INTEGRAL INEQUALITIES FOR SCHUR CONVEX FUNCTIONS ON SYMMETRIC AND CONVEX SETS IN LINEAR SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper, we establish some integral inequalities for Schur convex functions defined on symmetric and convex sets from a Cartesion product of linear spaces. Some applications related to the Hermite-Hadamard inequality for convex functions defined on real intervals are also provided.

1. Introduction

For any $x = (x_1, ..., x_n) \in \mathbb{R}^n$, let $x_{[1]} \geq ... \geq x_{[n]}$ denote the components of x in decreasing order, and let $x_{\downarrow} = \left(x_{[1]}, ..., x_{[n]}\right)$ denote the decreasing rearrangement of x. For x, $y \in \mathbb{R}^n$, $x \prec y$ if, by definition,

$$\begin{cases} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \; k = 1, ..., n-1; \\ \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}. \end{cases}$$

When $x \prec y$, x is said to be *majorized* by y (y *majorizes* x). This notation and terminology was introduced by Hardy, Littlewood and Pólya in 1934.

Functions that preserve the ordering of majorization are said to be Schur-convex. Perhaps "Schur-increasing" would be more appropriate, but the term "Schur-convex" is by now well entrenched in the literature, as mentioned in [8, p.80].

A real-valued function ϕ defined on a set $\mathcal{A} \subset \mathbb{R}^n$ is said to be *Schur-convex* on $\mathcal{A}$ if

$$x \prec y \text{ on } \mathcal{A} \Rightarrow \phi(x) \leq \phi(y). \tag{1.1}$$

If, in addition, $\phi(x) < \phi(y)$ whenever $x \prec y$ but x is not a permutation of y, then ϕ is said to be *strictly Schur-convex* on $\mathcal{A}$. If $\mathcal{A} = \mathbb{R}^n$, then ϕ is simply said to be Schur-convex or strictly Schur-convex.

For fundamental properties of Schur convexity see the monograph [8] and the references therein. For some recent results, see [3]-[5] and [9]-[11].

The following result is known in the literature as *Schur-Ostrowski theorem* [8, p. 84]:

Theorem 1. *Let $I \subset \mathbb{R}$ be an open interval and let $\phi : I^n \to \mathbb{R}$ be continuously differentiable. Necessary and sufficient conditions for ϕ to be Schur-convex on I^n are*

$$\phi \text{ is symmetric on } I^n, \tag{1.2}$$

1991 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Schur convex functions, Integral inequalities, Hermite-Hadamard inequality.

RGMIA Res. Rep. Coll. 22 (2019), Art. 74, 17 pp. Received 20/07/19

and for all $i \neq j$*, with* $i,\, j \in \{1, ..., n\}$,

$$(z_i - z_j)\left[\frac{\partial\phi(z)}{\partial x_i} - \frac{\partial\phi(z)}{\partial x_j}\right] \geq 0 \text{ for all } z \in I^n, \tag{1.3}$$

where $\frac{\partial\phi}{\partial x_k}$ *denotes the partial derivative of* ϕ *with respect to its* k*-th argument.*

Let $\mathcal{A} \subset \mathbb{R}^n$ be a set with the following properties:

(i) $\mathcal{A}$ is *symmetric* in the sense that $x \in \mathcal{A} \Rightarrow x\Pi \in \mathcal{A}$ for all permutations Π of the coordinates.

(ii) $\mathcal{A}$ is convex and has a nonempty interior.

We have the following result, [8, p. 85].

Theorem 2. *If* ϕ *is continuously differentiable on the interior of* $\mathcal{A}$ *and continuous on* $\mathcal{A}$*, then necessary and sufficient conditions for* ϕ *to be Schur-convex on* $\mathcal{A}$ *are*

$$\phi \text{ is symmetric on } \mathcal{A} \tag{1.4}$$

and

$$(z_1 - z_2)\left[\frac{\partial\phi(z)}{\partial x_1} - \frac{\partial\phi(z)}{\partial x_2}\right] \geq 0 \text{ for all } z \in \mathcal{A}. \tag{1.5}$$

Another interesting characterization of Schur convex functions ϕ on $\mathcal{A}$ was obtained by C. Stępniak in [12]:

Theorem 3. *Let* ϕ *be any function defined on a symmetric convex set* $\mathcal{A}$ *in* $\mathbb{R}^n$*. Then the function* ϕ *is Schur convex on* $\mathcal{A}$ *if and only if*

$$\phi\left(x_1, ..., x_i, ..., x_j, ..., x_n\right) = \phi\left(x_1, ..., x_j, ..., x_i, ..., x_n\right) \tag{1.6}$$

for all $(x_1, ..., x_n) \in \mathcal{A}$ *and* $1 \leq i < j \leq n$ *and*

$$\phi\left(\lambda x_1 + (1-\lambda)\, x_2, \lambda x_2 + (1-\lambda)\, x_1, x_3, ..., x_n\right) \leq \phi\left(x_1, ..., x_n\right) \tag{1.7}$$

for all $(x_1, ..., x_n) \in \mathcal{A}$ *and for all* $\lambda \in (0, 1)$,

It is well known that any symmetric convex function defined on a symmetric convex set $\mathcal{A}$ is Schur convex, [8, p. 97]. If the function $\phi : \mathcal{A} \to \mathbb{R}$ is symmetric and quasi-convex, namely

$$\phi\left(\alpha u + (1-\alpha)\, v\right) \leq \max\left\{\phi\left(u\right), \phi\left(v\right)\right\}$$

for all $\alpha \in [0, 1]$ and $u,\, v \in \mathcal{A}$, a symmetric convex set, then ϕ is Schur convex on $\mathcal{A}$ [8, p. 98].

Motivated by the above results, in this paper we establish some integral inequalities for Schur convex functions defined on symmetric and convex sets from a Cartesian product of linear spaces. Some applications related to the Hermite-Hadamard inequality for convex functions defined on real intervals are also provided.

2. Main Results

Let X be a linear space and $G \subset X^2 := X \times X$ a convex set. We say that G is *symmetric* if $(x, y) \in G$ implies that $(y, x) \in G$. If $D \subset X$ is a convex subset of X, then the Cartesian product $G := D^2 := D \times D$ is convex and symmetric in X^2.

Motivated by the characterization result of Stępniak above, we say that a function $f : G \to \mathbb{R}$ will be called *Schur convex* on the convex and symmetric set $G \subset X^2$ if

$$f\left(t\left(x, y\right) + (1-t)\left(y, x\right)\right) \leq f\left(x, y\right) \tag{2.1}$$

for all $(x,y) \in G$ and for all $t \in [0,1]$.

If $G = D^2$ then we recapture the general concept of Schur convexity introduced by Burai and Makó in 2016, [1].

We say that the function $f : G \to \mathbb{R}$ is symmetric on G if $f(x,y) = f(y,x)$ for all $(x,y) \in G$. If the function f is symmetric on G and the inequality holds for a given $t \in (0,1)$ and for all $(x,y) \in G$, then we say that f is *t-Schur convex* on G.

The following fact follows from the definition of Schur convex functions:

Proposition 1. *If $f : G \to \mathbb{R}$ is Schur convex on the convex and symmetric set $G \subset X^2$, then f is symmetric on G.*

Proof. If $(x,y) \in G$, then by (2.1) we get for $t = 0$ that $f(y,x) \leq f(x,y)$. If we replace x with y then we also get $f(x,y) \leq f(y,x)$ which shows that $f(x,y) = f(y,x)$ for all $(x,y) \in G$. □

For $(x,y) \in G$, as in [1], let us define the following auxiliary function $\varphi_{(x,y)} : [0,1] \to R$ by

$$\varphi_{f,(x,y)}(t) = f(t(x,y) + (1-t)(y,x)) = f(tx + (1-t)y, ty + (1-t)x). \tag{2.2}$$

The properties of this function are as follows:

Lemma 1. *Let $G \subset X^2$ be a convex and symmetric set and $f : G \to \mathbb{R}$ a symmetric function on G. Then f is Schur convex on G if and only if for all arbitrarily fixed $(x,y) \in G$ the function $\varphi_{f,(x,y)}$ is monotone decreasing on $[0,1/2)$, monotone increasing on $(1/2,1]$, and $\varphi_{f,(x,y)}$ has a global minimum at $1/2$.*

Proof. We give a similar prove to the one from [1].

Assume that f is Schur convex on G. Then for all $(u,v) \in G$ and $t \in [0,1]$ we have

$$f(t(u,v) + (1-t)(v,u)) \leq f(u,v). \tag{2.3}$$

Let $(x,y) \in G$ and for $0 \leq r < s < \frac{1}{2}$ and put $u = rx + (1-r)y$, $v = ry + (1-r)x$ and $t = \frac{s-r}{1-2r}$. Then $(u,v) = r(x,y) + (1-r)(y,x) \in G$ since G is symmetric and convex. By (2.3) we have

$$\begin{aligned} \varphi_{f,(x,y)}(r) &= f(r(x,y) + (1-r)(y,x)) = f(u,v) \\ &\geq f\left(\frac{s-r}{1-2r}(u,v) + \left(1 - \frac{s-r}{1-2r}\right)(v,u)\right) =: B. \end{aligned} \tag{2.4}$$

Observe that

$$\begin{aligned}
&\frac{s-r}{1-2r}(u,v)+\left(1-\frac{s-r}{1-2r}\right)(v,u)\\
&=\frac{s-r}{1-2r}\left[r(x,y)+(1-r)(y,x)\right]\\
&+\left(\frac{1-r-s}{1-2r}\right)\left[r(y,x)+(1-r)(x,y)\right]\\
&=\left[\left(\frac{s-r}{1-2r}\right)r+\left(\frac{1-r-s}{1-2r}\right)(1-r)\right](x,y)\\
&+\left[\frac{s-r}{1-2r}(1-r)+\left(\frac{1-r-s}{1-2r}\right)r\right](y,x)\\
&=\left(\frac{1-s-2r+2rs}{1-2r}\right)(x,y)+\left(\frac{s-2rs}{1-2r}\right)(y,x)\\
&=(1-s)(x,y)+s(y,x).
\end{aligned}$$

Then

$$B=f((1-s)(x,y)+s(y,x))=\varphi_{f,(x,y)}(s)$$

and by (2.4) we get that $\varphi_{f,(x,y)}(r)\geq\varphi_{f,(x,y)}(s)$ for $0\leq r<s<\frac{1}{2}$, which shows that the function $\varphi_{f,(x,y)}$ is monotone decreasing on $[0,1/2)$.

Observe that, by the symmetry of f on G, we have

$$\begin{aligned}
\varphi_{f,(x,y)}(1-t)&=f((1-t)(x,y)+t(y,x))\\
&=f((1-t)x+ty,(1-t)y+tx)\\
&=f((1-t)y+tx,(1-t)x+ty)\\
&=f(t(x,y)+(1-t)(y,x))=\varphi_{f,(x,y)}(t)
\end{aligned}$$

for all $t\in[0,1]$.

This shows that the function $\varphi_{f,(x,y)}$ is also monotone increasing on $(1/2,1]$.

From (2.3) we get for $t=\frac{1}{2}$ that

$$f\left(\frac{u+v}{2},\frac{u+v}{2}\right)\leq f(u,v) \tag{2.5}$$

for all $(u,v)\in G$. If $(x,y)\in G$ and we take $u=tx+(1-t)y$, $v=ty+(1-t)x$, $t\in[0,1]$ then $(u,v)=t(x,y)+(1-t)(y,x)\in G$, $\frac{u+v}{2}=\frac{x+y}{2}$ and by (2.5) we get $\varphi_{f,(x,y)}(1/2)\leq\varphi_{f,(x,y)}(t)$ for all $t\in[0,1]$, showing that $\varphi_{f,(x,y)}$ has a global minimum at $1/2$.

Now, for fixed $(x,y)\in G$, assume that the function $\varphi_{f,(x,y)}$ is monotone decreasing on $[0,1/2)$, monotone increasing on $(1/2,1]$, and has a global minimum at $1/2$.

Then for $t\in[0,1/2)$ we have

$$f(t(x,y)+(1-t)(y,x))=\varphi_{f,(x,y)}(t)\leq\varphi_{f,(x,y)}(0)=f(y,x)=f(x,y)$$

and for $t\in(1/2,1]$ we have

$$f(t(x,y)+(1-t)(y,x))=\varphi_{f,(x,y)}(t)\leq\varphi_{f,(x,y)}(1)=f(x,y).$$

Therefore, for all $t\in[0,1]$ we have $\varphi_{f,(x,y)}(t)\leq f(x,y)$, which shows that f is Schur convex on G. $\square$

We have the following weighted integral inequality:

Theorem 4. *Assume that the function $f : G \to \mathbb{R}$ is Schur convex on the convex and symmetric set $G \subset X^2$. Then for any Lebesgue integrable function $p : [0,1] \to [0,\infty)$ we have*

$$f\left(\frac{x+y}{2},\frac{x+y}{2}\right)\int_0^1 p(t)\,dt \le \int_0^1 f(t(x,y)+(1-t)(y,x))\,p(t)\,dt \le f(x,y)\int_0^1 p(t)\,dt \tag{2.6}$$

for all $(x,y) \in G$.

In particular, we have

$$f\left(\frac{x+y}{2},\frac{x+y}{2}\right) \le \int_0^1 f(t(x,y)+(1-t)(y,x))\,dt \le f(x,y) \tag{2.7}$$

for all $(x,y) \in G$.

Proof. Using Lemma 1 we have

$$f\left(\frac{x+y}{2},\frac{x+y}{2}\right) \le f(t(x,y)+(1-t)(y,x)) \le f(x,y)$$

for all $(x,y) \in G$ and $t \in [0,1]$.

If we multiply this inequality by $p(t) \ge 0$ and integrate on $[0,1]$ we deduce the desired result (2.6). □

If some monotonicity information is available for the function p we also have:

Theorem 5. *Assume that the function $f : G \to \mathbb{R}$ is Schur convex on the convex and symmetric set $G \subset X^2$. If $p : [0,1] \to \mathbb{R}$ is symmetric towards $1/2$, namely $p(1-t) = p(t)$ for all $t \in [0,1]$ and monotonic decreasing (increasing) on $[0,1/2]$, then*

$$\int_0^1 f(t(x,y)+(1-t)(y,x))\,p(t)\,dt \ge (\le) \int_0^1 p(t)\,dt \int_0^1 f(t(x,y)+(1-t)(y,x))\,dt. \tag{2.8}$$

Proof. Let $(x,y) \in G$. Since the functions $\varphi_{f,(x,y)}$ and p are symmetric on $[0,1]$, then

$$\int_0^1 f(t(x,y)+(1-t)(y,x))\,p(t)\,dt = 2\int_0^{1/2} f(t(x,y)+(1-t)(y,x))\,p(t)\,dt.$$

Assume that the functions $\varphi_{f,(x,y)}$ and p are both decreasing on $[0,1/2]$, then by Čebyšev's inequality for synchronous functions h, $g : [a,b] \to \mathbb{R}$

$$\frac{1}{b-a}\int_a^b h(t)\,g(t)\,dt \ge \frac{1}{b-a}\int_a^b h(t)\,dt \frac{1}{b-a}\int_a^b g(t)\,dt,$$

we have

$$2\int_0^{1/2} f(t(x,y)+(1-t)(y,x))\,p(t)\,dt \ge 2\int_0^{1/2} f(t(x,y)+(1-t)(y,x))\,dt \cdot 2\int_0^{1/2} p(t)\,dt \tag{2.9}$$

and since, by symmetry,

$$2\int_0^{1/2} f\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)dt=\int_0^1 f\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)dt$$

and

$$2\int_0^{1/2} p\left(t\right)dt=\int_0^1 p\left(t\right)dt$$

hence by (2.9) we get the desired result (2.8). □

The following Čebyšev's type inequality holds for two Schur convex functions:

Corollary 1. *Assume that the functions f, $g : G \to \mathbb{R}$ are Schur convex on the convex and symmetric set $G \subset X^2$. Then we have*

$$\begin{aligned}(2.10)\qquad &\int_0^1 f\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)g\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)dt\\ &\geq \int_0^1 g\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)dt\int_0^1 f\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)dt\end{aligned}$$

for all $(x,y)\in G$.

If one of the functions is Schur convex and the other Schur concave, then the sign of inequality reverses in (2.10).

We can prove the following refinement of (2.6):

Corollary 2. *Assume that the function $f : G \to \mathbb{R}$ is Schur convex on the convex and symmetric set $G \subset X^2$ and $p : [0,1] \to [0,\infty)$ is symmetric towards $1/2$ and positive.*

(i) *If p is decreasing on $[0,1/2]$, then*

$$\begin{aligned}(2.11)\qquad f\left(\frac{x+y}{2},\frac{x+y}{2}\right)&\leq \int_0^1 f\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)dt\\ &\leq \frac{1}{\int_0^1 p\left(t\right)dt}\int_0^1 f\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)p\left(t\right)dt\\ &\leq f\left(x,y\right)\end{aligned}$$

for all $(x,y)\in G$.

(ii) *If p is increasing on $[0,1/2]$, then*

$$\begin{aligned}(2.12)\qquad f\left(\frac{x+y}{2},\frac{x+y}{2}\right)&\leq \frac{1}{\int_0^1 p\left(t\right)dt}\int_0^1 f\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)p\left(t\right)dt\\ &\leq \int_0^1 f\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)dt\\ &\leq f\left(x,y\right)\end{aligned}$$

for all $(x,y)\in G$.

Proof. (i). From (2.8) we get

$$\frac{1}{\int_0^1 p\left(t\right)dt}\int_0^1 f\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)p\left(t\right)dt\geq\int_0^1 f\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)dt$$

and by (2.6) and (2.7) we get the desired result (2.11).

(ii). The proof goes in a similar way. □

Remark 1. *If we consider the weight $p(t) = \left|t - \frac{1}{2}\right|$, then $\int_0^1 p(t)\,dt = \frac{1}{4}$ and by (2.11) we get*

$$f\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \le \int_0^1 f(t(x,y) + (1-t)(y,x))\,dt \le 4\int_0^1 f(t(x,y) + (1-t)(y,x))\left|t - \frac{1}{2}\right| dt \le f(x,y) \tag{2.13}$$

for any function $f : G \to \mathbb{R}$ that is Schur convex on the convex and symmetric set $G \subset X^2$ and for all $(x,y) \in G$.

If we consider the weight $p(t) = t(1-t)$, then $\int_0^1 p(t)\,dt = \frac{1}{6}$ and by (2.12) we get

$$f\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \le \int_0^1 f(t(x,y) + (1-t)(y,x))\,dt \le 6\int_0^1 f(t(x,y) + (1-t)(y,x))\,t(1-t)\,dt \le f(x,y) \tag{2.14}$$

for any function $f : G \to \mathbb{R}$ that is Schur convex on the convex and symmetric set $G \subset X^2$ and for all $(x,y) \in G$.

We also have the following inequality for two functions:

Corollary 3. *Assume that the functions f, $g : G \to \mathbb{R}$ are Schur convex on the convex and symmetric set $G \subset X^2$ and g is nonnegative, then*

$$\begin{aligned} & f\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \\ & \le \int_0^1 f(t(x,y) + (1-t)(y,x))\,dt \\ & \le \frac{1}{\int_0^1 g(t(x,y) + (1-t)(y,x))\,dt} \\ & \times \int_0^1 f(t(x,y) + (1-t)(y,x))\,g(t(x,y) + (1-t)(y,x))\,dt \\ & \le f(x,y) \end{aligned} \tag{2.15}$$

for all $(x,y) \in G$.

If g is Schur concave and nonnegative on G, then

$$
\begin{aligned}
(2.16)\qquad & f\left(\frac{x+y}{2},\frac{x+y}{2}\right) \\
&\leq \frac{1}{\int_0^1 g\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)dt} \\
&\times \int_0^1 f\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right) g\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)dt \\
&\leq \int_0^1 f\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)dt \\
&\leq f\left(x,y\right)
\end{aligned}
$$

for all $(x,y)\in G$.

Recall the famous *Grüss' inequality* that provides an upper bound for the distance between the integral mean of the product and the product of integral means, more precisely

$$
\begin{aligned}
(2.17)\qquad & \left|\frac{1}{b-a}\int_a^b h\left(t\right)k\left(t\right)dt-\frac{1}{b-a}\int_a^b h\left(t\right)dt\frac{1}{b-a}\int_a^b k\left(t\right)dt\right| \\
&\leq \frac{1}{4}\left(M-m\right)\left(N-n\right)
\end{aligned}
$$

provided the functions h,k are measurable on $[a,b]$ and $-\infty<m\leq h\left(t\right)\leq M<\infty$, $-\infty<n\leq k\left(t\right)\leq N<\infty$, for almost every $t\in[a,b]$. The constant $\frac{1}{4}$ is best possible in (2.17).

Theorem 6. *Assume that the function $f:G\to\mathbb{R}$ is Schur convex on the convex and symmetric set $G\subset X^2$. If $p:[0,1]\to\mathbb{R}$ is symmetric towards $1/2$, namely $p\left(1-t\right)=p\left(t\right)$ for all $t\in[0,1]$ and monotonic decreasing on $[0,1/2]$ then*

$$
\begin{aligned}
(2.18)\qquad 0 &\leq \int_0^1 f\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)p\left(t\right)dt \\
&- \int_0^1 p\left(t\right)dt\int_0^1 f\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)dt \\
&\leq \frac{1}{4}\left[p\left(0\right)-p\left(\frac{1}{2}\right)\right]\left[f\left(x,y\right)-f\left(\frac{x+y}{2},\frac{x+y}{2}\right)\right]
\end{aligned}
$$

for all $(x,y)\in G$.

If p is monotonic increasing on $[0,1/2]$, then

$$
\begin{aligned}
(2.19)\qquad 0 &\leq \int_0^1 p\left(t\right)dt\int_0^1 f\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)dt \\
&- \int_0^1 f\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)p\left(t\right)dt \\
&\leq \frac{1}{4}\left[p\left(\frac{1}{2}\right)-p\left(0\right)\right]\left[f\left(x,y\right)-f\left(\frac{x+y}{2},\frac{x+y}{2}\right)\right]
\end{aligned}
$$

for all $(x,y)\in G$.

The proof follows by Gruss' inequality (2.17) written for $h\left(t\right)=p\left(t\right)$ and $k\left(t\right)=f\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)$, $t\in[0,1]$ and $(x,y)\in G$.

Corollary 4. *Assume that both functions $f,\ g : G \to \mathbb{R}$ are Schur convex on the convex and symmetric set $G \subset X^2$. Then we have*

$$0 \le \int_0^1 f\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right) g\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right) dt \\ - \int_0^1 g\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right) dt \int_0^1 f\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right) dt \\ \le \frac{1}{4}\left[g\left(x,y\right) - g\left(\frac{x+y}{2},\frac{x+y}{2}\right)\right]\left[f\left(x,y\right) - f\left(\frac{x+y}{2},\frac{x+y}{2}\right)\right]. \tag{2.20}$$

If f is Schur convex and g is Schur concave, then

$$0 \le \int_0^1 g\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right) dt \int_0^1 f\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right) dt \\ - \int_0^1 f\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right) g\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right) dt \\ \le \frac{1}{4}\left[g\left(\frac{x+y}{2},\frac{x+y}{2}\right) - g\left(x,y\right)\right]\left[f\left(x,y\right) - f\left(\frac{x+y}{2},\frac{x+y}{2}\right)\right]. \tag{2.21}$$

3. Examples for Functions of Two Real Variables

We assume in this section that G is a convex and symmetric subset of the two dimensional space $\mathbb{R}^2$ and $f : G \to \mathbb{R}$ is Schur convex on G. If $(a,b) \in G$ with $a < b$ and we put $u = (1-t)\,a + tb$, then $(1-t)\,b + ta = b + a - tb - (1-t)\,a = b + a - u$. We also assume that $w : [a,b] \to [0,\infty)$ is Lebesgue integrable on $[a,b]$ and symmetric on this interval, namely $w(b+a-u) = w(u)$ for all $u \in [a,b]$. Since $du = (b-a)\,dt$ then by taking $p(t) = w((1-t)\,a + tb)$, $t \in [0,1]$ we have by Theorem 4 that

$$f\left(\frac{a+b}{2},\frac{a+b}{2}\right) \le \frac{1}{\int_a^b w(u)\,du}\int_a^b f(u,a+b-u)\,w(u)\,du \le f(a,b). \tag{3.1}$$

In particular, we have

$$f\left(\frac{a+b}{2},\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_0^1 f(u,a+b-u)\,du \le f(a,b). \tag{3.2}$$

If we take $w(u) = \left|u - \frac{a+b}{2}\right|$, $u \in [a,b]$ in (3.1), then we get

$$f\left(\frac{a+b}{2},\frac{a+b}{2}\right) \le \frac{4}{(b-a)^2}\int_a^b f(u,a+b-u)\left|u - \frac{a+b}{2}\right| du \le f(a,b) \tag{3.3}$$

while for $w(u) = (u-a)(b-u)$, $u \in [a,b]$ we get

$$f\left(\frac{a+b}{2},\frac{a+b}{2}\right) \le \frac{6}{(b-a)^3}\int_a^b f(u,a+b-u)(u-a)(b-u)\,du \le f(a,b). \tag{3.4}$$

If we have two Schur convex functions $f,\ g : G \to \mathbb{R}$, then

$$\int_a^b f(u,a+b-u)\,g(u,a+b-u)\,dt \\ \ge \int_a^b f(u,a+b-u)\,du \int_a^b g(u,a+b-u)\,du. \tag{3.5}$$

If one function is Schur convex and the other is Schur concave, then the sign of inequality in (3.5) is reversed.

By utilising Corollary 2 we can improve the inequality (3.1) as follows:

Proposition 2. *Assume that the function $f : G \to \mathbb{R}$ is Schur convex on the convex and symmetric set $G \subset \mathbb{R}^2$, $(a, b) \in G$ with $a < b$ and $w : [a, b] \to [0, \infty)$ is Lebesgue integrable on $[a, b]$ and symmetric on $[a, b]$.*

(i) *If w is decreasing on $\left[a, \frac{a+b}{2}\right]$, then*

$$
\begin{aligned}
f\left(\frac{a+b}{2}, \frac{a+b}{2}\right) &\le \frac{1}{b-a}\int_a^b f(u, a+b-u)\,du \\
&\le \frac{1}{\int_a^b w(u)\,du}\int_a^b f(u, a+b-u)\,w(u)\,du \\
&\le f(a, b).
\end{aligned} \tag{3.6}
$$

(ii) *If w is increasing on $\left[a, \frac{a+b}{2}\right]$, then*

$$
\begin{aligned}
f\left(\frac{a+b}{2}, \frac{a+b}{2}\right) &\le \frac{1}{\int_a^b w(u)\,du}\int_a^b f(u, a+b-u)\,w(u)\,du \\
&\le \frac{1}{b-a}\int_a^b f(u, a+b-u)\,du \\
&\le f(a, b).
\end{aligned} \tag{3.7}
$$

If we take $w(u) = \left|u - \frac{a+b}{2}\right|$, $u \in [a, b]$ in (3.6), then we get

$$
\begin{aligned}
f\left(\frac{a+b}{2}, \frac{a+b}{2}\right) &\le \frac{1}{b-a}\int_a^b f(u, a+b-u)\,du \\
&\le \frac{4}{(b-a)^2}\int_a^b f(u, a+b-u)\left|u - \frac{a+b}{2}\right| du \\
&\le f(a, b).
\end{aligned} \tag{3.8}
$$

Also, if we choose $w(u) = (u-a)(b-u)$, $u \in [a, b]$ in (3.7), then we obtain

$$
\begin{aligned}
f\left(\frac{a+b}{2}, \frac{a+b}{2}\right) &\le \frac{6}{(b-a)^3}\int_a^b f(u, a+b-u)(u-a)(b-u)\,du \\
&\le \frac{1}{b-a}\int_a^b f(u, a+b-u)\,du \\
&\le f(a, b).
\end{aligned} \tag{3.9}
$$

From Theorem 6 we also have:

Proposition 3. *Assume that the function $f : G \to \mathbb{R}$ is Schur convex on the convex and symmetric set $G \subset \mathbb{R}^2$, $(a, b) \in G$ with $a < b$ and $w : [a, b] \to [0, \infty)$ is Lebesgue integrable on $[a, b]$ and symmetric on $[a, b]$.*

(i) *If w is decreasing on $\left[a, \frac{a+b}{2}\right]$, then*

$$
\begin{aligned}
0 &\le \int_a^b f(u, a+b-u)\,w(u)\,du - \int_a^b w(u)\,du \int_a^b f(u, a+b-u)\,du \\
&\le \frac{1}{4}\left[w(b) - w\left(\frac{a+b}{2}\right)\right]\left[f(a, b) - f\left(\frac{a+b}{2}, \frac{a+b}{2}\right)\right]
\end{aligned} \tag{3.10}
$$

(ii) *If w is increasing on* $\left[a, \frac{a+b}{2}\right]$, *then*

$$0 \leq \int_a^b w(u)\,du \int_a^b f(u, a+b-u)\,du - \int_a^b f(u, a+b-u)\,w(u)\,du \tag{3.11}$$
$$\leq \frac{1}{4}\left[w\left(\frac{a+b}{2}\right) - w(b)\right]\left[f(a,b) - f\left(\frac{a+b}{2}, \frac{a+b}{2}\right)\right].$$

From this proposition we derive the following reverse inequalities of (3.5).

Corollary 5. *Assume that the function f, $g : G \to \mathbb{R}$ are Schur convex on the convex and symmetric set $G \subset \mathbb{R}^2$, $(a,b) \in G$ with $a < b$. Then*

$$0 \leq \int_a^b f(u, a+b-u)\,g(u, a+b-u)\,du \tag{3.12}$$
$$- \int_a^b g(u, a+b-u)\,du \int_a^b f(u, a+b-u)\,du$$
$$\leq \frac{1}{4}\left[g(a,b) - g\left(\frac{a+b}{2}, \frac{a+b}{2}\right)\right]\left[f(a,b) - f\left(\frac{a+b}{2}, \frac{a+b}{2}\right)\right].$$

If $f : G \to \mathbb{R}$ is Schur convex and $f : G \to \mathbb{R}$ is Schur concave, then

$$0 \leq \int_a^b g(u, a+b-u)\,du \int_a^b f(u, a+b-u)\,du \tag{3.13}$$
$$- \int_a^b f(u, a+b-u)\,g(u, a+b-u)\,du$$
$$\leq \frac{1}{4}\left[g\left(\frac{a+b}{2}, \frac{a+b}{2}\right) - g(a,b)\right]\left[f(a,b) - f\left(\frac{a+b}{2}, \frac{a+b}{2}\right)\right].$$

4. Some Applications for Hermite-Hadamard Inequality

We recall the celebrated *Hermite-Hadamard inequality* for continuous convex functions h defined on a real interval I, which state that

$$h\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y h(t)\,dt \leq \frac{h(x)+h(y)}{2} \tag{4.1}$$

for all $x \neq y$, x, $y \in I$. For a monograph devoted to this inequality, see [6]. Many related results are also presented in the survey paper [4].

The following result concerning the Schur convexity of the integral mean was obtained by Elezović and Pečarić in [7]:

Theorem 7. *Let h be a continuous function on I. Then*

$$H(x,y) := \begin{cases} \frac{1}{y-x}\int_x^y h(t)\,dt, \text{ for } x \neq y,\ x,\ y \in I; \\ \\ h(x),\ \text{ for } y = x,\ x \in I, \end{cases}$$

is Schur convex (concave) on I^2 if and only if h is convex (concave) on I.

Let h be a continuous function on I. We have for $t\in[0,1]$, $t\neq 1/2$ that

$$\begin{aligned}
&H\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)\\
&=H\left(tx+\left(1-t\right)y,ty+\left(1-t\right)x\right)\\
&=\begin{cases}\frac{1}{tx+(1-t)y-ty-(1-t)x}\int_{ty+(1-t)x}^{tx+(1-t)y}h\left(s\right)ds, \text{ for } x\neq y,\ x,\ y\in I;\\ \\ h\left(x\right), \text{ for } y=x,\ x\in I,\end{cases}\\
&=\begin{cases}\frac{1}{(1-2t)(y-x)}\int_{ty+(1-t)x}^{tx+(1-t)y}h\left(s\right)ds, \text{ for } x\neq y,\ x,\ y\in I;\\ \\ h\left(x\right), \text{ for } y=x,\ x\in I.\end{cases}
\end{aligned}$$

For $t=1/2$ we have

$$H\left(\frac{x+y}{2},\frac{x+y}{2}\right)=h\left(\frac{x+y}{2}\right)$$

for x, $y\in I$.

Corollary 6. *Assume that h is continuous convex on I. Then we have the following refinement of the first Hermite-Hadamard inequality*

$$\begin{aligned}
h\left(\frac{x+y}{2}\right)&\leq\frac{1}{\left(1-2t\right)\left(y-x\right)}\int_{ty+(1-t)x}^{tx+(1-t)y}h\left(s\right)ds\\
&\leq\frac{1}{y-x}\int_{x}^{y}h\left(t\right)dt,
\end{aligned}\tag{4.2}$$

for all $x\neq y$, x, $y\in I$ and $t\in[0,1]$, $t\neq 1/2$.

Proof. Since h is continuous convex on I, hence by Theorem 7 we get that H is Schur convex on I^2. By utilising Lemma 1 we conclude that

$$H\left(\frac{x+y}{2},\frac{x+y}{2}\right)\leq H\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)\leq H\left(x,y\right)$$

for $t\in[0,1]$, and the inequality (4.2) is obtained. □

Assume that h is continuous on I. For $x\neq y$, x, $y\in I$, we consider the function $\psi_{h,(x,y)}:[0,1]\to\mathbb{R}$ defined by

$$\psi_{h,(x,y)}\left(t\right):=\begin{cases}\frac{1}{(1-2t)(y-x)}\int_{ty+(1-t)x}^{tx+(1-t)y}h\left(s\right)ds \text{ for } t\neq 1/2;\\ \\ h\left(\frac{x+y}{2}\right) \text{ for } t=1/2.\end{cases}$$

Remark 2. *Assume that h is continuous convex on I. For any Lebesgue integrable function $p:[0,1]\to[0,\infty)$ with $\int_0^1 p\left(t\right)dt>0$ we have from Theorem 4 that*

$$\begin{aligned}
h\left(\frac{x+y}{2}\right)&\leq\frac{1}{\left(y-x\right)\int_0^1 p\left(t\right)dt}\int_0^1 p\left(t\right)\psi_{h,(x,y)}\left(t\right)dt\\
&\leq\frac{1}{y-x}\int_x^y h\left(t\right)dt,
\end{aligned}\tag{4.3}$$

and, in particular,

$$h\left(\frac{x+y}{2}\right)\leq\frac{1}{y-x}\int_0^1\psi_{h,(x,y)}\left(t\right)dt\leq\frac{1}{y-x}\int_x^y h\left(t\right)dt,\tag{4.4}$$

for all $x \neq y$, $x, y \in I$.

We also have:

Corollary 7. *Assume that* h *is continuous convex on* I, *then the function* $\psi_{h,(x,y)}$ *is monotone decreasing on* $[0,1/2)$, *monotone increasing on* $(1/2,1]$, *and* $\psi_{h,(x,y)}$ *has a global minimum at* $1/2$.

The proof is obvious by Lemma 1.

If more assumptions are imposed on the weight p, then some better inequalities are obtained:

Corollary 8. *Assume that* h *is continuous convex on* I *and* $p : [0,1] \to [0,\infty)$ *is symmetric towards* $1/2$.

(i) *If* p *is decreasing on* $[0,1/2]$, *then*

$$h\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_0^1 \psi_{h,(x,y)}(t)\,dt \leq \frac{1}{(y-x)\int_0^1 p(t)\,dt}\int_0^1 p(t)\,\psi_{h,(x,y)}(t)\,dt \leq \frac{1}{y-x}\int_x^y h(t)\,dt \tag{4.5}$$

for all $x \neq y$, $x, y \in I$.

(ii) *If* p *is increasing on* $[0,1/2]$, *then*

$$h\left(\frac{x+y}{2}\right) \leq \frac{1}{(y-x)\int_0^1 p(t)\,dt}\int_0^1 p(t)\,\psi_{h,(x,y)}(t)\,dt \leq \frac{1}{y-x}\int_0^1 \psi_{h,(x,y)}(t)\,dt \leq \frac{1}{y-x}\int_x^y h(t)\,dt \tag{4.6}$$

for all $x \neq y$, $x, y \in I$.

Remark 3. *If we take* $p(t) = \left|t - \frac{1}{2}\right|$, $t \in [0,1]$ *in (4.5), then we get*

$$h\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_0^1 \psi_{h,(x,y)}(t)\,dt \leq \frac{4}{y-x}\int_0^1 \left|t-\frac{1}{2}\right| \psi_{h,(x,y)}(t)\,dt \leq \frac{1}{y-x}\int_x^y h(t)\,dt \tag{4.7}$$

and for $p(t) = t(1-t)$, $t \in [0,1]$ *in (4.6) we obtain*

$$h\left(\frac{x+y}{2}\right) \leq \frac{6}{y-x}\int_0^1 t(1-t)\,\psi_{h,(x,y)}(t)\,dt \leq \frac{1}{y-x}\int_0^1 \psi_{h,(x,y)}(t)\,dt \leq \frac{1}{y-x}\int_x^y h(t)\,dt. \tag{4.8}$$

Finally, we can state

Corollary 9. *Assume that* h *is continuous convex on* I *and* $p : [0,1] \to [0,\infty)$ *is symmetric towards* $1/2$. *If* p *is monotonic decreasing on* $[0,1/2]$ *then*

$$0 \leq \int_0^1 \psi_{h,(x,y)}(t)\,p(t)\,dt - \int_0^1 p(t)\,dt \int_0^1 \psi_{h,(x,y)}(t)\,dt \leq \frac{1}{4}\left[p(1) - p\left(\frac{1}{2}\right)\right]\left[\frac{1}{y-x}\int_x^y h(t)\,dt - h\left(\frac{x+y}{2}\right)\right] \tag{4.9}$$

for all $x \neq y$, $x, y \in I$.

If we take in (4.9) $p(t)=\left|t-\frac{1}{2}\right|$, $t\in[0,1]$, then we obtain the inequality

$$0\le\int_0^1\psi_{h,(x,y)}(t)\left|t-\frac{1}{2}\right|dt-\frac{1}{4}\int_0^1\psi_{h,(x,y)}(t)\,dt\le\frac{1}{8}\left[\frac{1}{y-x}\int_x^y h(t)\,dt-h\left(\frac{x+y}{2}\right)\right]\tag{4.10}$$

provided that h is continuous convex on I and $x\neq y$, x, $y\in I$.

In [2] Chu et al. obtained the following results:

Theorem 8. *Suppose $h:I\to\mathbb{R}$ is a continuous function. Function*

$$M(x,y):=\begin{cases}\frac{1}{y-x}\int_x^y h(t)\,dt-h\left(\frac{x+y}{2}\right), & (x,y)\in I^2,\ x\neq y\\ 0, & (x,y)\in I^2,\ x=y\end{cases}$$

is Schur-convex (Schur-concave) on I^2 if and only if h is convex (concave) on I. Furthermore, function

$$T(x,y):=\begin{cases}\frac{h(x)+h(y)}{2}-\frac{1}{y-x}\int_x^y h(t)\,dt, & (x,y)\in I^2,\ x\neq y\\ 0, (x,y)\in I^2,\ x=y\end{cases}$$

is Schur-convex (Schur-concave) on I^2 if and only if h is convex (concave) on I.

Observe that for $t\in[0,1]$, $t\neq 1/2$ we have

$$\begin{aligned}&T(t(x,y)+(1-t)(y,x))\\&=T(tx+(1-t)y,ty+(1-t)x)\\&=\begin{cases}\frac{h(tx+(1-t)y)+h(ty+(1-t)x)}{2}\\ -\frac{1}{(1-2t)(y-x)}\int_{ty+(1-t)x}^{tx+(1-t)y}h(s)\,ds, & (x,y)\in I^2,\ x\neq y\\ 0, & (x,y)\in I^2,\ x=y.\end{cases}\end{aligned}$$

For $t=\frac{1}{2}$ we have

$$T\left(\frac{x+y}{2},\frac{x+y}{2}\right)=0,\ (x,y)\in I^2.$$

We have:

Corollary 10. *Assume that h is continuous convex on I. Then we have the following refinement of the second Hermite-Hadamard inequality*

$$\begin{aligned}0&\le\frac{h(tx+(1-t)y)+h(ty+(1-t)x)}{2}\\&\quad-\frac{1}{(1-2t)(y-x)}\int_{ty+(1-t)x}^{tx+(1-t)y}h(s)\,ds\\&\le\frac{h(x)+h(y)}{2}-\frac{1}{y-x}\int_x^y h(t)\,dt,\end{aligned}\tag{4.11}$$

for all $x\neq y$, x, $y\in I$ and $t\in[0,1]$, $t\neq 1/2$.

Proof. Since h is continuous convex on I, hence by Theorem 8 we get that H is Schur convex on I^2. By utilising Lemma 1 we conclude that

$$T\left(\frac{x+y}{2},\frac{x+y}{2}\right) \le T\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right) \le T\left(x,y\right)$$

for $t\in[0,1]$, and the inequality (4.11) is obtained. □

With the notations above, we have for $x\neq y$, x, $y\in I$ and $t\in[0,1]$, $t\neq 1/2$ let put

$$\begin{aligned}\delta_{h,(x,y)}\left(t\right) &:= T\left(t\left(x,y\right)+\left(1-t\right)\left(y,x\right)\right)\\ &= \frac{h\left(tx+\left(1-t\right)y\right)+h\left(ty+\left(1-t\right)x\right)}{2}-\psi_{h,(x,y)}\left(t\right)\end{aligned} \tag{4.12}$$

and

$$\delta_{h,(x,y)}\left(\frac{1}{2}\right) := 0$$

From Lemma 1 we have:

Corollary 11. *Assume that h is continuous convex on I and $x\neq y$, x, $y\in I$. Then the function the function $\delta_{h,(x,y)}$ is nonnegative, monotone decreasing on $[0,1/2)$, monotone increasing on $(1/2,1]$, and $\delta_{h,(x,y)}$ has a global minimum at $1/2$.*

We also have, by utilising Theorem 4:

Corollary 12. *Assume that h is continuous convex on I and $x\neq y$, x, $y\in I$. Then for any Lebesgue integrable function $p:[0,1]\to[0,\infty)$ we have*

$$\begin{aligned}0 &\le \int_0^1 \frac{h\left(tx+\left(1-t\right)y\right)+h\left(ty+\left(1-t\right)x\right)}{2}p\left(t\right)dt\\ &\quad -\int_0^1 \psi_{h,(x,y)}\left(t\right)p\left(t\right)dt\\ &\le \left[\frac{h\left(x\right)+h\left(y\right)}{2}-\frac{1}{y-x}\int_x^y h\left(t\right)dt\right]\int_0^1 p\left(t\right)dt.\end{aligned} \tag{4.13}$$

In particular, we have the following refinement of the second Hermite-Hadamard inequality

$$\begin{aligned}0 &\le \frac{1}{y-x}\int_x^y h\left(t\right)dt-\int_0^1 \psi_{h,(x,y)}\left(t\right)dt\\ &\le \frac{h\left(x\right)+h\left(y\right)}{2}-\frac{1}{y-x}\int_x^y h\left(t\right)dt.\end{aligned} \tag{4.14}$$

If more conditions are assumed for the weight p, then we also have:

Corollary 13. *Assume that h is continuous convex on I and $x\neq y$, x, $y\in I$ and $p:[0,1]\to[0,\infty)$ is symmetric towards $1/2$ and positive.*

(i) *If p is decreasing on $[0,1/2]$, then*

$$0 \leq \frac{1}{y-x}\int_x^y h(t)\,dt - \int_0^1 \psi_{h,(x,y)}(t)\,dt \tag{4.15}$$
$$\leq \frac{1}{\int_0^1 p(t)\,dt}\int_0^1 h(tx+(1-t)y)\,p(t)\,dt$$
$$-\frac{1}{\int_0^1 p(t)\,dt}\int_0^1 \psi_{h,(x,y)}(t)\,p(t)\,dt$$
$$\leq \frac{h(x)+h(y)}{2} - \frac{1}{y-x}\int_x^y h(t)\,dt$$

for all $(x,y)\in G$.

(ii) *If p is increasing on $[0,1/2]$, then*

$$0 \leq \frac{1}{\int_0^1 p(t)\,dt}\int_0^1 h(tx+(1-t)y)\,p(t)\,dt \tag{4.16}$$
$$-\frac{1}{\int_0^1 p(t)\,dt}\int_0^1 \psi_{h,(x,y)}(t)\,p(t)\,dt$$
$$\leq \frac{1}{y-x}\int_x^y h(t)\,dt - \int_0^1 \psi_{h,(x,y)}(t)\,dt$$
$$\leq \frac{h(x)+h(y)}{2} - \frac{1}{y-x}\int_x^y h(t)\,dt.$$

The proof follows by Corollary 2.

Corollary 14. *Assume that h is continuous convex on I and $x\neq y$, x, $y\in I$ and $p:[0,1]\to[0,\infty)$ is symmetric towards $1/2$ and positive. If $p:[0,1]\to\mathbb{R}$ is symmetric towards $1/2$ and monotonic decreasing on $[0,1/2]$ then*

$$0 \leq \int_0^1 h(tx+(1-t)y)\,p(t)\,dt - \int_0^1 \psi_{h,(x,y)}(t)\,p(t)\,dt \tag{4.17}$$
$$-\int_0^1 p(t)\,dt\left[\frac{1}{y-x}\int_x^y h(t)\,dt - \int_0^1 \psi_{h,(x,y)}(t)\,dt\right]$$
$$\leq \frac{1}{4}\left[p(1)-p\left(\frac{1}{2}\right)\right]\left[\frac{h(x)+h(y)}{2} - \frac{1}{y-x}\int_x^y h(t)\,dt\right].$$

If we take in (4.17) $p(t)=\left|t-\frac{1}{2}\right|$, $t\in[0,1]$, then we obtain the inequality

$$0 \leq \int_0^1 h(tx+(1-t)y)\left|t-\frac{1}{2}\right|dt - \int_0^1 \psi_{h,(x,y)}(t)\left|t-\frac{1}{2}\right|dt \tag{4.18}$$
$$-\frac{1}{4}\left[\frac{1}{y-x}\int_x^y h(t)\,dt - \int_0^1 \psi_{h,(x,y)}(t)\,dt\right]$$
$$\leq \frac{1}{8}\left[\frac{h(x)+h(y)}{2} - \frac{1}{y-x}\int_x^y h(t)\,dt\right].$$

References

[1] P. Burai and J. Makó, On certain Schur-convex functions, *Publ. Math. Debrecen*, **89** (3) (2016), 307-319.

[2] Y. Chu, G. Wang, X. Zhang, Schur convexity and Hadamard's inequality, *Math. Inequal. Appl.* **13** (4) (2010) 725–731.

[3] V. Čuljak, A remark on Schur-convexity of the mean of a convex function. *J. Math. Inequal.* **9** (2015), No. 4, 1133–1142.

[4] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results. *Aust. J. Math. Anal. Appl.* **14** (2017), No. 1, Art. 1, 283 pp. [Online https://ajmaa.org/cgi-bin/paper.pl?string=v14n1/V14I1P1.tex].

[5] S. S. Dragomir and K. Nikodem, Functions generating (m, M, Ψ)-Schur-convex sums. *Aequationes Math.* **93** (2019), No. 1, 79–90.

[6] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, 2000. [Online https://rgmia.org/monographs/hermite_hadamard.html].

[7] N. Elezović and J. Pečarić, A note on Schur convex fuctions, *Rocky Mountain J. Math.* **30** (2000), No. 3, 853-956.

[8] A. W. Marshall, I. Olkin and B. C. Arnold, *Inequalities: Theory of Majorization and Its Applications,* Second Edition, Springer New York Dordrecht Heidelberg London, 2011.

[9] K. Nikodem, T. Rajba and S. Wąsowicz, *Functions generating strongly Schur-convex sums.* Inequalities and applications 2010, 175–182, Internat. Ser. Numer. Math., 161, Birkhäuser/Springer, Basel, 2012.

[10] J. Qi and W. Wang, Schur convex functions and the Bonnesen style isoperimetric inequalities for planar convex polygons. *J. Math. Inequal.* **12** (2018), no. 1, 23–29.

[11] H.-N. Shi and J. Zhang, Compositions involving Schur harmonically convex functions. *J. Comput. Anal. Appl.* **22** (2017), no. 5, 907–922.

[12] C. Stępniak, An effective characterization of Schur-convex functions with applications, *Journal of Convex Analysis*, **14** (2007), No. 1, 103–108.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa