GLOBAL CONVEXITY OF THE WEIGHTED INTEGRAL MEAN OF FUNCTIONS DEFINED ON CONVEX SETS IN LINEAR SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For a Lebesgue integrable function $p:[0,1]\to[0,\infty)$ we consider the function $F_p:C^2\to\mathbb{R}$ defined by

$$F_p(x,y):=\int_0^1 f((1-t)x+ty)\,p(t)\,dt,$$

where $f:C\to\mathbb{R}$ is convex on the convex subset C of a linear space X. In this paper we investigate the global convexity and Schur convexity of the function F_p on C^2 and provide some applications for norms and convex functions of a real variable defined on an interval.

1. Introduction

For any $x=(x_1,...,x_n)\in\mathbb{R}^n$, let $x_{[1]}\geq...\geq x_{[n]}$ denote the components of x in decreasing order, and let $x_{\downarrow}=\left(x_{[1]},...,x_{[n]}\right)$ denote the decreasing rearrangement of x. For x, $y\in\mathbb{R}^n$, $x\prec y$ if, by definition,

$$\begin{cases}\sum_{i=1}^{k}x_{[i]}\leq\sum_{i=1}^{k}y_{[i]},\ k=1,...,n-1;\\ \sum_{i=1}^{n}x_{[i]}=\sum_{i=1}^{n}y_{[i]}.\end{cases}$$

When $x\prec y$, x is said to be *majorized* by y (y *majorizes* x). This notation and terminology was introduced by Hardy, Littlewood and Pólya in 1934.

A real-valued function ϕ defined on a set $\mathcal{A}\subset\mathbb{R}^n$ is said to be *Schur-convex* on $\mathcal{A}$ if

$$x\prec y\text{ on }\mathcal{A}\Rightarrow\phi(x)\leq\phi(y).\tag{1.1}$$

If, in addition, $\phi(x)<\phi(y)$ whenever $x\prec y$ but x is not a permutation of y, then ϕ is said to be *strictly Schur-convex* on $\mathcal{A}$. If $\mathcal{A}=\mathbb{R}^n$, then ϕ is simply said to be Schur-convex or strictly Schur-convex.

For fundamental properties of Schur convexity see the monograph [9] and the references therein. For some recent results, see [3]-[6] and [10]-[12].

The following result is known in the literature as *Schur-Ostrowski theorem* [9, p. 84]:

Theorem 1. *Let $I\subset\mathbb{R}$ be an open interval and let $\phi:I^n\to\mathbb{R}$ be continuously differentiable. Necessary and sufficient conditions for ϕ to be Schur-convex on I^n*

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are

$$\phi \text{ is symmetric on } I^n, \tag{1.2}$$

and for all $i \neq j$, *with* $i, j \in \{1, ..., n\}$,

$$(z_i - z_j)\left[\frac{\partial \phi(z)}{\partial x_i} - \frac{\partial \phi(z)}{\partial x_j}\right] \geq 0 \text{ for all } z \in I^n, \tag{1.3}$$

where $\frac{\partial \phi}{\partial x_k}$ *denotes the partial derivative of* ϕ *with respect to its* k*-th argument.*

The above condition is not sufficiently general for all applications because the domain of ϕ may not be a Cartesian product.

Let $\mathcal{A} \subset \mathbb{R}^n$ be a set with the following properties:

(i) $\mathcal{A}$ is *symmetric* in the sense that $x \in \mathcal{A} \Rightarrow x\Pi \in \mathcal{A}$ for all permutations Π of the coordinates.

(ii) $\mathcal{A}$ is convex and has a nonempty interior.

We have the following result, [9, p. 85].

Theorem 2. *If* ϕ *is continuously differentiable on the interior of* $\mathcal{A}$ *and continuous on* $\mathcal{A}$, *then necessary and sufficient conditions for* ϕ *to be Schur-convex on* $\mathcal{A}$ *are*

$$\phi \text{ is symmetric on } \mathcal{A} \tag{1.4}$$

and

$$(z_1 - z_2)\left[\frac{\partial \phi(z)}{\partial x_1} - \frac{\partial \phi(z)}{\partial x_2}\right] \geq 0 \text{ for all } z \in \mathcal{A}. \tag{1.5}$$

Another interesting characterization of Schur convex functions ϕ on $\mathcal{A}$ was obtained by C. Stępniak in [13]:

Theorem 3. *Let* ϕ *be any function defined on a symmetric convex set* $\mathcal{A}$ *in* $\mathbb{R}^n$. *Then the function* ϕ *is Schur convex on* $\mathcal{A}$ *if and only if*

$$\phi(x_1, ..., x_i, ..., x_j, ..., x_n) = \phi(x_1, ..., x_j, ..., x_i, ..., x_n) \tag{1.6}$$

for all $(x_1, ..., x_n) \in \mathcal{A}$ *and* $1 \leq i < j \leq n$ *and*

$$\phi(\lambda x_1 + (1-\lambda) x_2, \lambda x_2 + (1-\lambda) x_1, x_3, ..., x_n) \leq \phi(x_1, ..., x_n) \tag{1.7}$$

for all $(x_1, ..., x_n) \in \mathcal{A}$ *and for all* $\lambda \in (0, 1)$,

It is well known that any symmetric convex function defined on a symmetric convex set $\mathcal{A}$ is Schur convex, [9, p. 97]. If the function $\phi : \mathcal{A} \to \mathbb{R}$ is symmetric and quasi-convex, namely

$$\phi(\alpha u + (1-\alpha) v) \leq \max\{\phi(u), \phi(v)\}$$

for all $\alpha \in [0, 1]$ and $u, v \in \mathcal{A}$, a symmetric convex set, then ϕ is Schur convex on $\mathcal{A}$ [9, p. 98].

The following result concerning the Schur convexity of the integral mean was obtained by Elezović and Pečarić in [8]:

Theorem 4. *Let* h *be a continuous function on* I. *Then*

$$H(x, y) := \begin{cases} \frac{1}{y-x}\int_x^y h(t)\, dt, \text{ for } x \neq y,\ x,\ y \in I; \\ \\ h(x), \text{ for } y = x,\ x \in I, \end{cases} \tag{1.8}$$

is Schur convex (concave) on I^2 *if and only if* h *is convex (concave) on* I.

Three year later, in 2003, Wulbert, [14], improved the above result by showing that *the integral mean H defined in (1.8) is in fact convex on I^2* if f is convex on I.

We consider the function $f : C \to \mathbb{R}$ defined on the convex subset C of the linear space X and for each $(x,y) \in C^2 := C \times C$ we introduce the auxiliary function $\varphi_{(x,y)} : [0,1] \to \mathbb{R}$ defined by

$$\varphi_{(x,y)}(t) := f((1-t)x + ty).$$

It is well known that the function f is convex on C if and only if for each $(x,y) \in C^2$ the auxiliary function $\varphi_{(x,y)}$ is convex on $[0,1]$.

By utilising the classical Hermite-Hadamard inequality for the convex function $\varphi_{(x,y)}$ on $[0,1]$ we then have

$$f\left(\frac{x+y}{2}\right) \le \int_0^1 f((1-t)x + ty)\,dt \le \frac{f(x)+f(y)}{2} \tag{1.9}$$

for all $(x,y) \in C^2$.

For a non-negative Lebesgue integrable function $p : [0,1] \to [0,\infty)$ we consider the function $F_p : C^2 \to \mathbb{R}$ defined by

$$F_p(x,y) := \int_0^1 f((1-t)x + ty)\,p(t)\,dt$$

where $f : C \to \mathbb{R}$ is convex on the convex subset C of a linear space X.

We observe that

$$F_p(x,x) := f(x)\int_0^1 p(t)\,dt \text{ for all } x \in C.$$

Motivated by the above results, in this paper we investigate the global convexity and Schur convexity of the function F_p and provide some applications for convex functions of a real variable defined on an interval.

2. Main Results

We start with the following simple fact:

Proposition 1. *If p is symmetric and Lebesgue integrable in $[0,1]$, namely $p(1-t) = p(t)$ for all $t \in [0,1]$, then $F_p(x,y) = F_p(y,x)$ for all $(x,y) \in C^2$, i.e. F_p is symmetric on C^2.*

Proof. Observe that

$$F_p(y,x) = \int_0^1 f((1-t)y + tx)\,p(t)\,dt.$$

By changing the variable $s = 1-t$, $t \in [0,1]$, then

$$\begin{aligned}\int_0^1 f((1-t)y + tx)\,p(t)\,dt &= \int_0^1 f(sy + (1-s)x)\,p(1-s)\,ds \\ &= \int_0^1 f(sy + (1-s)x)\,p(s)\,ds = F_p(x,y),\end{aligned}$$

which proves the claim. □

For $p \equiv 1$, if we put

$$F(x,y) := \int_0^1 f((1-t)x+ty)\,dt,$$

then, obviously, F is also symmetric on C^2.

Theorem 5. *Let $f : C \to \mathbb{R}$ be a convex function on C and $p : [0,1] \to [0,\infty)$ a Lebesgue integrable function on $[0,1]$, then F_p is convex on C^2. If p is symmetric on $[0,1]$, then F_p is convex and symmetric on C^2. In particular, F is convex and symmetric on C^2.*

Proof. Let $(x,y), (u,v) \in C^2$ and $\alpha \in [0,1]$. Then

$$\begin{aligned}
&F_p(\alpha(x,y)+(1-\alpha)(u,v))\\
&= F_p(\alpha x+(1-\alpha)u, \alpha y+(1-\alpha)v)\\
&= \int_0^1 f((1-t)(\alpha x+(1-\alpha)u)+t(\alpha y+(1-\alpha)v))\,p(t)\,dt\\
&= \int_0^1 f(\alpha((1-t)x+ty)+(1-\alpha)((1-t)u+tv))\,p(t)\,dt\\
&\le \int_0^1 \{\alpha f((1-t)x+ty)+(1-\alpha)f((1-t)u+tv)\}\,p(t)\,dt\\
&\text{(by the convexity of } f)\\
&= \alpha\int_0^1 f((1-t)x+ty)\,p(t)\,dt+(1-\alpha)\int_0^1 f((1-t)u+tv)\,p(t)\,dt\\
&= \alpha F_p(x,y)+(1-\alpha)F_p(u,v),
\end{aligned}$$

which proves the convexity of F_p on C^2. □

Since F_p is convex on C^2 we have

$$F_p(\alpha(x,y)+(1-\alpha)(y,x)) \le \alpha F_p(x,y)+(1-\alpha)F_p(y,x)$$

for $\alpha \in [0,1]$, namely

$$\begin{aligned}
&\int_0^1 f(\alpha((1-t)x+ty)+(1-\alpha)((1-t)y+tx))\,p(t)\,dt\\
&\le \alpha\int_0^1 f((1-t)x+ty)\,p(t)\,dt+(1-\alpha)\int_0^1 f((1-t)y+tx)\,p(t)\,dt.
\end{aligned}\tag{2.1}$$

Since

$$\begin{aligned}
&\int_0^1 f(\alpha((1-t)x+ty)+(1-\alpha)((1-t)y+tx))\,p(t)\,dt\\
&= \int_0^1 f[(\alpha(1-t)+(1-\alpha)t)x+(\alpha t+(1-\alpha)(1-t))y]\,p(t)\,dt
\end{aligned}$$

and

$$\int_0^1 f((1-t)y+tx)\,p(t)\,dt = \int_0^1 f(sy+(1-s)x)\,p(1-s)\,ds$$

hence by (2.1) we get

$$\begin{aligned}
&\int_0^1 f\left[\left(\alpha\left(1-t\right)+\left(1-\alpha\right)t\right)x+\left(\alpha t+\left(1-\alpha\right)\left(1-t\right)\right)y\right]p\left(t\right)dt \\
&\leq \int_0^1 f\left(\left(1-t\right)x+ty\right)\left[\alpha p\left(t\right)+\left(1-\alpha\right)p\left(1-t\right)\right]dt
\end{aligned} \tag{2.2}$$

for all $\alpha \in [0,1]$.

In particular, if we take $\alpha = \frac{1}{2}$ in (2.2), then we get

$$f\left(\frac{x+y}{2}\right)\int_0^1 p\left(t\right)dt \leq \int_0^1 f\left(\left(1-t\right)x+ty\right)\left[\frac{p\left(t\right)+p\left(1-t\right)}{2}\right]dt. \tag{2.3}$$

We can improve the inequality (2.3) as follows:

Corollary 1. *Let $f : C \to \mathbb{R}$ be a convex function on C and $p : [0,1] \to [0,\infty)$ a Lebesgue integrable function on $[0,1]$, then*

$$\begin{aligned}
&f\left(\frac{x+y}{2}\right)\int_0^1 p\left(t\right)dt \\
&\leq \int_0^1\left(\int_0^1 f\left[\left(\alpha\left(1-t\right)+\left(1-\alpha\right)t\right)x+\left(\alpha t+\left(1-\alpha\right)\left(1-t\right)\right)y\right]p\left(t\right)dt\right)d\alpha \\
&\leq \int_0^1 f\left(\left(1-t\right)x+ty\right)\left[\frac{p\left(t\right)+p\left(1-t\right)}{2}\right]dt \leq \frac{f\left(x\right)+f\left(y\right)}{2}\int_0^1 p\left(t\right)dt.
\end{aligned} \tag{2.4}$$

If p is symmetric on $[0,1]$, then we have the following refinement of Fejér's inequality

$$\begin{aligned}
&f\left(\frac{x+y}{2}\right)\int_0^1 p\left(t\right)dt \\
&\leq \int_0^1\left(\int_0^1 f\left[\left(\alpha\left(1-t\right)+\left(1-\alpha\right)t\right)x+\left(\alpha t+\left(1-\alpha\right)\left(1-t\right)\right)y\right]p\left(t\right)dt\right)d\alpha \\
&\leq \int_0^1 f\left(\left(1-t\right)x+ty\right)p\left(t\right)dt \leq \frac{f\left(x\right)+f\left(y\right)}{2}\int_0^1 p\left(t\right)dt.
\end{aligned} \tag{2.5}$$

Proof. If we use the Hermite-Hadamard inequality for the convex function F_p and the points (x,y) and (y,x) we also get

$$\begin{aligned}
F_p\left(\frac{(x,y)+(y,x)}{2}\right) &\leq \int_0^1 F_p\left(\alpha\left(x,y\right)+\left(1-\alpha\right)\left(y,x\right)\right)d\alpha \\
&\leq \frac{F_p\left(x,y\right)+F_p\left(y,x\right)}{2}
\end{aligned}$$

that is equivalent to

$$\begin{aligned}
&f\left(\frac{x+y}{2}\right)\int_0^1 p\left(t\right)dt \\
&\leq \int_0^1\left(\int_0^1 f\left[\left(\alpha\left(1-t\right)+\left(1-\alpha\right)t\right)x+\left(\alpha t+\left(1-\alpha\right)\left(1-t\right)\right)y\right]p\left(t\right)dt\right)d\alpha \\
&\leq \int_0^1 f\left(\left(1-t\right)x+ty\right)\left[\frac{p\left(t\right)+p\left(1-t\right)}{2}\right]dt.
\end{aligned}$$

By the convexity of f we also have

$$\frac{f\left((1-t)x+ty\right)+f\left(tx+(1-t)y\right)}{2}\leq\frac{f(x)+f(y)}{2} \tag{2.6}$$

for all $t\in[0,1]$ and $(x,y)\in C^2$.

If we multiply (2.6) by $\frac{p(t)+p(1-t)}{2}$ and integrate on $[0,1]$ we get

$$\begin{aligned}\int_0^1 \frac{f\left((1-t)x+ty\right)+f\left(tx+(1-t)y\right)}{2}\cdot\frac{p(t)+p(1-t)}{2}dt \\ \leq\frac{f(x)+f(y)}{2}\int_0^1\frac{p(t)+p(1-t)}{2}dt.\end{aligned} \tag{2.7}$$

Since

$$\int_0^1 f\left(tx+(1-t)y\right)\frac{p(t)+p(1-t)}{2}dt=\int_0^1 f\left((1-t)x+ty\right)\frac{p(t)+p(1-t)}{2}dt$$

and

$$\int_0^1\frac{p(t)+p(1-t)}{2}dt=\int_0^1 p(t)\,dt,$$

hence by (2.7) we get the last part of (2.4). □

Some simple nonnegative symmetric weights are $p(t)=\left|t-\frac{1}{2}\right|$ and $p(t)=t(1-t)$, $t\in[0,1]$. So, for convex functions $f:C\to\mathbb{R}$, the mappings

$$F_{\left|\cdot-\frac{1}{2}\right|}(x,y):=\int_0^1 f\left((1-t)x+ty\right)\left|t-\frac{1}{2}\right|dt$$

and

$$F_{\cdot(1-\cdot)}(x,y):=\int_0^1 f\left((1-t)x+ty\right)t(1-t)\,dt$$

are convex and symmetric on C^2.

Since

$$\int_0^1\left|t-\frac{1}{2}\right|dt=\frac{1}{4}\text{ and }\int_0^1 t(1-t)\,dt=\frac{1}{6},$$

hence from the inequality (2.5) we have

$$\begin{aligned}&f\left(\frac{x+y}{2}\right)\\ &\leq 4\int_0^1\left(\int_0^1 f\left[\left(\alpha(1-t)+(1-\alpha)t\right)x+\left(\alpha t+(1-\alpha)(1-t)\right)y\right]\left|t-\frac{1}{2}\right|dt\right)d\alpha\\ &\qquad\leq 4\int_0^1 f\left((1-t)x+ty\right)\left|t-\frac{1}{2}\right|dtdt\leq\frac{f(x)+f(y)}{2}\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}&f\left(\frac{x+y}{2}\right)\\ &\leq 6\int_0^1\left(\int_0^1 f\left[\left(\alpha(1-t)+(1-\alpha)t\right)x+\left(\alpha t+(1-\alpha)(1-t)\right)y\right]t(1-t)\,dt\right)d\alpha\\ &\qquad\leq 6\int_0^1 f\left((1-t)x+ty\right)t(1-t)\,dtdt\leq\frac{f(x)+f(y)}{2}\end{aligned} \tag{2.9}$$

for any convex function $f : C \to \mathbb{R}$ and $(x, y) \in C^2$.

Let $(X, \|\cdot\|)$ be a normed space. The function $f(x) = \|x\|^r$, $r \geq 1$ is convex on X. Assume that $p : [0,1] \to [0, \infty)$ is a Lebesgue integrable symmetric function on $[0,1]$. If we define

$$N_{r,p}(x, y) := \int_0^1 \|(1-t)x + ty\|^r p(t)\,dt \tag{2.10}$$

then we can state the following results:

Proposition 2. *The function $N_{r,p}(\cdot,\cdot)$ defined by (2.10) is convex and symmetric on X^2. We also have the norm inequalities*

$$\begin{aligned} &\left\|\frac{x+y}{2}\right\|^r \int_0^1 p(t)\,dt \\ &\leq \int_0^1 \left(\int_0^1 \|(\alpha(1-t) + (1-\alpha)t)x + (\alpha t + (1-\alpha)(1-t))y\|^r p(t)\,dt\right) d\alpha \\ &\qquad \leq \int_0^1 \|(1-t)x + ty\|^r p(t)\,dt \leq \frac{\|x\|^r + \|y\|^r}{2} \int_0^1 p(t)\,dt \end{aligned} \tag{2.11}$$

for all $(x, y) \in X^2$.

From (2.11) we can derive the following norm inequalities

$$\begin{aligned} &\left\|\frac{x+y}{2}\right\|^r \\ &\leq \int_0^1 \left(\int_0^1 \|(\alpha(1-t) + (1-\alpha)t)x + (\alpha t + (1-\alpha)(1-t))y\|^r dt\right) d\alpha \\ &\leq \int_0^1 \|(1-t)x + ty\|^r dt \leq \frac{\|x\|^r + \|y\|^r}{2}, \end{aligned} \tag{2.12}$$

$$\begin{aligned} &\left\|\frac{x+y}{2}\right\|^r \\ &\leq 4\int_0^1 \left(\int_0^1 \|(\alpha(1-t) + (1-\alpha)t)x + (\alpha t + (1-\alpha)(1-t))y\|^r \left|t - \frac{1}{2}\right| dt\right) d\alpha \\ &\qquad \leq 4\int_0^1 \|(1-t)x + ty\|^r \left|t - \frac{1}{2}\right| dt \leq \frac{\|x\|^r + \|y\|^r}{2} \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} &\left\|\frac{x+y}{2}\right\|^r \\ &\leq 6\int_0^1 \left(\int_0^1 \|(\alpha(1-t) + (1-\alpha)t)x + (\alpha t + (1-\alpha)(1-t))y\|^r t(1-t)\,dt\right) d\alpha \\ &\qquad \leq 6\int_0^1 \|(1-t)x + ty\|^r t(1-t)\,dt \leq \frac{\|x\|^r + \|y\|^r}{2} \end{aligned} \tag{2.14}$$

for all $(x, y) \in X^2$.

3. Schur Convexity on Linear Spaces

Let X be a linear space and $G \subset X^2 := X \times X$ a convex set. We say that G is *symmetric* if $(x, y) \in G$ implies that $(y, x) \in G$. If $D \subset X$ is a convex subset of X, then the Cartesian product $G := D^2 := D \times D$ is convex and symmetric in X^2.

Motivated by the characterization result of Stępniak above, we say that a function $\phi : G \to \mathbb{R}$ will be called *Schur convex* on the convex and symmetric set $G \subset X^2$ if

$$\phi(s(x, y) + (1 - s)(y, x)) \leq \phi(x, y) \tag{3.1}$$

for all $(x, y) \in G$ and for all $s \in [0, 1]$.

If $G = D^2$, then we recapture the general concept of Schur convexity introduced by Burai and Makó in 2016, [1].

We say that the function $\phi : G \to \mathbb{R}$ is symmetric on G if $\phi(x, y) = \phi(y, x)$ for all $(x, y) \in G$.

The following fact follows from the definition of Schur convex functions:

Proposition 3. *If $\phi : G \to \mathbb{R}$ is Schur convex on the convex and symmetric set $G \subset X^2$, then ϕ is symmetric on G.*

Proof. If $(x, y) \in G$, then by (3.1) we get for $s = 0$ that $\phi(y, x) \leq \phi(x, y)$. If we replace x with y then we also get $\phi(x, y) \leq \phi(y, x)$ which shows that $\phi(x, y) = \phi(y, x)$ for all $(x, y) \in G$. □

The following result provides many examples of Schur convex functions on Cartesian products of convex subsets in linear spaces.

Theorem 6. *Let $f : C \to \mathbb{R}$ be a convex function on C and $p : [0, 1] \to [0, \infty)$ a Lebesgue integrable and symmetric function on $[0, 1]$, then F_p is Schur convex on C^2.*

Proof. From Theorem 5 we have that F_p is convex and symmetric, then for all $(x, y) \in C^2$ and $s \in [0, 1]$

$$\begin{aligned} F_p(s(x, y) + (1 - s)(y, x)) &\leq sF_p(x, y) + (1 - s) F_p(y, x) \\ &= sF_p(x, y) + (1 - s) F_p(x, y) = F_p(x, y), \end{aligned}$$

which shows that F_p is Schur convex on C^2. □

For $(x, y) \in G$, as in [1], let us define the following auxiliary function $\varphi_{(x,y)} : [0, 1] \to R$ by

$$\varphi_{\phi,(x,y)}(s) = \phi(s(x, y) + (1 - s)(y, x)) = \phi(sx + (1 - s) y, sy + (1 - s) x),$$

where $\phi : G \to \mathbb{R}$.

The properties of this function are as follows:

Lemma 1. *Let $G \subset X^2$ be a convex and symmetric set and $\phi : G \to \mathbb{R}$ a symmetric function on G. Then ϕ is Schur convex on G if and only if for all arbitrarily fixed $(x, y) \in G$ the function $\varphi_{\phi,(x,y)}$ is monotone decreasing on $[0, 1/2)$, monotone increasing on $(1/2, 1]$, and $\varphi_{\phi,(x,y)}$ has a global minimum at $1/2$.*

For the proof in the case when $G = D^2$, see [1]. The proof in the slightly more general case of symmetric subsets G in X^2 is given in [5].

Corollary 2. *Let $f : C \to \mathbb{R}$ be a convex function on C and $p : [0,1] \to [0,\infty)$ a Lebesgue integrable and symmetric function on $[0,1]$, and consider the function $\varphi_{F_p,(x,y)} : [0,1] \to \mathbb{R}$ defined by*

$$\begin{aligned}\varphi_{F_p,(x,y)}(s) &:= F_p(sx + (1-s)y, sy + (1-s)x) \\ &= \int_0^1 f[(s(1-t) + (1-s)t)x + (st + (1-s)(1-t))y]\,p(t)\,dt,\end{aligned}$$

where $(x,y) \in D^2$. Then the function $\varphi_{F_p,(x,y)}$ is monotone decreasing on $[0,1/2)$, monotone increasing on $(1/2,1]$, and $\varphi_{F_p,(x,y)}$ has a global minimum at $1/2$. We also have the inequality

$$f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt \le \varphi_{F_p,(x,y)}(s) \le \int_0^1 f[(1-t)x + ty]\,p(t)\,dt \tag{3.2}$$

for all $s \in [0,1]$ and for $(x,y) \in D^2$.

If we consider another function $g : C \to \mathbb{R}$, then, as above, $G_p : C^2 \to \mathbb{R}$ will stand for

$$G_p(x,y) := \int_0^1 g((1-t)x + ty)\,p(t)\,dt.$$

We also have the following result:

Theorem 7. *Let $f, g : C \to \mathbb{R}$ be two convex functions on C and $p : [0,1] \to [0,\infty)$ a Lebesgue integrable and symmetric function on $[0,1]$. Then we have*

$$\begin{aligned}0 &\le \int_0^1 F_p(sx + (1-s)y, sy + (1-s)x) \\ &\quad \times G_p(sx + (1-s)y, sy + (1-s)x)\,ds \\ &\quad - \int_0^1 F_p(sx + (1-s)y, sy + (1-s)x)\,ds \\ &\quad \times \int_0^1 G_p(sx + (1-s)y, sy + (1-s)x)\,ds \\ &\le \frac{1}{4}\left(\int_0^1 f[(1-t)x + ty]\,p(t)\,dt - f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt\right) \\ &\quad \times \left(\int_0^1 g[(1-t)x + ty]\,p(t)\,dt - g\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt\right)\end{aligned} \tag{3.3}$$

for all $(x,y) \in D^2$.

Proof. Let $(x,y) \in D^2$. Since the functions $\varphi_{F_p,(x,y)}$ and $\varphi_{G_p,(x,y)}$ are symmetric on $[0,1]$, then

$$\int_0^1 \varphi_{F_p,(x,y)}(s)\,\varphi_{G_p,(x,y)}(s)\,dt = 2\int_0^{1/2} \varphi_{F_p,(x,y)}(s)\,\varphi_{G_p,(x,y)}(s)\,dt,$$

$$\int_0^1 \varphi_{F_p,(x,y)}(s)\,dt = 2\int_0^{1/2} \varphi_{F_p,(x,y)}(s)\,dt$$

and

$$\int_0^1 \varphi_{G_p,(x,y)}(s)\,dt = 2\int_0^{1/2} \varphi_{G_p,(x,y)}(s)\,dt.$$

By Čebyšev's inequality for synchronous functions $h,\ g:[a,b]\to\mathbb{R}$, we recall that

$$\frac{1}{b-a}\int_a^b h(t)\,g(t)\,dt \geq \frac{1}{b-a}\int_a^b h(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt,$$

we have

$$2\int_0^{1/2}\varphi_{F_p,(x,y)}(s)\,\varphi_{G_p,(x,y)}(s)\,dt \geq 2\int_0^{1/2}\varphi_{F_p,(x,y)}(s)\,dt\cdot 2\int_0^{1/2}\varphi_{G_p,(x,y)}(s)\,dt,$$

which proves the first inequality in (3.3).

Now, recall Grüss' inequality that provides an upper bound for the distance between the integral mean of the product and the product of integral means, more precisely

$$\left|\frac{1}{b-a}\int_a^b h(t)\,k(t)\,dt - \frac{1}{b-a}\int_a^b h(t)\,dt\frac{1}{b-a}\int_a^b k(t)\,dt\right| \leq \frac{1}{4}(M-m)(N-n) \tag{3.4}$$

provided the functions h, k are measurable on $[a,b]$ and $-\infty<m\leq h(t)\leq M<\infty$, $-\infty<n\leq k(t)\leq N<\infty$, for almost every $t\in[a,b]$. The constant $\frac{1}{4}$ is best possible in (3.4).

Since we have

$$f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt \leq \varphi_{F_p,(x,y)}(s) \leq \int_0^1 f[(1-t)x+ty]\,p(t)\,dt$$

and

$$g\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt \leq \varphi_{G_p,(x,y)}(s) \leq \int_0^1 g[(1-t)x+ty]\,p(t)\,dt,$$

then by Grüss' inequality for $h(s)=\varphi_{F_p,(x,y)}(s)$ and $k(s)=\varphi_{G_p,(x,y)}(s)$, $s\in[0,1]$ we get the second part of (3.3). □

Corollary 3. *Let $f,\ g:C\to\mathbb{R}$ be two convex functions on C. Then we have*

$$\begin{aligned} 0 &\leq \int_0^1 F(sx+(1-s)y, sy+(1-s)x)\,G(sx+(1-s)y, sy+(1-s)x)\,ds \\ &\quad - \int_0^1 F(sx+(1-s)y, sy+(1-s)x)\,ds \\ &\quad \times \int_0^1 G(sx+(1-s)y, sy+(1-s)x)\,ds \\ &\leq \frac{1}{4}\left(\int_0^1 f[(1-t)x+ty]\,dt - f\left(\frac{x+y}{2}\right)\right) \\ &\quad \times \left(\int_0^1 g[(1-t)x+ty]\,dt - g\left(\frac{x+y}{2}\right)\right) \end{aligned} \tag{3.5}$$

for all $(x,y)\in D^2$.

4. Applications for Functions of a Real Variable

Assume that f is a continuous function on the interval I and $x,\, y \in I$. Also, let $p : [0,1] \to [0,\infty)$ be a Lebesgue integrable function on $[0,1]$. If we consider

$$F_p(x,y) := \int_0^1 f\left((1-t)\, x + ty\right) p(t)\, dt$$

then

$$F_p(x,x) := f(x) \int_0^1 p(t)\, dt \text{ for } x \in I$$

and if $x \neq y$, then by the change of the variable $u = (1-t)\, x + ty$, we have $du = (y-x)\, dt$, $t = \frac{u-x}{y-x}$ and

$$F_p(x,y) := \frac{1}{y-x} \int_x^y f(u)\, p\left(\frac{u-x}{y-x}\right) du.$$

Therefore, we can consider the function of two variables $F_p : I^2 \to \mathbb{R}$ defined by

$$F_p(x,y) := \begin{cases} \frac{1}{y-x} \int_x^y f(u)\, p\left(\frac{u-x}{y-x}\right) du, & (x,y) \in I^2,\ x \neq y, \\ \\ f(x) \int_0^1 p(t)\, dt, & (x,y) \in I^2,\ x \neq y. \end{cases} \tag{4.1}$$

In particular, we can consider the functions F, $F_{\left|\cdot - \frac{1}{2}\right|}$, $F_{\cdot(1-\cdot)} : I^2 \to \mathbb{R}$ defined by

$$F(x,y) := \begin{cases} \frac{1}{y-x} \int_x^y f(u)\, du, & (x,y) \in I^2,\ x \neq y, \\ \\ f(x), & (x,y) \in I^2,\ x \neq y, \end{cases}$$

$$F_{\left|\cdot - \frac{1}{2}\right|}(x,y) := \begin{cases} \frac{1}{(y-x)^2} \int_x^y f(u) \left|u - \frac{x+y}{2}\right| du, & (x,y) \in I^2,\ x \neq y, \\ \\ \frac{1}{4} f(x), & (x,y) \in I^2,\ x \neq y, \end{cases}$$

and

$$F_{\cdot(1-\cdot)}(x,y) := \begin{cases} \frac{1}{(y-x)^3} \int_x^y f(u)\,(u-x)\,(y-u)\, du, & (x,y) \in I^2,\ x \neq y, \\ \\ \frac{1}{6} f(x), & (x,y) \in I^2,\ x \neq y. \end{cases}$$

By utilising Theorem 5 we can state the following two-variables convexity result:

Proposition 4. *Assume that f is a continuous function on the interval I and let $p : [0,1] \to [0,\infty)$ be a Lebesgue integrable function on $[0,1]$. If f is convex on I, then the function F_p defined in (4.1) is globally convex on I^2. Moreover, if p is symmetric on $[0,1]$, then F_p is convex and symmetric on I^2 and, a fortiori, F_p is Schur convex on I^2.*

From this proposition we obtain for $p \equiv 1$ Wulbert's result from [14] concerning the convexity of the integral mean F. We also obtain the result of Elezović and Pečarić concerning the Schur convexity of F obtained in [8]. In addition, we also observe that the functions $F_{\left|\cdot - \frac{1}{2}\right|}$ and $F_{\cdot(1-\cdot)}$ are globally convex and symmetric on I^2 and therefore Schur convex on I^2, providing a new and reach source of Schur convex functions related to the integral mean.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa