( \cos\psi \cos\varphi - \frac{\cos\psi \sin\varphi + \sin\psi}{2} \right) \right. \right. \\ &\times f(R\cos\psi\cos\varphi, R\cos\psi\sin\varphi, R\sin\psi) \cos^2\psi \cos\varphi d\psi d\varphi \\ &+ \int_{S(O,R)} \left( \cos\psi \sin\varphi - \frac{\cos\psi\cos\varphi + \sin\psi}{2} \right) \\ &\times f(R\cos\psi\cos\varphi, R\cos\psi\sin\varphi, R\sin\psi) \cos^2\psi \sin\varphi d\psi d\varphi \\ &- \int_{S(O,R)} \left( \sin\psi - \frac{\cos\psi\cos\varphi + \cos\psi\sin\varphi}{2} \right) \\ &\left. \times f(R\cos\psi\cos\varphi, R\cos\psi\sin\varphi, R\sin\psi) \sin\psi \cos\psi d\psi d\varphi \right] \\ &\left. - \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} f(r\cos\psi\cos\varphi, r\cos\psi\sin\varphi, r\sin\psi) r^2 \cos\psi dr d\psi d\varphi \right| \\ &\leq \frac{4}{45}\pi R^5 (L_{12} + L_{23} + L_{13}). \end{aligned} \tag{5.3}$$

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