SCHUR CONVEXITY OF FUNCTIONS ASSOCIATED TO FEJÉR'S INEQUALITY FOR CONVEX FUNCTIONS IN LINEAR SPACES

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Abstract. For a non-negative Lebesgue integrable and symmetric function $p:[0,1]\to[0,\infty)$ we consider the functions $M_p, T_p : C^2 \to \mathbb{R}$ defined by

$$M_p(x,y) := \int_0^1 f((1-t)x+ty)\,p(t)\,dt - f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt$$

and

$$T_p(x,y) := \frac{f(x)+f(y)}{2}\int_0^1 p(t)\,dt - \int_0^1 f((1-t)x+ty)\,p(t)\,dt,$$

where $f : C \to \mathbb{R}$ is convex on the convex subset C of a linear space X.

In this paper we show, among others, that M_p and T_p are Schur convex on C^2. Applications for norms and convex functions of a real variable are also given.

1. Introduction

For any $x=(x_1,...,x_n)\in\mathbb{R}^n$, let $x_{[1]}\geq ...\geq x_{[n]}$ denote the components of x in decreasing order, and let $x_{\downarrow}=\left(x_{[1]},...,x_{[n]}\right)$ denote the decreasing rearrangement of x. For $x, y\in\mathbb{R}^n$, $x\prec y$ if, by definition,

$$\begin{cases} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \ k=1,...,n-1; \\ \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}. \end{cases}$$

When $x\prec y$, x is said to be *majorized* by y (y *majorizes* x). This notation and terminology was introduced by Hardy, Littlewood and Pólya in 1934.

A real-valued function ϕ defined on a set $\mathcal{A}\subset\mathbb{R}^n$ is said to be *Schur-convex* on $\mathcal{A}$ if

$$x\prec y \text{ on } \mathcal{A} \Rightarrow \phi(x)\leq\phi(y). \tag{1.1}$$

If, in addition, $\phi(x)<\phi(y)$ whenever $x\prec y$ but x is not a permutation of y, then ϕ is said to be *strictly Schur-convex* on $\mathcal{A}$. If $\mathcal{A}=\mathbb{R}^n$, then ϕ is simply said to be Schur-convex or strictly Schur-convex.

For fundamental properties of Schur convexity see the monograph [11] and the references therein. For some recent results, see [3]-[7] and [12]-[14].

The following result is known in the literature as *Schur-Ostrowski theorem* [11, p. 84]:

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Theorem 1. *Let $I \subset \mathbb{R}$ be an open interval and let $\phi : I^n \to \mathbb{R}$ be continuously differentiable. Necessary and sufficient conditions for ϕ to be Schur-convex on I^n are*

$$\phi \text{ is symmetric on } I^n, \tag{1.2}$$

and for all $i \neq j$, with $i, j \in \{1, ..., n\}$,

$$(z_i - z_j)\left[\frac{\partial \phi(z)}{\partial x_i} - \frac{\partial \phi(z)}{\partial x_j}\right] \geq 0 \text{ for all } z \in I^n, \tag{1.3}$$

where $\frac{\partial \phi}{\partial x_k}$ denotes the partial derivative of ϕ with respect to its k-th argument.

Let $\mathcal{A} \subset \mathbb{R}^n$ be a set with the following properties:

(i) $\mathcal{A}$ is *symmetric* in the sense that $x \in \mathcal{A} \Rightarrow x\Pi \in \mathcal{A}$ for all permutations Π of the coordinates.

(ii) $\mathcal{A}$ is convex and has a nonempty interior.

We have the following result, [11, p. 85].

Theorem 2. *If ϕ is continuously differentiable on the interior of $\mathcal{A}$ and continuous on $\mathcal{A}$, then necessary and sufficient conditions for ϕ to be Schur-convex on $\mathcal{A}$ are*

$$\phi \text{ is symmetric on } \mathcal{A} \tag{1.4}$$

and

$$(z_1 - z_2)\left[\frac{\partial \phi(z)}{\partial x_1} - \frac{\partial \phi(z)}{\partial x_2}\right] \geq 0 \text{ for all } z \in \mathcal{A}. \tag{1.5}$$

Another interesting characterization of Schur convex functions ϕ on $\mathcal{A}$ was obtained by C. Stępniak in [15]:

Theorem 3. *Let ϕ be any function defined on a symmetric convex set $\mathcal{A}$ in $\mathbb{R}^n$. Then the function ϕ is Schur convex on $\mathcal{A}$ if and only if*

$$\phi(x_1, ..., x_i, ..., x_j, ..., x_n) = \phi(x_1, ..., x_j, ..., x_i, ..., x_n) \tag{1.6}$$

for all $(x_1, ..., x_n) \in \mathcal{A}$ and $1 \leq i < j \leq n$ and

$$\phi(\lambda x_1 + (1 - \lambda) x_2, \lambda x_2 + (1 - \lambda) x_1, x_3, ..., x_n) \leq \phi(x_1, ..., x_n) \tag{1.7}$$

for all $(x_1, ..., x_n) \in \mathcal{A}$ and for all $\lambda \in (0, 1)$,

It is well known that any symmetric convex function defined on a symmetric convex set $\mathcal{A}$ is Schur convex, [11, p. 97]. If the function $\phi : \mathcal{A} \to \mathbb{R}$ is symmetric and quasi-convex, namely

$$\phi(\alpha u + (1 - \alpha) v) \leq \max\{\phi(u), \phi(v)\}$$

for all $\alpha \in [0, 1]$ and $u, v \in \mathcal{A}$, a symmetric convex set, then ϕ is Schur convex on $\mathcal{A}$ [11, p. 98].

The following result concerning the Schur convexity of the integral mean was obtained by Elezović and Pečarić in [9]:

Theorem 4. *Let h be a continuous function on I. Then*

$$H(x, y) := \begin{cases} \frac{1}{y-x}\int_x^y h(t)\,dt, \text{ for } x \neq y, \ x, \ y \in I; \\ \\ h(x), \ \text{ for } y = x, \ x \in I, \end{cases} \tag{1.8}$$

is Schur convex (concave) on I^2 if and only if h is convex (concave) on I.

Three year later, in 2003, Wulbert, [16], improved the above result by showing that *the integral mean H defined in (1.8) is in fact convex on I^2* if f is convex on I.

In 2010, Chu et al. [2] obtained the following result concerning the difference functions associated to the Hermite-Hadamard inequalities:

Theorem 5. *Let h be a continuous function on I. Then the functions*

$$F(x,y) := \begin{cases} \frac{1}{y-x}\int_x^y h(t)\,dt - f\left(\frac{x+y}{2}\right), \text{ for } x \neq y,\ x,\ y \in I; \\ \\ 0,\ \text{for } y = x,\ x \in I, \end{cases}$$

and

$$G(x,y) := \begin{cases} \frac{f(x)+f(y)}{2} - \frac{1}{y-x}\int_x^y h(t)\,dt, \text{ for } x \neq y,\ x,\ y \in I; \\ \\ 0,\ \text{for } y = x,\ x \in I, \end{cases}$$

are Schur convex (concave) on I^2 if and only if h is convex (concave) on I.

In 1906, Fejér [10], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

Theorem 6 (Fejér's Inequality)**.** *Consider the integral $\int_a^b h(x)\,p(x)\,dx$, where h is a convex function in the interval (a,b) and g is a positive function in the same interval such that*

$$p(x) = p(a+b-x),\ x \in [a,b],$$

i.e., $y = p(x)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b),0\right)$ and is normal to the x-axis. Under those conditions the following inequalities are valid:

$$h\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \leq \int_a^b h(x)\,p(x)\,dx \leq \frac{h(a)+h(b)}{2}\int_a^b p(x)\,dx. \tag{1.9}$$

If h is concave on (a,b), then the inequalities reverse in (1.9).

We consider the function $f : C \to \mathbb{R}$ defined on the convex subset C of the linear space X and for each $(x,y) \in C^2 := C \times C$ we introduce the auxiliary function $\varphi_{(x,y)} : [0,1] \to \mathbb{R}$ defined by

$$\varphi_{(x,y)}(t) := f((1-t)x + ty).$$

It is well known that the function f is convex on C if and only if for each $(x,y) \in C^2$ the auxiliary function $\varphi_{(x,y)}$ is convex on $[0,1]$.

By utilising the classical Fejér's inequality for the convex function $\varphi_{(x,y)}$ on $[0,1]$ we then have for an integrable non-negative weight p that is symmetric, i.e. $p(1-t) = p(t)$ for all $t \in [0,1]$,

$$\begin{aligned} f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt &\leq \int_0^1 f((1-t)x+ty)\,p(t)\,dt \\ &\leq \frac{f(x)+f(y)}{2}\int_0^1 p(t)\,dt \end{aligned} \tag{1.10}$$

for all $(x,y) \in C^2$.

If $(X, \|\cdot\|)$ is a normed space and $r \geq 1$, then from (1.10) we get the norm inequalities

$$\left\|\frac{x+y}{2}\right\|^{r} \int_{0}^{1} p(t)\, dt \leq \int_{0}^{1} \|(1-t)x+ty\|^{r} p(t)\, dt \leq \frac{\|x\|^{r}+\|y\|^{r}}{2} \int_{0}^{1} p(t)\, dt, \tag{1.11}$$

for all $(x, y) \in X^{2}$ and an integrable non-negative weight p that is symmetric on $[0,1]$.

For a non-negative Lebesgue integrable and symmetric function $p : [0,1] \to [0, \infty)$ we consider the functions M_p, $T_p : C^2 \to \mathbb{R}$ defined by

$$M_p(x, y) := \int_{0}^{1} f((1-t)x+ty)\, p(t)\, dt - f\left(\frac{x+y}{2}\right) \int_{0}^{1} p(t)\, dt \tag{1.12}$$

and

$$T_p(x, y) := \frac{f(x)+f(y)}{2} \int_{0}^{1} p(t)\, dt - \int_{0}^{1} f((1-t)x+ty)\, p(t)\, dt \tag{1.13}$$

where $f : C \to \mathbb{R}$ is convex on the convex subset C of a linear space X.

We observe that

$$M_p(x, x) = T_p(x, x) = f(x) \int_{0}^{1} p(t)\, dt \text{ for all } x \in C.$$

Motivated by the above results, in this paper we investigate, among others, the Schur convexity of the functions M_p and T_p and provide some applications for norms and convex functions of a real variable defined on an interval.

2. Schur Convexity on Linear Spaces

Let X be a linear space and $G \subset X^2 := X \times X$ a convex set. We say that G is *symmetric* if $(x, y) \in G$ implies that $(y, x) \in G$. If $D \subset X$ is a convex subset of X, then the Cartesian product $G := D^2 := D \times D$ is convex and symmetric in X^2.

Motivated by the characterization result of Stępniak above, we say that a function $\phi : G \to \mathbb{R}$ will be called *Schur convex* on the convex and symmetric set $G \subset X^2$ if

$$\phi(s(x, y) + (1-s)(y, x)) \leq \phi(x, y) \tag{2.1}$$

for all $(x, y) \in G$ and for all $s \in [0,1]$.

If $G = D^2$, then we recapture the general concept of Schur convexity introduced by Burai and Makó in 2016, [1].

We say that the function $\phi : G \to \mathbb{R}$ is symmetric on G if $\phi(x, y) = \phi(y, x)$ for all $(x, y) \in G$.

If $\phi : G \to \mathbb{R}$ is *Schur convex* on the convex and symmetric set $G \subset X^2$, then ϕ is symmetric on G. Indeed, if $(x, y) \in G$, then by (2.1) we get for $s = 0$ that $\phi(y, x) \leq \phi(x, y)$. If we replace x with y then we also get $\phi(x, y) \leq \phi(y, x)$ which shows that $\phi(x, y) = \phi(y, x)$ for all $(x, y) \in G$.

Now, for a convex function $f : C \to \mathbb{R}$ and a $t \in [0,1]$ define the functions M_t, $T_t : C^2 \to \mathbb{R}$

$$M_t(x, y) := \frac{1}{2}\left[f((1-t)x+ty) + f((1-t)y+tx)\right] - f\left(\frac{x+y}{2}\right) \geq 0$$

and

$$T_t(x,y) := \frac{f(x)+f(y)}{2} - \frac{1}{2}\left[f((1-t)x+ty) + f((1-t)y+tx)\right] \geq 0.$$

We have the following result concerning the Schur convexity of M_t.

Theorem 7. *Let $f : C \to \mathbb{R}$ be a convex function on the convex set C in X. For all $t \in [0,1]$, $t \neq \frac{1}{2}$ the function M_t is Schur convex on C^2.*

Proof. Let $(x,y) \in C^2$ and $s \in [0,1]$. Then

$$\begin{aligned}
&M_t(s(x,y)+(1-s)(y,x)) \\
&= M_t(sx+(1-s)y, sy+(1-s)x) \\
&= \frac{1}{2}f((1-t)(sx+(1-s)y)+t(sy+(1-s)x)) \\
&+ \frac{1}{2}f((1-t)(sy+(1-s)x)+t(sx+(1-s)y)) \\
&- f\left(\frac{sx+(1-s)y+sy+(1-s)x}{2}\right) \\
&= \frac{1}{2}f(s((1-t)x+ty)+(1-s)((1-t)y+tx)) \\
&+ \frac{1}{2}f(s((1-t)y+tx)+(1-s)((1-t)x+ty)) - f\left(\frac{x+y}{2}\right).
\end{aligned}$$

By the convexity of f we have

$$\begin{aligned}
&f(s((1-t)x+ty)+(1-s)((1-t)y+tx)) \\
&\leq sf((1-t)x+ty)+(1-s)f((1-t)y+tx)
\end{aligned}$$

and

$$\begin{aligned}
&f(s((1-t)y+tx)+(1-s)((1-t)x+ty)) \\
&\leq sf((1-t)y+tx)+(1-s)f((1-t)x+ty).
\end{aligned}$$

for all $(x,y) \in C^2$ and $s \in [0,1]$.

If we add these two inequalities and divide by 2 we get

$$\begin{aligned}
&\frac{1}{2}f(s((1-t)x+ty)+(1-s)((1-t)y+tx)) \\
&+ \frac{1}{2}f(s((1-t)y+tx)+(1-s)((1-t)x+ty)) \\
&\leq \frac{1}{2}\left[f((1-t)y+tx)+f((1-t)x+ty)\right]
\end{aligned}$$

for all $(x,y) \in C^2$ and $s \in [0,1]$.

Therefore

$$\begin{aligned}
&M_t(s(x,y)+(1-s)(y,x)) \\
&\leq \frac{1}{2}\left[f((1-t)y+tx)+f((1-t)x+ty)\right] - f\left(\frac{x+y}{2}\right) \\
&= M_t(x,y)
\end{aligned}$$

for all $(x,y) \in C^2$ and $s \in [0,1]$, which shows that M_t is Schur convex on C^2. $\square$

For a convex function $f : C \to \mathbb{R}$ and $q : [0,1] \to [0,\infty)$ a Lebesgue integrable function we consider the function $M_{\breve{q}} : C^2 \to [0,\infty)$ defined by

$$\begin{aligned} M_{\breve{q}}(x,y) &:= \int_0^1 M_t(x,y)\, q(t)\, dt \\ &= \frac{1}{2}\int_0^1 [f((1-t)x+ty) + f((1-t)y+tx)]\, q(t)\, dt \\ &- f\left(\frac{x+y}{2}\right)\int_0^1 q(t)\, dt \\ &= \int_0^1 f((1-t)x+ty)\, \breve{q}(t)\, dt - f\left(\frac{x+y}{2}\right)\int_0^1 q(t)\, dt, \end{aligned}$$

where

$$\breve{q}(t) := \frac{1}{2}[q(t) + q(1-t)], t \in [0,1].$$

Corollary 1. *Let $f : C \to \mathbb{R}$ be a convex function on C and $q : [0,1] \to [0,\infty)$ a Lebesgue integrable function on $[0,1]$, then $M_{\breve{q}}$ is Schur convex on C^2.*

Proof. Let $(x,y) \in C^2$ and $s \in [0,1]$. By the Schur convexity of M_t for all $t \in [0,1]$, we have

$$\begin{aligned} M_{\breve{q}}(s(x,y) + (1-s)(y,x)) &= \int_0^1 M_t(s(x,y) + (1-s)(y,x))\, q(t)\, dt \\ &\le \int_0^1 M_t(x,y)\, q(t)\, dt = M_{\breve{q}}(x,y), \end{aligned}$$

which proves the Schur convexity of $M_{\breve{q}}$. □

Corollary 2. *Let $f : C \to \mathbb{R}$ be a convex function on C and $p : [0,1] \to [0,\infty)$ a Lebesgue integrable symmetric function on $[0,1]$, then M_p is Schur convex on C^2.*

We denote by $[x,y]$ the closed segment defined by $\{(1-s)x + sy,\ s \in [0,1]\}$. We also define the functional

$$\Psi_{f,t}(x,y) := (1-t) f(x) + t f(y) - f((1-t)x + ty) \ge 0 \tag{2.2}$$

where $x, y \in C$ and $t \in [0,1]$.

In [4] we obtained among others the following result :

Lemma 1. *Let $f : C \subset X \to \mathbb{R}$ be a convex function on the convex set C. Then for each $x, y \in C$ and $z \in [x,y]$ we have*

$$(0 \le)\, \Psi_{f,t}(x,z) + \Psi_{f,t}(z,y) \le \Psi_{f,t}(x,y) \tag{2.3}$$

for each $t \in [0,1]$, i.e., the functional $\Psi_{f,t}(\cdot,\cdot)$ is superadditive as a function of interval.

If $z, u \in [x,y]$, then

$$(0 \le)\, \Psi_{f,t}(z,u) \le \Psi_{f,t}(x,y) \tag{2.4}$$

for each $t \in [0,1]$, i.e., the functional $\Psi_f(\cdot,\cdot)$ is nondecreasing as a function of interval.

By utilising this lemma we can prove the following result as well:

Theorem 8. *Let $f : C \to \mathbb{R}$ be a convex function on the convex set C in X. For all $t \in (0,1)$, the function T_t is Schur convex on C^2.*

Proof. Let $(x,y) \in C^2$ with $x \neq y$ and $s \in [0,1]$. Then

$$\begin{aligned}
&T_t\left(s\left(x,y\right)+\left(1-s\right)\left(y,x\right)\right)\\
&= T_t\left(sx+\left(1-s\right)y, sy+\left(1-s\right)x\right)\\
&= \frac{f\left(sx+\left(1-s\right)y\right)+f\left(sy+\left(1-s\right)x\right)}{2}\\
&- \frac{1}{2}f\left(\left(1-t\right)\left(sx+\left(1-s\right)y\right)+t\left(sy+\left(1-s\right)x\right)\right)\\
&- \frac{1}{2}f\left(\left(1-t\right)\left(sy+\left(1-s\right)x\right)+t\left(sx+\left(1-s\right)y\right)\right).
\end{aligned}$$

From (2.4) we have for z, $u \in [x,y]$

$$\Psi_{f,t}\left(z,u\right) \leq \Psi_{f,t}\left(x,y\right) \text{ and } \Psi_{f,1-t}\left(z,u\right) \leq \Psi_{f,1-t}\left(x,y\right),$$

which, by addition gives that

$$\Psi_{f,t}\left(z,u\right)+\Psi_{f,1-t}\left(z,u\right) \leq \Psi_{f,t}\left(x,y\right)+\Psi_{f,1-t}\left(x,y\right)$$

namely

$$\begin{aligned}
&\left(1-t\right)f\left(z\right)+tf\left(u\right)-f\left(\left(1-t\right)z+tu\right)\\
&+tf\left(z\right)+\left(1-t\right)f\left(u\right)-f\left(tz+\left(1-t\right)u\right)\\
&\leq \left(1-t\right)f\left(x\right)+tf\left(y\right)-f\left(\left(1-t\right)x+ty\right)\\
&+tf\left(x\right)+\left(1-t\right)f\left(y\right)-f\left(tx+\left(1-t\right)y\right),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&f\left(z\right)+f\left(u\right)-f\left(\left(1-t\right)z+tu\right)-f\left(tz+\left(1-t\right)u\right)\\
&\leq f\left(x\right)+f\left(y\right)-f\left(\left(1-t\right)x+ty\right)-f\left(tx+\left(1-t\right)y\right)
\end{aligned} \tag{2.5}$$

for all z, $u \in [x,y]$.

If we take $z = sx+(1-s)y$ and $u = sy+(1-s)x$, with $s \in [0,1]$ then z, $u \in [x,y]$ and by (2.5) we get

$$\begin{aligned}
&f\left(sx+\left(1-s\right)y\right)+f\left(sy+\left(1-s\right)x\right)\\
&-f\left(\left(1-t\right)\left(sx+\left(1-s\right)y\right)+t\left(sy+\left(1-s\right)x\right)\right)\\
&-f\left(\left(1-t\right)\left(sy+\left(1-s\right)x\right)+t\left(sx+\left(1-s\right)y\right)\right)\\
&\leq f\left(x\right)+f\left(y\right)-f\left(\left(1-t\right)x+ty\right)-f\left(tx+\left(1-t\right)y\right).
\end{aligned}$$

This inequality is equivalent to

$$T_t\left(s\left(x,y\right)+\left(1-s\right)\left(y,x\right)\right) \leq T_t\left(x,y\right)$$

for all $(x,y) \in C^2$ and $s \in [0,1]$. This proves the Schur convexity of T_t. □

Remark 1. *Since both M_t and T_t are Schur convex when f is convex on C it follows that the sum, namely the Jensen's functional*

$$J\left(x,y\right) := \frac{f\left(x\right)+f\left(y\right)}{2}-f\left(\frac{x+y}{2}\right)$$

is also Schur Convex on C^2.

In the case of normed spaces $(X,\|\cdot\|)$, if we put

$$J_r\left(x,y\right) := \frac{\|x\|^r+\|y\|^r}{2}-\left\|\frac{x+y}{2}\right\|^r, \ r \geq 1,$$

then we conclude that J_r is Schur convex on X^2.

For a convex function $f : C \to \mathbb{R}$ and $q : [0,1] \to [0,\infty)$ a Lebesgue integrable function we consider the function $T_{\breve{q}} : C^2 \to [0,\infty)$ defined by

$$\begin{aligned} T_{\breve{q}}(x,y) &:= \int_0^1 T_t(x,y)\, q(t)\, dt \\ &= \frac{f(x)+f(y)}{2} \int_0^1 q(t)\, dt \\ &- \frac{1}{2} \int_0^1 [f((1-t)x+ty) + f((1-t)y+tx)]\, q(t)\, dt \\ &= \frac{f(x)+f(y)}{2} \int_0^1 q(t)\, dt - \int_0^1 f((1-t)x+ty)\, \breve{q}(t)\, dt. \end{aligned}$$

Corollary 3. *Let $f : C \to \mathbb{R}$ be a convex function on C and $q : [0,1] \to [0,\infty)$ a Lebesgue integrable function on $[0,1]$, then $T_{\breve{q}}$ is Schur convex on C^2. In particular, if $p : [0,1] \to [0,\infty)$ is a Lebesgue integrable symmetric function on $[0,1]$, then T_p is Schur convex on C^2.*

If $(X, \|\cdot\|)$ is a normed linear space, $r \geq 1$ and $p : [0,1] \to [0,\infty)$ is a Lebesgue integrable symmetric function on $[0,1]$, then the functions

$$M_{r,p}(x,y) := \int_0^1 \|(1-t)x+ty\|^r p(t)\, dt - \left\| \frac{x+y}{2} \right\|^r \int_0^1 p(t)\, dt \tag{2.6}$$

and

$$T_{r,p}(x,y) := \frac{\|x\|^r + \|y\|^r}{2} \int_0^1 p(t)\, dt - \int_0^1 \|(1-t)x+ty\|^r p(t)\, dt \tag{2.7}$$

are Schur convex on X^2.

In particular,

$$M_r(x,y) := \int_0^1 \|(1-t)x+ty\|^r dt - \left\| \frac{x+y}{2} \right\|^r \tag{2.8}$$

and

$$T_r(x,y) := \frac{\|x\|^r + \|y\|^r}{2} - \int_0^1 \|(1-t)x+ty\|^r dt \tag{2.9}$$

are Schur convex on X^2.

If we take $p \equiv 1$ and consider the functions

$$M(x,y) := \int_0^1 f((1-t)x+ty)\, dt - f\left(\frac{x+y}{2}\right)$$

and

$$T(x,y) := \frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)x+ty)\, dt$$

then we conclude that M and T are *Schur convex functions on* C^2 if f is convex on C. This result generalizes the result of Chu et al. [2] that was proved in the case of convex functions defined on real intervals.

Also, if we consider the symmetric weights $p_1(t) = \left|t - \frac{1}{2}\right|$ and $p_2(t) = t(1-t)$, $t \in [0,1]$, then

$$M_{\left|\cdot-\frac{1}{2}\right|}(x,y) := \int_0^1 f((1-t)x+ty)\left|t-\frac{1}{2}\right| dt - \frac{1}{4} f\left(\frac{x+y}{2}\right)$$

and

$$M_{\cdot(1-\cdot)}(x,y) := \int_0^1 f((1-t)x+ty)\, t(1-t)\, dt - \frac{1}{6} f\left(\frac{x+y}{2}\right)$$

are Schur convex on C^2 if f is convex on C.

The trapezoid functions

$$T_{\left|\cdot-\frac{1}{2}\right|}(x,y) := \frac{f(x)+f(y)}{8} - \int_0^1 f((1-t)x+ty)\left|t-\frac{1}{2}\right| dt$$

and

$$T_{\cdot(1-\cdot)}(x,y) := \frac{f(x)+f(y)}{12} - \int_0^1 f((1-t)x+ty)\, t(1-t)\, dt$$

are also *Schur convex on* C^2 if f is convex on C.

3. Examples for Functions of a Real Variable

Assume that f is a continuous function on the interval I and $x, y \in I$. Also, let $p : [0,1] \to [0,\infty)$ be a Lebesgue integrable function on $[0,1]$. If we consider the functions

$$M_p(x,y) := \int_0^1 f((1-t)x+ty)\, p(t)\, dt - f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\, dt$$

and

$$T_p(x,y) := \frac{f(x)+f(y)}{2}\int_0^1 p(t)\, dt - \int_0^1 f((1-t)x+ty)\, p(t)\, dt$$

then

$$M_p(x,x) = T_p(x,x) = 0 \text{ for } x \in I.$$

If $x \neq y$, then by the change of the variable $u = (1-t)x + ty$, we have $du = (y-x)\,dt$, $t = \frac{u-x}{y-x}$, and we can consider the functions of two variables M_p, $T_p : I^2 \to \mathbb{R}$ defined by

$$M_p(x,y) := \begin{cases} \frac{1}{y-x}\int_x^y f(u)\, p\left(\frac{u-x}{y-x}\right) du - f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\, dt, \\ (x,y) \in I^2,\ x \neq y, \\ \\ 0,\ \ (x,y) \in I^2,\ x \neq y \end{cases} \tag{3.1}$$

and

$$T_p(x,y) := \begin{cases} \frac{f(x)+f(y)}{2}\int_0^1 p(t)\, dt - \frac{1}{y-x}\int_x^y f(u)\, p\left(\frac{u-x}{y-x}\right) du \\ (x,y) \in I^2,\ x \neq y, \\ \\ 0,\ \ (x,y) \in I^2,\ x \neq y. \end{cases} \tag{3.2}$$

In particular, we have the functions $M, T : I^2 \to \mathbb{R}$ introduced in [2] and defined by

$$M(x,y) := \begin{cases} \frac{1}{y-x}\int_x^y f(u)\,du - f\left(\frac{x+y}{2}\right), & (x,y) \in I^2,\ x \neq y, \\ \\ 0, & (x,y) \in I^2,\ x \neq y, \end{cases}$$

and

$$T(x,y) := \begin{cases} \frac{f(x)+f(y)}{2} - \frac{1}{y-x}\int_x^y f(u)\,du, & (x,y) \in I^2,\ x \neq y, \\ \\ 0, & (x,y) \in I^2,\ x \neq y. \end{cases}$$

We can also consider the weighted functions defined on I^2

$$M_{\left|\cdot-\frac{1}{2}\right|}(x,y) := \begin{cases} \frac{1}{(y-x)^2}\int_x^y f(u)\left|u-\frac{x+y}{2}\right|du - \frac{1}{4}f\left(\frac{x+y}{2}\right), \\ (x,y) \in I^2,\ x \neq y, \\ \\ 0,\ (x,y) \in I^2,\ x \neq y, \end{cases}$$

$$T_{\left|\cdot-\frac{1}{2}\right|}(x,y) := \begin{cases} \frac{f(x)+f(y)}{8} - \frac{1}{(y-x)^2}\int_x^y f(u)\left|u-\frac{x+y}{2}\right|du, \\ (x,y) \in I^2,\ x \neq y, \\ \\ 0,\ (x,y) \in I^2,\ x \neq y, \end{cases}$$

$$M_{\cdot(1-\cdot)}(x,y) := \begin{cases} \frac{1}{(y-x)^3}\int_x^y f(u)(u-x)(y-u)\,du - \frac{1}{6}f\left(\frac{x+y}{2}\right), \\ (x,y) \in I^2,\ x \neq y, \\ \\ 0,\ (x,y) \in I^2,\ x \neq y, \end{cases}$$

and

$$T_{\cdot(1-\cdot)}(x,y) := \begin{cases} \frac{f(x)+f(y)}{12} - \frac{1}{(y-x)^3}\int_x^y f(u)(u-x)(y-u)\,du, \\ (x,y) \in I^2,\ x \neq y, \\ \\ 0,\ (x,y) \in I^2,\ x \neq y. \end{cases}$$

By utilising Corollary 2 and Corollary 3 we can state the following Schur convexity result:

Proposition 1. *Assume that f is a convex function on the interval I and let $p : [0,1] \to [0,\infty)$ be a Lebesgue integrable symmetric function on $[0,1]$. Then the functions M_p and T_p are Schur convex on I^2.*

In the case $p \equiv 1$ and f is convex on I, we obtain the fact that the functions M and T are Schur convex on I^2, established by Chu et al. in [2]. The functions $M_{\left|\cdot-\frac{1}{2}\right|}$, $T_{\left|\cdot-\frac{1}{2}\right|}$, $M_{\cdot(1-\cdot)}$ and $T_{\cdot(1-\cdot)}$ defined above are also Schur convex on I^2, provided that f is convex on I.

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