INTEGRAL INEQUALITIES FOR SCHUR CONVEX FUNCTIONS
ON SYMMETRIC AND CONVEX SETS IN LINEAR SPACES

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Abstract. In this paper, we establish some integral inequalities for Schur convex functions defined on symmetric and convex sets from a Cartesian product of linear spaces. Some applications related to the Hermite-Hadamard inequality for convex functions defined on real intervals are also provided.

1. Introduction

For any $x = (x_1, ..., x_n) \in \mathbb{R}^n$, let $x_{[1]} \geq ... \geq x_{[n]}$ denote the components of $x$ in decreasing order, and let $x_1 = (x_{[1]}, ..., x_{[n]})$ denote the decreasing rearrangement of $x$. For $x, y \in \mathbb{R}^n$, $x \prec y$ if, by definition,

$$\left\{\begin{array}{l}
\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}, \quad k = 1, ..., n-1; \\
\sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]}.
\end{array}\right.$$ 

When $x \prec y$, $x$ is said to be majorized by $y$ ($y$ majorizes $x$). This notation and terminology was introduced by Hardy, Littlewood and Pólya in 1934.

Functions that preserve the ordering of majorization are said to be Schur-convex. Perhaps “Schur-increasing” would be more appropriate, but the term “Schur-convex” is by now well entrenched in the literature, as mentioned in [8, p.80].

A real-valued function $\phi$ defined on a set $\mathcal{A} \subset \mathbb{R}^n$ is said to be Schur-convex on $\mathcal{A}$ if

$$x \prec y \text{ on } \mathcal{A} \implies \phi(x) \leq \phi(y).$$

If, in addition, $\phi(x) < \phi(y)$ whenever $x \prec y$ but $x$ is not a permutation of $y$, then $\phi$ is said to be strictly Schur-convex on $\mathcal{A}$. If $\mathcal{A} = \mathbb{R}^n$, then $\phi$ is simply said to be Schur-convex or strictly Schur-convex.

For fundamental properties of Schur convexity see the monograph [8] and the references therein. For some recent results, see [3]-[5] and [9]-[11].

The following result is known in the literature as Schur-Ostrowski theorem [8, p. 84]:

**Theorem 1.** Let $I \subset \mathbb{R}$ be an open interval and let $\phi : I^n \to \mathbb{R}$ be continuously differentiable. Necessary and sufficient conditions for $\phi$ to be Schur-convex on $I^n$ are

$$\phi \text{ is symmetric on } I^n,$$

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and for all $i \neq j$, with $i, j \in \{1, ..., n\},$

\[(z_i - z_j) \left[ \frac{\partial \phi(z)}{\partial x_i} - \frac{\partial \phi(z)}{\partial x_j} \right] \geq 0 \text{ for all } z \in \mathbb{R}^n,\]

where $\frac{\partial \phi}{\partial x_k}$ denotes the partial derivative of $\phi$ with respect to its $k$-th argument.

Let $\mathcal{A} \subset \mathbb{R}^n$ be a set with the following properties:

(i) $\mathcal{A}$ is symmetric in the sense that $x \in \mathcal{A} \Rightarrow x\Pi \in \mathcal{A}$ for all permutations $\Pi$ of the coordinates.

(ii) $\mathcal{A}$ is convex and has a nonempty interior.

We have the following result, [8, p. 85].

**Theorem 2.** If $\phi$ is continuously differentiable on the interior of $\mathcal{A}$ and continuous on $\mathcal{A}$, then necessary and sufficient conditions for $\phi$ to be Schur-convex on $\mathcal{A}$ are

\[(z_1 - z_2) \left[ \frac{\partial \phi(z)}{\partial x_1} - \frac{\partial \phi(z)}{\partial x_2} \right] \geq 0 \text{ for all } z \in \mathcal{A}.\]

Another interesting characterization of Schur convex functions $\phi$ on $\mathcal{A}$ was obtained by C. Stępiński in [12]:

**Theorem 3.** Let $\phi$ be any function defined on a symmetric convex set $\mathcal{A}$ in $\mathbb{R}^n$. Then the function $\phi$ is Schur convex on $\mathcal{A}$ if and only if

\[\phi(x_1, ..., x_i, ..., x_j, ..., x_n) = \phi(x_1, ..., x_j, ..., x_i, ..., x_n)\]

for all $(x_1, ..., x_n) \in \mathcal{A}$ and $1 \leq i < j \leq n$ and

\[\phi(\lambda x_1 + (1 - \lambda)x_2, \lambda x_2 + (1 - \lambda)x_1, x_3, ..., x_n) \leq \phi(x_1, ..., x_n)\]

for all $(x_1, ..., x_n) \in \mathcal{A}$ and for all $\lambda \in (0, 1)$.

It is well known that any symmetric convex function defined on a symmetric convex set $\mathcal{A}$ is Schur convex, [8, p. 97]. If the function $\phi : \mathcal{A} \rightarrow \mathbb{R}$ is symmetric and quasi-convex, namely

\[\phi(\alpha u + (1 - \alpha)v) \leq \max \{\phi(u), \phi(v)\}\]

for all $\alpha \in [0, 1]$ and $u, v \in \mathcal{A}$, a symmetric convex set, then $\phi$ is Schur convex on $\mathcal{A}$ [8, p. 98].

Motivated by the above results, in this paper we establish some integral inequalities for Schur convex functions defined on symmetric and convex sets from a Cartesian product of linear spaces. Some applications related to the Hermite-Hadamard inequality for convex functions defined on real intervals are also provided.

2. **Main Results**

Let $X$ be a linear space and $G \subset X^2 := X \times X$ a convex set. We say that $G$ is symmetric if $(x, y) \in G$ implies that $(y, x) \in G$. If $D \subset X$ is a convex subset of $X$, then the Cartesian product $G := D^2 := D \times D$ is convex and symmetric in $X^2$.

Motivated by the characterization result of Stępiński above, we say that a function $f : G \rightarrow \mathbb{R}$ will be called Schur convex on the convex and symmetric set $G \subset X^2$ if

\[f(t(x, y) + (1 - t)(y, x)) \leq f(x, y)\]
for all \((x, y) \in G\) and for all \(t \in [0, 1]\).

If \(G = D^2\) then we recapture the general concept of Schur convexity introduced by Burai and Makó in 2016, [1].

We say that the function \(f : G \rightarrow \mathbb{R}\) is symmetric on \(G\) if \(f(x, y) = f(y, x)\) for all \((x, y) \in G\). If the function \(f\) is symmetric on \(G\) and the inequality holds for a given \(t \in (0, 1)\) and for all \((x, y) \in G\), then we say that \(f\) is \(t\)-Schur convex on \(G\).

The following fact follows from the definition of Schur convex functions:

**Proposition 1.** If \(f : G \rightarrow \mathbb{R}\) is Schur convex on the convex and symmetric set \(G \subset X^2\), then \(f\) is symmetric on \(G\).

**Proof.** If \((x, y) \in G\), then by (2.1) we get for \(t = 0\) that \(f(y, x) \leq f(x, y)\). If we replace \(x\) with \(y\) then we also get \(f(x, y) \leq f(y, x)\) which shows that \(f(x, y) = f(y, x)\) for all \((x, y) \in G\).

For \((x, y) \in G\), as in [1], let us define the following auxiliary function \(\varphi_{(x,y)} : [0, 1] \rightarrow \mathbb{R}\) by

\[
\varphi_{f,(x,y)}(t) = f\left( t (x, y) + (1 - t) (y, x) \right) = f\left( tx + (1 - t) y, ty + (1 - t) x \right).
\]

The properties of this function are as follows:

**Lemma 1.** Let \(G \subset X^2\) be a convex and symmetric set and \(f : G \rightarrow \mathbb{R}\) a symmetric function on \(G\). Then \(f\) is Schur convex on \(G\) if and only if for all arbitrarily fixed \((x, y) \in G\) the function \(\varphi_{f,(x,y)}\) is monotone decreasing on \([0, 1/2]\), monotone increasing on \((1/2, 1]\), and \(\varphi_{f,(x,y)}\) has a global minimum at \(1/2\).

**Proof.** We give a similar prove to the one from [1]. Assume that \(f\) is Schur convex on \(G\). Then for all \((u, v) \in G\) and \(t \in [0, 1]\) we have

\[
f\left( t (u, v) + (1 - t) (v, u) \right) \leq f\left( u, v \right).
\]

Let \((x, y) \in G\) and for \(0 \leq r < s < \frac{1}{2}\) and put \(u = rx + (1 - r) y\), \(v = ry + (1 - r) x\) and \(t = \frac{s - r}{1 - 2r}\). Then \((u, v) = r(x, y) + (1 - r)(y, x) \in G\) since \(G\) is symmetric and convex. By (2.3) we have

\[
\varphi_{f,(x,y)}(r) = f\left( r (x, y) + (1 - r) (y, x) \right) = f\left( u, v \right)
\]

\[
\geq f\left( \frac{s - r}{1 - 2r} (u, v) + \left(1 - \frac{s - r}{1 - 2r}\right) (v, u) \right) =: B.
\]
Observe that
\[
\frac{s-r}{1-2r} (u,v) + \left(1 - \frac{s-r}{1-2r}\right) (v,u)
\]
\[= \frac{s-r}{1-2r} [r(x,y) + (1-r)(y,x)]
\]
\[+ \left(\frac{1-r-s}{1-2r}\right) [r(y,x) + (1-r)(x,y)]
\]
\[= \left[\left(\frac{s-r}{1-2r}\right) r + \left(\frac{1-r-s}{1-2r}\right) (1-r)\right] (x,y)
\]
\[+ \left[\frac{s-r}{1-2r} (1-r) + \left(\frac{1-r-s}{1-2r}\right) r\right] (y,x)
\]
\[= \left(\frac{1-s-2r+2rs}{1-2r}\right) (x,y) + \left(\frac{s-2rs}{1-2r}\right) (y,x)
\]
\[= (1-s)(x,y) + s(y,x).
\]

Then
\[B = f ((1-s)(x,y) + s(y,x)) = \varphi_{f,(x,y)} (s)
\]
and by (2.4) we get that \(\varphi_{f,(x,y)} (r) \geq \varphi_{f,(x,y)} (s)\) for \(0 \leq r < s < \frac{1}{2}\), which shows that the function \(\varphi_{f,(x,y)}\) is monotone decreasing on \([0,1/2]\).

Observe that, by the symmetry of \(f\) on \(G\), we have
\[\varphi_{f,(x,y)} (1-t) = f \left( ((1-t)(x,y) + t(y,x) \right)
\]
\[= f ( ((1-t)x + ty, (1-t)y + tx) \right)
\]
\[= f ( ((1-t)y + tx, (1-t)x + ty) \right)
\]
\[= f (t(x,y) + (1-t)(y,x)) = \varphi_{f,(x,y)} (t)
\]
for all \(t \in [0,1]\).

This shows that the function \(\varphi_{f,(x,y)}(\cdot)\) is also monotone increasing on \((1/2,1]\).

From (2.3) we get for \(t = \frac{1}{2}\) that
\[(2.5) \quad f \left(\frac{u+v}{2}, \frac{u+v}{2}\right) \leq f (u,v)
\]
for all \((u,v) \in G\). If \((x,y) \in G\) and we take \(u = tx + (1-t)y, v = ty + (1-t)x, t \in [0,1]\) then \((u,v) = t(x,y) + (1-t)(y,x) \in G\), \(\frac{u+v}{2} = \frac{x+y}{2}\) and by (2.5) we get \(\varphi_{f,(x,y)}(1/2) \leq \varphi_{f,(x,y)} (t)\) for all \(t \in [0,1]\), showing that \(\varphi_{f,(x,y)}\) has a global minimum at \(1/2\).

Now, for fixed \((x,y) \in G\), assume that the function \(\varphi_{f,(x,y)}(\cdot)\) is monotone decreasing on \([0,1/2]\), monotone increasing on \((1/2,1]\), and has a global minimum at \(1/2\).

Then for \(t \in [0,1/2]\) we have
\[f (t(x,y) + (1-t)(y,x)) = \varphi_{f,(x,y)} (t) \leq \varphi_{f,(x,y)} (0) = f (y,x) = f (x,y)
\]
and for \(t \in (1/2,1]\) we have
\[f (t(x,y) + (1-t)(y,x)) = \varphi_{f,(x,y)} (t) \leq \varphi_{f,(x,y)} (1) = f (x,y).
\]

Therefore, for all \(t \in [0,1]\) we have \(\varphi_{f,(x,y)} (t) \leq f (x,y)\), which shows that \(f\) is Schur convex on \(G\). \(\square\)
We have the following weighted integral inequality:

**Theorem 4.** Assume that the function $f : G \to \mathbb{R}$ is Schur convex on the convex and symmetric set $G \subset X^2$. Then for any Lebesgue integrable function $p : [0, 1] \to [0, \infty)$ we have

$$f \left( \frac{x+y}{2}, \frac{x+y}{2} \right) \int_0^1 p(t) \, dt \leq \int_0^1 \left( f(t(x,y) + (1-t)(y,x)) \right) p(t) \, dt$$

for all $(x, y) \in G$.

In particular, we have

$$f \left( \frac{x+y}{2}, \frac{x+y}{2} \right) \leq \int_0^1 f(t(x,y) + (1-t)(y,x)) \, dt \leq f(x, y)$$

for all $(x, y) \in G$.

**Proof.** Using Lemma 1 we have

$$f \left( \frac{x+y}{2}, \frac{x+y}{2} \right) \leq f(t(x,y) + (1-t)(y,x)) \leq f(x, y)$$

for all $(x, y) \in G$ and $t \in [0, 1]$.

If we multiply this inequality by $p(t) \geq 0$ and integrate on $[0, 1]$ we deduce the desired result (2.6).

If some monotonicity information is available for the function $p$ we also have:

**Theorem 5.** Assume that the function $f : G \to \mathbb{R}$ is Schur convex on the convex and symmetric set $G \subset X^2$. If $p : [0, 1] \to \mathbb{R}$ is symmetric towards $1/2$, namely $p(1-t) = p(t)$ for all $t \in [0, 1]$ and monotonic decreasing (increasing) on $[0, 1/2]$, then

$$\left( \int_0^1 f(t(x,y) + (1-t)(y,x)) \, dt \right) p(t) \, dt \geq \left( \int_0^1 f(t(x,y) + (1-t)(y,x)) \, dt \right) p(t) \, dt.$$

**Proof.** Let $(x, y) \in G$. Since the functions $\varphi_{f,(x,y)}$ and $p$ are symmetric on $[0, 1]$, then

$$\int_0^1 f(t(x,y) + (1-t)(y,x)) \, p(t) \, dt = 2 \int_0^{1/2} f(t(x,y) + (1-t)(y,x)) \, p(t) \, dt.$$

Assume that the functions $\varphi_{f,(x,y)}$ and $p$ are both decreasing on $[0, 1/2]$, then by Chebyshev’s inequality for synchronous functions $h, g : [a, b] \to \mathbb{R}$

$$\frac{1}{b-a} \int_a^b h(t) \, g(t) \, dt \geq \frac{1}{b-a} \int_a^b h(t) \, dt \cdot \frac{1}{b-a} \int_a^b g(t) \, dt,$$

we have

$$\int_0^{1/2} f(t(x,y) + (1-t)(y,x)) \, p(t) \, dt \geq 2 \int_0^{1/2} f(t(x,y) + (1-t)(y,x)) \, dt \cdot 2 \int_0^{1/2} p(t) \, dt$$

$$\geq 2 \int_0^{1/2} f(t(x,y) + (1-t)(y,x)) \, dt \cdot 2 \int_0^{1/2} p(t) \, dt.$$
and since, by symmetry,
\[ 2 \int_0^{1/2} f(t(x,y) + (1-t)(y,x)) dt = \int_0^1 f(t(x,y) + (1-t)(y,x)) dt \]
and
\[ 2 \int_0^{1/2} p(t) dt = \int_0^1 p(t) dt \]
hence by (2.9) we get the desired result (2.8).

The following Čebyšev’s type inequality holds for two Schur convex functions:

**Corollary 1.** Assume that the functions \( f, g : G \to \mathbb{R} \) are Schur convex on the convex and symmetric set \( G \subset X^2 \). Then we have

\[
\int_0^1 f(t(x,y) + (1-t)(y,x)) g(t(x,y) + (1-t)(y,x)) dt \\
\geq \int_0^1 g(t(x,y) + (1-t)(y,x)) dt \int_0^1 f(t(x,y) + (1-t)(y,x)) dt
\]

for all \((x,y) \in G\).

If one of the functions is Schur convex and the other Schur concave, then the sign of inequality reverses in (2.10).

We can prove the following refinement of (2.6):

**Corollary 2.** Assume that the function \( f : G \to \mathbb{R} \) is Schur convex on the convex and symmetric set \( G \subset X^2 \) and \( p : [0,1] \to [0,\infty) \) is symmetric towards \( 1/2 \) and positive.

(i) If \( p \) is decreasing on \([0,1/2]\), then

\[
f\left( \frac{x+y}{2}, \frac{x+y}{2} \right) \leq \int_0^1 f(t(x,y) + (1-t)(y,x)) dt \\
\leq \frac{1}{\int_0^1 p(t) dt} \int_0^1 f(t(x,y) + (1-t)(y,x)) p(t) dt \\
\leq f(x,y)
\]

for all \((x,y) \in G\).

(ii) If \( p \) is increasing on \([0,1/2]\), then

\[
f\left( \frac{x+y}{2}, \frac{x+y}{2} \right) \leq \frac{1}{\int_0^1 p(t) dt} \int_0^1 f(t(x,y) + (1-t)(y,x)) p(t) dt \\
\leq \int_0^1 f(t(x,y) + (1-t)(y,x)) dt \\
\leq f(x,y)
\]

for all \((x,y) \in G\).

**Proof.** (i). From (2.8) we get

\[
\frac{1}{\int_0^1 p(t) dt} \int_0^1 f(t(x,y) + (1-t)(y,x)) p(t) dt \geq \int_0^1 f(t(x,y) + (1-t)(y,x)) dt
\]

and by (2.6) and (2.7) we get the desired result (2.11).

(ii). The proof goes in a similar way.  

\[ \square \]
Remark 1. If we consider the weight \( p(t) = |t - \frac{1}{2}| \), then \( \int_0^1 p(t)\,dt = \frac{1}{4} \) and by (2.11) we get

\[
\left( \frac{x+y}{2}, \frac{x+y}{2} \right) \leq \int_0^1 f(t(x,y) + (1-t)(y,x))\,dt
\]
\[
\leq 4 \int_0^1 f(t(x,y) + (1-t)(y,x)) \left| t - \frac{1}{2} \right| \,dt
\leq f(x,y)
\]

for any function \( f : G \to \mathbb{R} \) that is Schur convex on the convex and symmetric set \( G \subset X^2 \) and for all \( (x,y) \in G \).

If we consider the weight \( p(t) = t(1-t) \), then \( \int_0^1 p(t)\,dt = \frac{1}{6} \) and by (2.12) we get

\[
\left( \frac{x+y}{2}, \frac{x+y}{2} \right) \leq \int_0^1 f(t(x,y) + (1-t)(y,x))\,dt
\]
\[
\leq 6 \int_0^1 f(t(x,y) + (1-t)(y,x)) t(1-t)\,dt
\leq f(x,y)
\]

for any function \( f : G \to \mathbb{R} \) that is Schur convex on the convex and symmetric set \( G \subset X^2 \) and for all \( (x,y) \in G \).

We also have the following inequality for two functions:

Corollary 3. Assume that the functions \( f, g : G \to \mathbb{R} \) are Schur convex on the convex and symmetric set \( G \subset X^2 \) and \( g \) is nonnegative, then

\[
\left( \frac{x+y}{2}, \frac{x+y}{2} \right)
\]
\[
\leq \int_0^1 f(t(x,y) + (1-t)(y,x))\,dt
\]
\[
\leq \frac{1}{\int_0^1 g(t(x,y) + (1-t)(y,x))\,dt}
\]
\[
\times \int_0^1 f(t(x,y) + (1-t)(y,x))g(t(x,y) + (1-t)(y,x))\,dt
\leq f(x,y)
\]

for all \( (x,y) \in G \).
If \( g \) is Schur concave and nonnegative on \( G \), then

\[
(f(x, y) + (1 - t)(y, x)) \leq \frac{1}{1 - a} \int_a^b h(t) k(t) dt - \frac{1}{1 - a} \int_a^b h(t) dt \frac{1}{1 - a} \int_a^b k(t) dt \\
\leq \frac{1}{4} (M - m) (N - n)
\]

for all \((x, y) \in G\).

Recall the famous Grüss’ inequality that provides an upper bound for the distance between the integral mean of the product and the product of integral means, more precisely

\[
\left| \frac{1}{b-a} \int_a^b h(t) k(t) dt - \frac{1}{b-a} \int_a^b h(t) dt \frac{1}{b-a} \int_a^b k(t) dt \right| \\
\leq \frac{1}{4} (M - m) (N - n)
\]

provided the functions \( h, k \) are measurable on \([a, b]\) and \(-\infty < m \leq h(t) \leq M \leq \infty, -\infty < n \leq k(t) \leq N < \infty\), for almost every \( t \in [a, b] \). The constant \( \frac{1}{4} \) is best possible in (2.17).

**Theorem 6.** Assume that the function \( f : G \to \mathbb{R} \) is Schur convex on the convex and symmetric set \( G \subset X^2 \). If \( p : [0, 1] \to \mathbb{R} \) is symmetric towards \( 1/2 \), namely \( p(1 - t) = p(t) \) for all \( t \in [0, 1] \) and monotonic decreasing on \([0, 1/2]\) then

\[
0 \leq \int_0^1 f(t(x, y) + (1 - t)(y, x)) p(t) dt \\
- \int_0^1 p(t) dt \int_0^1 f(t(x, y) + (1 - t)(y, x)) dt \\
\leq \frac{1}{4} \left[ p(0) - p\left(\frac{1}{2}\right) \right] \left[ f(x, y) - f\left(\frac{x + y}{2}, \frac{x + y}{2}\right) \right]
\]

for all \((x, y) \in G\).

If \( p \) is monotonic increasing on \([0, 1/2]\), then

\[
0 \leq \int_0^1 p(t) dt \int_0^1 f(t(x, y) + (1 - t)(y, x)) dt \\
- \int_0^1 f(t(x, y) + (1 - t)(y, x)) p(t) dt \\
\leq \frac{1}{4} \left[ p\left(\frac{1}{2}\right) - p(0) \right] \left[ f(x, y) - f\left(\frac{x + y}{2}, \frac{x + y}{2}\right) \right]
\]

for all \((x, y) \in G\).

The proof follows by Gruss’ inequality (2.17) written for \( h(t) = p(t) \) and \( k(t) = f(t(x, y) + (1 - t)(y, x)) \), \( t \in [0, 1] \) and \((x, y) \in G\).
Corollary 4. Assume that both functions \( f, g : G \to \mathbb{R} \) are Schur convex on the convex and symmetric set \( G \subset X^2 \). Then we have

\[
0 \leq \int_0^1 f(t(x, y) + (1 - t)(y, x)) \, g(t(x, y) + (1 - t)(y, x)) \, dt
\]

\[
- \int_0^1 g(t(x, y) + (1 - t)(y, x)) \, \int_0^1 f(t(x, y) + (1 - t)(y, x)) \, dt
\]

\[
\leq \frac{1}{4} \left[ g(x, y) - g\left( \frac{x + y}{2}, \frac{x + y}{2} \right) \right] \left[ f(x, y) - f\left( \frac{x + y}{2}, \frac{x + y}{2} \right) \right].
\]

If \( f \) is Schur convex and \( g \) is Schur concave, then

\[
0 \leq \int_0^1 g(t(x, y) + (1 - t)(y, x)) \, \int_0^1 f(t(x, y) + (1 - t)(y, x)) \, dt
\]

\[
- \int_0^1 f(t(x, y) + (1 - t)(y, x)) \, g(t(x, y) + (1 - t)(y, x)) \, dt
\]

\[
\leq \frac{1}{4} \left[ g\left( \frac{x + y}{2}, \frac{x + y}{2} \right) - g(x, y) \right] \left[ f(x, y) - f\left( \frac{x + y}{2}, \frac{x + y}{2} \right) \right].
\]

3. Examples for Functions of Two Real Variables

We assume in this section that \( G \) is a convex and symmetric subset of the two dimensional space \( \mathbb{R}^2 \) and \( f : G \to \mathbb{R} \) is Schur convex on \( G \). If \((a, b) \in G \) with \( a < b \) and we put \( u = (1 - t) a + tb \), then \((1 - t) b + ta = b + a - t(a - b) \). We also assume that \( w : [a, b] \to [0, \infty) \) is Lebesgue integrable on \([a, b]\) and symmetric on this interval, namely \( w(b + a - u) = w(u) \) for all \( u \in [a, b] \). Since \( du = (b - a) \, dt \) then by taking \( p(t) = w((1 - t)a + tb) \), \( t \in [0, 1] \) we have by Theorem 4 that

\[
f\left( \frac{a + b}{2}, \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(u, a + b - u) \, w(u) \, du \leq f(a, b).
\]

In particular, we have

\[
f\left( \frac{a + b}{2}, \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_0^1 f(u, a + b - u) \, du \leq f(a, b).
\]

If we take \( w(u) = |u - \frac{a + b}{2}| \), \( u \in [a, b] \) in (3.1), then we get

\[
f\left( \frac{a + b}{2}, \frac{a + b}{2} \right) \leq \frac{4}{(b - a)^2} \int_a^b f(u, a + b - u) \, \left| u - \frac{a + b}{2} \right| \, du \leq f(a, b)
\]

while for \( w(u) = (u - a)(b - u) \), \( u \in [a, b] \) we get

\[
f\left( \frac{a + b}{2}, \frac{a + b}{2} \right) \leq \frac{6}{(b - a)^2} \int_a^b f(u, a + b - u) \, (u - a)(b - u) \, du \leq f(a, b).
\]

If we have two Schur convex functions \( f, g : G \to \mathbb{R} \), then

\[
\int_a^b f(u, a + b - u) \, g(u, a + b - u) \, du \geq \int_a^b f(u, a + b - u) \, du \int_a^b g(u, a + b - u) \, du.
\]
If one function is Schur convex and the other is Schur concave, then the sign of inequality in (3.5) is reversed.

By utilising Corollary 2 we can improve the inequality (3.1) as follows:

**Proposition 2.** Assume that the function $f : G \rightarrow \mathbb{R}$ is Schur convex on the convex and symmetric set $G \subset \mathbb{R}^2$, $(a, b) \in G$ with $a < b$ and $w : [a, b] \rightarrow [0, \infty)$ is Lebesgue integrable on $[a, b]$ and symmetric on $[a, b]$.

(i) If $w$ is decreasing on $[a, \frac{a+b}{2}]$, then

$$f \left( \frac{a+b}{2}, \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(u, a+b-u) \, du$$

$$\leq \frac{1}{\int_a^b w(u) \, du} \int_a^b f(u, a+b-u) \, w(u) \, du$$

$$\leq f(a, b).$$

(ii) If $w$ is increasing on $[a, \frac{a+b}{2}]$, then

$$f \left( \frac{a+b}{2}, \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(u, a+b-u) \, w(u) \, du$$

$$\leq \frac{1}{b-a} \int_a^b f(u, a+b-u) \, du$$

$$\leq f(a, b).$$

If we take $w(u) = \left| u - \frac{a+b}{2} \right|$, $u \in [a, b]$ in (3.6), then we get

$$f \left( \frac{a+b}{2}, \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(u, a+b-u) \, du$$

$$\leq \frac{4}{(b-a)^2} \int_a^b f(u, a+b-u) \left| u - \frac{a+b}{2} \right| \, du$$

$$\leq f(a, b).$$

Also, if we choose $w(u) = (u-a)(b-u)$, $u \in [a, b]$ in (3.7), then we obtain

$$f \left( \frac{a+b}{2}, \frac{a+b}{2} \right) \leq \frac{6}{(b-a)^2} \int_a^b f(u, a+b-u) (u-a)(b-u) \, du$$

$$\leq \frac{1}{b-a} \int_a^b f(u, a+b-u) \, du$$

$$\leq f(a, b).$$

From Theorem 6 we also have:

**Proposition 3.** Assume that the function $f : G \rightarrow \mathbb{R}$ is Schur convex on the convex and symmetric set $G \subset \mathbb{R}^2$, $(a, b) \in G$ with $a < b$ and $w : [a, b] \rightarrow [0, \infty)$ is Lebesgue integrable on $[a, b]$ and symmetric on $[a, b]$.

(i) If $w$ is decreasing on $[a, \frac{a+b}{2}]$, then

$$0 \leq \int_a^b f(u, a+b-u) \, w(u) \, du - \int_a^b w(u) \, du \int_a^b f(u, a+b-u) \, du$$

$$\leq \frac{1}{4} \left[ w(b) - w \left( \frac{a+b}{2} \right) \right] \left[ f(a, b) - f \left( \frac{a+b}{2}, \frac{a+b}{2} \right) \right]$$
(ii) If \( w \) is increasing on \([a, \frac{a+b}{2}]\), then

\[
0 \leq \int_a^b w(u) du \int_a^b f(u, a + b - u) du - \int_a^b f(u, a + b - u) w(u) du \leq \frac{1}{4} \left[ w \left( \frac{a+b}{2} \right) - w(b) \right] \left[ f(a, b) - f \left( \frac{a+b}{2}, \frac{a+b}{2} \right) \right].
\]

From this proposition we derive the following reverse inequalities of (3.5).

**Corollary 5.** Assume that the function \( f, g : G \to \mathbb{R} \) are Schur convex on the convex and symmetric set \( G \subset \mathbb{R}^2 \), \((a, b) \in G\) with \( a < b \). Then

\[
0 \leq \int_a^b f(u, a + b - u) g(u, a + b - u) du - \int_a^b g(u, a + b - u) f(u, a + b - u) du \leq \frac{1}{4} \left[ g(a, b) - g \left( \frac{a+b}{2}, \frac{a+b}{2} \right) \right] \left[ f(a, b) - f \left( \frac{a+b}{2}, \frac{a+b}{2} \right) \right].
\]

If \( f : G \to \mathbb{R} \) is Schur convex and \( f : G \to \mathbb{R} \) is Schur concave, then

\[
0 \leq \int_a^b g(u, a + b - u) du \int_a^b f(u, a + b - u) du - \int_a^b f(u, a + b - u) g(u, a + b - u) du \leq \frac{1}{4} \left[ g \left( \frac{a+b}{2}, \frac{a+b}{2} \right) - g(a, b) \right] \left[ f(a, b) - f \left( \frac{a+b}{2}, \frac{a+b}{2} \right) \right].
\]

### 4. Some Applications for Hermite-Hadamard Inequality

We recall the celebrated *Hermite-Hadamard inequality* for continuous convex functions \( h \) defined on a real interval \( I \), which state that

\[
h \left( \frac{x+y}{2} \right) \leq \frac{1}{y-x} \int_x^y h(t) dt \leq \frac{h(x) + h(y)}{2}
\]

for all \( x \neq y, x, y \in I \). For a monograph devoted to this inequality, see [6]. Many related results are also presented in the survey paper [4].

The following result concerning the Schur convexity of the integral mean was obtained by Elezović and Pečarić in [7]:

**Theorem 7.** Let \( h \) be a continuous function on \( I \). Then

\[
H(x, y) := \begin{cases} 
\frac{1}{y-x} \int_x^y h(t) dt, & \text{for } x \neq y, \ x, \ y \in I; \\
h(x), & \text{for } y = x, \ x \in I,
\end{cases}
\]

is Schur convex (concave) on \( I^2 \) if and only if \( h \) is convex (concave) on \( I \).
Let $h$ be a continuous function on $I$. We have for $t \in [0, 1], t \neq 1/2$ that
\[
H(t(x, y) + (1 - t)(y, x)) = H(tx + (1 - t)y, ty + (1 - t)x)
\]
\[
= \begin{cases}
\frac{1}{tx+(1-t)y-ty-(1-t)x} \int_{ty+(1-t)x}^{tx+(1-t)y} h(s) \, ds, & \text{for } x \neq y, \; x, y \in I; \\
h(x), & \text{for } y = x, \; x \in I,
\end{cases}
\]
\[
= \begin{cases}
\frac{1}{(1-2t)(y-x)} \int_{ty+(1-t)x}^{tx+(1-t)y} h(s) \, ds, & \text{for } x \neq y, \; x, y \in I; \\
h(x), & \text{for } y = x, \; x \in I.
\end{cases}
\]

For $t = 1/2$ we have
\[
H\left(\frac{x + y}{2}, \frac{x + y}{2}\right) = h\left(\frac{x + y}{2}\right)
\]
for $x, y \in I$.

**Corollary 6.** Assume that $h$ is continuous convex on $I$. Then we have the following refinement of the first Hermite-Hadamard inequality
\[
h\left(\frac{x + y}{2}\right) \leq \frac{1}{(1-2t)(y-x)} \int_{tx+(1-t)y}^{tx+(1-t)y} h(s) \, ds 
\]
\[
\leq \frac{1}{y-x} \int_{x}^{y} h(t) \, dt,
\]
for all $x \neq y, \; x, y \in I$ and $t \in [0, 1], \; t \neq 1/2$.

**Proof.** Since $h$ is continuous convex on $I$, hence by Theorem 7 we get that $H$ is Schur convex on $I^2$. By utilising Lemma 1 we conclude that
\[
H\left(\frac{x + y}{2}, \frac{x + y}{2}\right) \leq H\left(t(x, y) + (1 - t)(y, x)\right) \leq H(x, y)
\]
for $t \in [0, 1]$, and the inequality (4.2) is obtained. \qed

Assume that $h$ is continuous on $I$. For $x \neq y, \; x, y \in I$, we consider the function $\psi_{h,(x,y)} : [0, 1] \to \mathbb{R}$ defined by
\[
\psi_{h,(x,y)}(t) := \begin{cases}
\frac{1}{(1-2t)(y-x)} \int_{tx+(1-t)y}^{tx+(1-t)y} h(s) \, ds & \text{for } t \neq 1/2; \\
h\left(\frac{x+y}{2}\right) & \text{for } t = 1/2.
\end{cases}
\]

**Remark 2.** Assume that $h$ is continuous convex on $I$. For any Lebesgue integrable function $p : [0, 1] \to [0, \infty)$ with $\int_{0}^{1} p(t) \, dt > 0$ we have from Theorem 4 that
\[
h\left(\frac{x + y}{2}\right) \leq \frac{1}{(y-x)} \int_{0}^{1} p(t) \, dt \int_{0}^{1} \psi_{h,(x,y)}(t) \, dt
\]
\[
\leq \frac{1}{y-x} \int_{x}^{y} h(t) \, dt,
\]
and, in particular,
\[
h\left(\frac{x + y}{2}\right) \leq \frac{1}{y-x} \int_{0}^{1} \psi_{h,(x,y)}(t) \, dt \leq \frac{1}{y-x} \int_{x}^{y} h(t) \, dt,
\]
for all $x \neq y, x, y \in I$.

We also have:

**Corollary 7.** Assume that $h$ is continuous convex on $I$, then the function $\psi_{h,(x,y)}$ is monotone decreasing on $[0,1/2)$, monotone increasing on $(1/2,1]$, and $\psi_{h,(x,y)}$ has a global minimum at 1/2.

The proof is obvious by Lemma 1.

If more assumptions are imposed on the weight $p$, then some better inequalities are obtained:

**Corollary 8.** Assume that $h$ is continuous convex on $I$ and $p : [0,1] \rightarrow [0,\infty)$ is symmetric towards 1/2.

(i) If $p$ is decreasing on $[0,1/2]$, then

$$h \left( \frac{x+y}{2} \right) \leq \frac{1}{y-x} \int_0^1 \psi_{h,(x,y)}(t) \, dt$$

$$\leq \frac{1}{y-x} \int_0^1 \int_0^1 p(t) \psi_{h,(x,y)}(t) \, dt \leq \frac{1}{y-x} \int_x^y h(t) \, dt$$

for all $x \neq y, x, y \in I$.

(ii) If $p$ is increasing on $[0,1/2]$, then

$$h \left( \frac{x+y}{2} \right) \leq \frac{1}{y-x} \int_0^1 p(t) \psi_{h,(x,y)}(t) \, dt$$

$$\leq \frac{1}{y-x} \int_0^1 \psi_{h,(x,y)}(t) \, dt \leq \frac{1}{y-x} \int_x^y h(t) \, dt$$

for all $x \neq y, x, y \in I$.

**Remark 3.** If we take $p(t) = \left| t - \frac{1}{2} \right|$, $t \in [0,1]$ in (4.5), then we get

$$h \left( \frac{x+y}{2} \right) \leq \frac{1}{y-x} \int_0^1 \psi_{h,(x,y)}(t) \, dt$$

$$\leq \frac{4}{y-x} \int_0^1 \left| t - \frac{1}{2} \right| \psi_{h,(x,y)}(t) \, dt \leq \frac{1}{y-x} \int_x^y h(t) \, dt$$

and for $p(t) = t(1-t)$, $t \in [0,1]$ in (4.6) we obtain

$$h \left( \frac{x+y}{2} \right) \leq \frac{6}{y-x} \int_0^1 t(1-t) \psi_{h,(x,y)}(t) \, dt$$

$$\leq \frac{1}{y-x} \int_0^1 \psi_{h,(x,y)}(t) \, dt \leq \frac{1}{y-x} \int_x^y h(t) \, dt.$$
If we take in (4.9) \( p(t) = \left| t - \frac{1}{2} \right|, \ t \in [0,1] \), then we obtain the inequality

\[
0 \leq \int_0^1 \psi_{h,(x,y)} (t) \left| t - \frac{1}{2} \right| dt - \frac{1}{4} \int_0^1 \psi_{h,(x,y)} (t) dt \\
\leq \frac{1}{8} \left[ \frac{1}{y - x} \int_x^y h(t) dt - h \left( \frac{x + y}{2} \right) \right]
\]

provided that \( h \) is continuous convex on \( I \) and \( x \neq y, \ x, y \in I \).

In [2] Chu et al. obtained the following results:

**Theorem 8.** Suppose \( h : I \rightarrow \mathbb{R} \) is a continuous function. Function

\[
M(x,y) := \begin{cases} 
\frac{1}{y-x} \int_x^y h(t) dt - h \left( \frac{x + y}{2} \right) , & (x,y) \in I^2, \ x \neq y \\
0, \ (x,y) \in I^2, \ x = y
\end{cases}
\]

is Schur-convex (Schur-concave) on \( I^2 \) if and only if \( h \) is convex (concave) on \( I \). Furthermore, function

\[
T(x,y) := \begin{cases} 
\frac{h(x)+h(y)}{2} - \frac{1}{y-x} \int_x^y h(t) dt , & (x,y) \in I^2, \ x \neq y \\
0, \ (x,y) \in I^2, \ x = y
\end{cases}
\]

is Schur-convex (Schur-concave) on \( I^2 \) if and only if \( h \) is convex (concave) on \( I \).

Observe that for \( t \in [0,1], \ t \neq 1/2 \) we have

\[
T(t(x,y) + (1-t)(y,x)) = T(tx + (1-t)y,ty + (1-t)x) \\
= \frac{h(tx+(1-t)y)+h(ty+(1-t)x)}{2} - \frac{1}{(1-2t)(y-x)} \int_{ty+(1-t)x}^{tx+(1-t)y} h(s) ds , \ (x,y) \in I^2, \ x \neq y
\]

\[
= 0, \ (x,y) \in I^2, \ x = y.
\]

For \( t = \frac{1}{2} \) we have

\[
T\left( \frac{x+y}{2}, \frac{x+y}{2} \right) = 0, \ (x,y) \in I^2.
\]

We have:

**Corollary 10.** Assume that \( h \) is continuous convex on \( I \). Then we have the following refinement of the second Hermite-Hadamard inequality

\[
0 \leq \frac{h(tx+(1-t)y)+h(ty+(1-t)x)}{2} \\
- \frac{1}{(1-2t)(y-x)} \int_{ty+(1-t)x}^{tx+(1-t)y} h(s) ds \\
\leq \frac{h(x)+h(y)}{2} - \frac{1}{y-x} \int_x^y h(t) dt,
\]

for all \( x \neq y, \ x, y \in I \) and \( t \in [0,1], \ t \neq 1/2 \).
Proof. Since $h$ is continuous convex on $I$, hence by Theorem 8 we get that $H$ is Schur convex on $I^2$. By utilising Lemma 1 we conclude that

$$T\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \leq T(t(x,y) + (1-t)(y,x)) \leq T(x,y)$$

for $t \in [0,1]$, and the inequality (4.11) is obtained. \qed

With the notations above, we have for $x \neq y$, $x, y \in I$ and $t \in [0,1]$, $t \neq 1/2$ let put

$$\delta_{h,(x,y)}(t) := T(t(x,y) + (1-t)(y,x)) - \frac{h(tx + (1-t)y) + h(ty + (1-t)x)}{2} - \psi_{h,(x,y)}(t)$$

and

$$\delta_{h,(x,y)}\left(\frac{1}{2}\right) := 0$$

From Lemma 1 we have:

**Corollary 11.** Assume that $h$ is continuous convex on $I$ and $x \neq y$, $x, y \in I$. Then the function the function $\delta_{h,(x,y)}$ is nonnegative, monotone decreasing on $[0,1/2)$, monotone increasing on $(1/2,1]$, and $\delta_{h,(x,y)}$ has a global minimum at $1/2$.

We also have, by utilising Theorem 4:

**Corollary 12.** Assume that $h$ is continuous convex on $I$ and $x \neq y$, $x, y \in I$. Then for any Lebesgue integrable function $p : [0,1] \rightarrow [0,\infty)$ we have

$$0 \leq \int_0^1 \frac{h(tx + (1-t)y) + h(ty + (1-t)x)}{2} p(t) dt$$

$$- \int_0^1 \psi_{h,(x,y)}(t) p(t) dt$$

$$\leq \left[ \frac{h(x) + h(y)}{2} - \frac{1}{y-x} \int_x^y h(t) dt \right] \int_0^1 p(t) dt.$$ 

In particular, we have the following refinement of the second Hermite-Hadamard inequality

$$0 \leq \frac{1}{y-x} \int_x^y h(t) dt - \int_0^1 \psi_{h,(x,y)}(t) dt$$

$$\leq \frac{h(x) + h(y)}{2} - \frac{1}{y-x} \int_x^y h(t) dt.$$ 

If more conditions are assumed for the weight $p$, then we also have:

**Corollary 13.** Assume that $h$ is continuous convex on $I$ and $x \neq y$, $x, y \in I$ and $p : [0,1] \rightarrow [0,\infty)$ is symmetric towards $1/2$ and positive.
(i) If \( p \) is decreasing on \([0, 1/2]\), then

\[
0 \leq \frac{1}{y-x} \int_{x}^{y} h(t) \, dt - \int_{0}^{1} \psi_{h,(x,y)}(t) \, dt \\
\leq \frac{1}{\int_{0}^{1} p(t) \, dt} \int_{0}^{1} h(tx + (1-t)y) \, dt \\
- \frac{1}{\int_{0}^{1} p(t) \, dt} \int_{0}^{1} \psi_{h,(x,y)}(t) \, dt \\
\leq \frac{h(x) + h(y)}{2} - \frac{1}{y-x} \int_{x}^{y} h(t) \, dt
\]

for all \((x, y) \in G\).

(ii) If \( p \) is increasing on \([0, 1/2]\), then

\[
0 \leq \frac{1}{\int_{0}^{1} p(t) \, dt} \int_{0}^{1} h(tx + (1-t)y) \, dt \\
- \frac{1}{\int_{0}^{1} p(t) \, dt} \int_{0}^{1} \psi_{h,(x,y)}(t) \, dt \\
\leq \frac{1}{y-x} \int_{x}^{y} h(t) \, dt - \int_{0}^{1} \psi_{h,(x,y)}(t) \, dt \\
\leq \frac{h(x) + h(y)}{2} - \frac{1}{y-x} \int_{x}^{y} h(t) \, dt.
\]

The proof follows by Corollary 2.

**Corollary 14.** Assume that \( h \) is continuous convex on \( I \) and \( x \neq y, x, y \in I \) and \( p : [0, 1] \rightarrow [0, \infty) \) is symmetric towards \( 1/2 \) and positive. If \( p : [0, 1] \rightarrow \mathbb{R} \) is symmetric towards \( 1/2 \) and monotonic decreasing on \([0, 1/2]\) then

\[
0 \leq \int_{0}^{1} h(tx + (1-t)y) \, dt - \int_{0}^{1} \psi_{h,(x,y)}(t) \, dt \\
- \int_{0}^{1} p(t) \, dt \left[ \frac{1}{y-x} \int_{x}^{y} h(t) \, dt - \int_{0}^{1} \psi_{h,(x,y)}(t) \, dt \right] \\
\leq \frac{1}{4} \left[ p(1) - p \left( \frac{1}{2} \right) \right] \left[ \frac{h(x) + h(y)}{2} - \frac{1}{y-x} \int_{x}^{y} h(t) \, dt \right].
\]

If we take in (4.17) \( p(t) = |t - \frac{1}{2}|, t \in [0, 1] \), then we obtain the inequality

\[
0 \leq \int_{0}^{1} h(tx + (1-t)y) \left| t - \frac{1}{2} \right| \, dt - \int_{0}^{1} \psi_{h,(x,y)}(t) \left| t - \frac{1}{2} \right| \, dt \\
- \frac{1}{4} \left[ \frac{1}{y-x} \int_{x}^{y} h(t) \, dt - \int_{0}^{1} \psi_{h,(x,y)}(t) \, dt \right] \\
\leq \frac{1}{8} \left[ \frac{h(x) + h(y)}{2} - \frac{1}{y-x} \int_{x}^{y} h(t) \, dt \right].
\]

**References**

INTEGRAL INEQUALITIES FOR SCHUR CONVEX FUNCTIONS


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