h-CONVEXITY OF THE WEIGHTED INTEGRAL MEAN OF FUNCTIONS DEFINED ON CONVEX SETS IN LINEAR SPACES

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ABSTRACT. For a Lebesgue integrable function $p:[0,1]\to[0,\infty)$ we consider the function $F_p: C^2 \to \mathbb{R}$ defined by

$$F_p(x,y) := \int_0^1 f((1-t)x+ty)\,p(t)\,dt,$$

where $f: C\to\mathbb{R}$ is h-convex and hemi-Lebesgue integrable on the convex subset C of a linear space X. In this paper we investigate the h-global convexity of the function F_p, establish some Hermite-Hadamard type inequalities and provide some applications for some classical examples of h-convex functions that are available in the literature.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a)f\left(\frac{a+b}{2}\right) \le \int_a^b f(x)dx \le (b-a)\frac{f(a)+f(b)}{2}, \quad a,\ b\in\mathbb{R},\ a<b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [42]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [5]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [42]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

For related results, see [10]-[19], [22]-[25], [32]-[35] and [45].

Let X be a vector space over the real or complex number field $\mathbb{K}$ and $x,\ y\in X,\ x\neq y$. Define the segment

$$[x,y] := \{(1-t)x+ty,\ t\in[0,1]\}.$$

We consider the function $f:[x,y]\to\mathbb{R}$ and the associated function

$$g(x,y):[0,1]\to\mathbb{R},\ g(x,y)(t) := f[(1-t)x+ty],\ t\in[0,1].$$

Note that f is convex on $[x,y]$ if and only if $g(x,y)$ is convex on $[0,1]$.

For any convex function defined on a segment $[x,y]\subset X$, we have the *Hermite-Hadamard integral inequality* (see [20, p. 2], [21, p. 2])

$$f\left(\frac{x+y}{2}\right) \le \int_0^1 f[(1-t)x+ty]dt \le \frac{f(x)+f(y)}{2}, \tag{1.2}$$

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which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x,y):[0,1]\to\mathbb{R}$.

Since $f(x)=\|x\|^p$ ($x\in X$ and $1\le p<\infty$) is a convex function, then for any x, $y\in X$ we have the following norm inequality from (1.2) (see [46, p. 106])

$$\left\|\frac{x+y}{2}\right\|^p\le\int_0^1\|(1-t)x+ty\|^p dt\le\frac{\|x\|^p+\|y\|^p}{2}. \tag{1.3}$$

For a Lebesgue integrable function $p:[0,1]\to[0,\infty)$ we consider the function $F_p:C^2\to\mathbb{R}$ defined by

$$F_p(x,y):=\int_0^1 f((1-t)x+ty)\,p(t)\,dt,$$

where $f:C\to\mathbb{R}$ is h-convex and hemi-Lebesgue integrable on the convex subset C of a linear space X.

In this paper we investigate the h-global convexity of the function F_p, establish some Hermite-Hadamard type inequalities and provide some applications for some classical examples of h-convex functions that are available in the literature.

2. h-Convex Functions on Linear Spaces

We recall here some concepts of convexity that are well known in the literature. Let I be an interval in $\mathbb{R}$.

Definition 1 ([37]). *We say that $f:I\to\mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all x, $y\in I$ and $t\in(0,1)$ we have*

$$f(tx+(1-t)y)\le\frac{1}{t}f(x)+\frac{1}{1-t}f(y). \tag{2.1}$$

Some further properties of this class of functions can be found in [28], [29], [31], [43], [46] and [47]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

The above concept can be extended for functions $f:C\subseteq X\to[0,\infty)$ where C is a convex subset of the real or complex linear space X and the inequality (2.1) is satisfied for any vectors x, $y\in C$ and $t\in(0,1)$. If the function $f:C\subseteq X\to\mathbb{R}$ is non-negative and convex, then it is of Godunova-Levin type.

Definition 2 ([31]). *We say that a function $f:I\to\mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all x, $y\in I$ and $t\in[0,1]$ we have*

$$f(tx+(1-t)y)\le f(x)+f(y). \tag{2.2}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx+(1-t)y)\le\max\{f(x),f(y)\} \tag{2.3}$$

for all x, $y\in I$ and $t\in[0,1]$.

For some results on P-functions see [31] and [44] while for quasi convex functions, the reader can consult [30].

If $f : C \subseteq X \to [0, \infty)$, where C is a convex subset of the real or complex linear space X, then we say that it is of P-type (or quasi-convex) if the inequality (2.2) (or (2.3)) holds true for $x, y \in C$ and $t \in [0, 1]$.

Definition 3 ([7])**.** *Let s be a real number, $s \in (0, 1]$. A function $f : [0, \infty) \to [0, \infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if*

$$f(tx + (1 - t)y) \leq t^s f(x) + (1 - t)^s f(y)$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

For some properties of this class of functions see [1], [2], [7], [8], [26], [27], [38], [40] and [49].

The concept of Breckner s-convexity can be similarly extended for functions defined on convex subsets of linear spaces.

It is well known that if $(X, \|\cdot\|)$ is a normed linear space, then the function $f(x) = \|x\|^p, p \geq 1$ is convex on X.

Utilising the elementary inequality $(a + b)^s \leq a^s + b^s$ that holds for any $a, b \geq 0$ and $s \in (0, 1]$, we have for the function $g(x) = \|x\|^s$ that

$$\begin{aligned} g(tx + (1 - t)y) &= \|tx + (1 - t)y\|^s \leq (t\|x\| + (1 - t)\|y\|)^s \\ &\leq (t\|x\|)^s + [(1 - t)\|y\|]^s \\ &= t^s g(x) + (1 - t)^s g(y) \end{aligned}$$

for any $x, y \in X$ and $t \in [0, 1]$, which shows that g is Breckner s-convex on X.

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}$, $(0, 1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 4 ([52])**.** *Let $h : J \to [0, \infty)$ with h not identical to 0. We say that $f : I \to [0, \infty)$ is an h-convex function if for all $x, y \in I$ we have*

$$f(tx + (1 - t)y) \leq h(t) f(x) + h(1 - t) f(y) \tag{2.4}$$

for all $t \in (0, 1)$.

For some results concerning this class of functions see [52], [6], [41], [50], [48] and [51].

This concept can be extended for functions defined on convex subsets of linear spaces in the same way as above replacing the interval I be the corresponding convex subset C of the linear space X.

We can introduce now another class of functions.

Definition 5. *We say that the function $f : C \subseteq X \to [0, \infty)$ is of s-Godunova-Levin type, with $s \in [0, 1]$, if*

$$f(tx + (1 - t)y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1 - t)^s} f(y), \tag{2.5}$$

for all $t \in (0, 1)$ and $x, y \in C$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(C)$ the class of s-Godunova-Levin functions defined on C, then we obviously have

$$P(C) = Q_0(C) \subseteq Q_{s_1}(C) \subseteq Q_{s_2}(C) \subseteq Q_1(C) = Q(C)$$

for $0 \leq s_1 \leq s_2 \leq 1$.

We have the following generalization of the Hermite-Hadamard inequality for h-convex functions defined on convex subsets of linear spaces [24].

Theorem 1. *Assume that the function $f : C \subseteq X \to [0, \infty)$ is an h-convex function with $h \in L[0, 1]$. Let $y, x \in C$ with $y \neq x$ and assume that the mapping $[0, 1] \ni t \mapsto f[(1 - t)x + ty]$ is Lebesgue integrable on $[0, 1]$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]\,dt \leq [f(x) + f(y)] \int_0^1 h(t)\,dt. \tag{2.6}$$

Remark 1. *If $f : I \to [0, \infty)$ is an h-convex function on an interval I of real numbers with $h \in L[0, 1]$ and $f \in L[a, b]$ with $a, b \in I$, $a < b$, then from (2.6) we get the Hermite-Hadamard type inequality obtained by Sarikaya et al. in* [48]

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \int_a^b f(u)\,du \leq [f(a) + f(b)] \int_0^1 h(t)\,dt.$$

If we write (2.6) for $h(t) = t$, then we get the classical Hermite-Hadamard inequality for convex functions 1.2.

If we write (2.6) for the case of P-type functions $f : C \to [0, \infty)$, i.e., $h(t) = 1$, $t \in [0, 1]$, then we get the inequality

$$\frac{1}{2} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]\,dt \leq f(x) + f(y), \tag{2.7}$$

that has been obtained for functions of real variable in [31].

If f is Breckner s-convex on C, for $s \in (0, 1)$, then by taking $h(t) = t^s$ in (2.6) we get

$$2^{s-1} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]\,dt \leq \frac{f(x) + f(y)}{s+1}, \tag{2.8}$$

that was obtained for functions of a real variable in [26].

Since the function $g(x) = \|x\|^s$ is Breckner s-convex on on the normed linear space X, $s \in (0, 1)$, then for any $x, y \in X$ we have

$$\frac{1}{2} \|x + y\|^s \leq \int_0^1 \|(1-t)x + ty\|^s\,dt \leq \frac{\|x\|^s + \|x\|^s}{s+1}. \tag{2.9}$$

If $f : C \to [0, \infty)$ is of s-Godunova-Levin type, with $s \in [0, 1)$, then

$$\frac{1}{2^{s+1}} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]\,dt \leq \frac{f(x) + f(y)}{1-s}. \tag{2.10}$$

We notice that for $s = 1$ the first inequality in (2.10) still holds, i.e.

$$\frac{1}{4} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]\,dt. \tag{2.11}$$

The case for functions of real variables was obtained for the first time in [31].

Theorem 2. *Assume that the function $f : C \subseteq X \to [0, \infty)$ is an h-convex function with $h \in L[0, 1]$. Let $y, x \in C$ with $y \neq x$ and assume that the mapping $[0, 1] \ni t \mapsto$*

$f[(1-t)x+ty]$ is Lebesgue integrable on $[0,1]$. If $p:[0,1]\to[0,\infty)$ is Lebesgue integrable, then

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt \le \int_0^1 [f((1-t)x+ty)]\,\breve{p}(t)\,dt \le [f(x)+f(y)]\int_0^1 h(t)\,\breve{p}(t)\,dt \tag{2.12}$$

where $\breve{p}(t):=\frac{1}{2}[p(t)+p(1-t)]$, $t\in[0,1]$.

Proof. By the h-convexity of f we have

$$f(tx+(1-t)y)\le h(t)f(x)+h(1-t)f(y)$$

for any $t\in[0,1]$.

We also have

$$f((1-t)x+ty)\le h(1-t)f(x)+h(t)f(y)$$

for any $t\in[0,1]$.

If we add these two inequalities, we get

$$\frac{1}{2}[f(tx+(1-t)y)+f((1-t)x+ty)] \le \frac{1}{2}[h(t)+h(1-t)][f(x)+f(y)], \tag{2.13}$$

for any $t\in[0,1]$.

If we multiply (2.13) by $p(t)\ge 0$ and integrate on $[0,1]$ we get

$$\frac{1}{2}\int_0^1 [f(tx+(1-t)y)+f((1-t)x+ty)]\,p(t)\,dt \le \frac{1}{2}[f(x)+f(y)]\int_0^1 [h(t)+h(1-t)]\,p(t)\,dt. \tag{2.14}$$

By using the change of variable $s=1-t$, $t\in[0,1]$ we have

$$\int_0^1 f((1-t)x+ty)\,p(t)\,dt=\int_0^1 f(sx+(1-s)y)\,p(1-s)\,dt$$

and

$$\int_0^1 h(1-t)\,p(t)\,dt=\int_0^1 h(s)\,p(1-s)\,dt$$

and by (2.14) we get

$$\int_0^1 [f(tx+(1-t)y)]\,\breve{p}(t)\,dt\le[f(x)+f(y)]\int_0^1 h(t)\,\breve{p}(t)\,dt.$$

From the h-convexity of f we have

$$f\left(\frac{z+w}{2}\right)\le h\left(\frac{1}{2}\right)[f(z)+f(w)] \tag{2.15}$$

for any z, $w\in C$.

If we take in (2.15) $z=tx+(1-t)y$ and $w=(1-t)x+ty$, then we get

$$f\left(\frac{x+y}{2}\right)\le h\left(\frac{1}{2}\right)[f(tx+(1-t)y)+f((1-t)x+ty)] \tag{2.16}$$

for any $t\in[0,1]$.

If we multiply (2.16) by $p(t) \geq 0$ and integrate on $[0,1]$ we get

$$\begin{aligned}(2.17)\qquad & \frac{1}{2} f\left(\frac{x+y}{2}\right) \int_0^1 p(t)\,dt \\ & \leq h\left(\frac{1}{2}\right) \int_0^1 \left[\frac{f(tx+(1-t)y)+f((1-t)x+ty)}{2}\right] p(t)\,dt,\end{aligned}$$

which proves the first part of (2.12). □

Corollary 1. *With the assumptions of Theorem 2 and if p is symmetric, namely $p(1-t) = p(t)$ for $t \in [0,1]$, then*

$$\begin{aligned}(2.18)\qquad & \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{x+y}{2}\right) \int_0^1 p(t)\,dt \\ & \leq \int_0^1 [f((1-t)x+ty)]\,p(t)\,dt \leq [f(x)+f(y)] \int_0^1 h(t)\,p(t)\,dt.\end{aligned}$$

Remark 2. *If $f : I \to [0,\infty)$ is an h-convex function on an interval I of real numbers with $h \in L[0,1]$ and $f \in L[a,b]$ with a, $b \in I$, $a < b$. If p is Lebesgue integrable and symmetric on $[0,1]$, namely $p(1-t) = p(t)$ for $t \in [0,1]$, then*

$$\begin{aligned}(2.19)\qquad & \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \int_0^1 p(t)\,dt \\ & \leq \int_0^1 [f((1-t)a+tb)]\,p(t)\,dt \leq [f(a)+f(b)] \int_0^1 h(t)\,p(t)\,dt.\end{aligned}$$

If we change the variable $x = (1-t)a + tb$, $t \in [0,1]$ then by (2.19) we get

$$\begin{aligned}(2.20)\qquad & \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \int_0^1 p(t)\,dt \\ & \leq \frac{1}{b-a} \int_a^b f(x)\,p\left(\frac{x-a}{b-a}\right) dx \leq [f(a)+f(b)] \int_0^1 h(t)\,p(t)\,dt.\end{aligned}$$

If we put $w : [a,b] \to [0,\infty)$, $w(x) = p\left(\frac{x-a}{b-a}\right)$ then from (2.20) we recapture the result from [6]

$$\begin{aligned}(2.21)\qquad & \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \int_a^b w(x)\,dx \\ & \leq \int_a^b f(x)\,w(x)\,dx \leq [f(a)+f(b)] \int_0^1 h(t)\,w((1-t)a+tb)\,dt,\end{aligned}$$

where $f : I \to [0,\infty)$ is an h-convex function on an interval I of real numbers with $h \in L[0,1]$, $f \in L[a,b]$ and $w(x) = w(a+b-x)$, $x \in [a,b]$, $w \geq 0$ and Lebesgue integrable on $[a,b]$.

In what follows we assume that p is Lebesgue integrable and symmetric on $[0,1]$.

If we write (2.18) for $h(t) = t$, then we get the classical Hermite-Hadamard-Fejér's inequality for convex functions $f : C \to \mathbb{R}$ defined on convex subsets C of linear spaces

$$f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt \le \int_0^1 [f((1-t)x+ty)]\,p(t)\,dt \le [f(x)+f(y)]\int_0^1 tp(t)\,dt, \tag{2.22}$$

where $x,\ y\in C$.

If we write (2.18) for the case of P-type functions $f:C\to[0,\infty)$, i.e., $h(t)=1, t\in[0,1]$, then we get the inequality

$$\frac{1}{2}f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt \le \int_0^1 [f((1-t)x+ty)]\,p(t)\,dt \le [f(x)+f(y)]\int_0^1 p(t)\,dt \tag{2.23}$$

where $x,\ y\in C$.

If f is Breckner s-convex on C, for $s\in(0,1)$, then by taking $h(t)=t^s$ in (2.18) we get

$$2^{s-1}f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt \le \int_0^1 [f((1-t)x+ty)]\,p(t)\,dt \le [f(x)+f(y)]\int_0^1 t^s p(t)\,dt, \tag{2.24}$$

where $x,\ y\in C$.

Since the function $g(x)=\|x\|^s$ is Breckner s-convex on the normed linear space X, $s\in(0,1)$, then for any $x,\ y\in X$ we have

$$\begin{aligned}\frac{1}{2}\|x+y\|^s\int_0^1 p(t)\,dt &\le \int_0^1 \|(1-t)x+ty\|^s\,p(t)\,dt \\ &\le [\|x\|^s+\|x\|^s]\int_0^1 t^s p(t)\,dt.\end{aligned} \tag{2.25}$$

If $f:C\to[0,\infty)$ is of s-Godunova-Levin type, with $s\in[0,1)$, then by taking $h(t)=\frac{1}{t^s}$

$$\begin{aligned}\frac{1}{2^{s+1}}f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt &\le \int_0^1 f[(1-t)x+ty]\,p(t)\,dt \\ &\le [f(x)+f(y)]\int_0^1 \frac{1}{t^s}p(t)\,dt\end{aligned} \tag{2.26}$$

where $x,\ y\in C$.

We notice that for $s=1$ we get

$$\begin{aligned}\frac{1}{4}f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt &\le \int_0^1 f[(1-t)x+ty]\,p(t)\,dt \\ &\le [f(x)+f(y)]\int_0^1 \frac{1}{t}p(t)\,dt,\end{aligned} \tag{2.27}$$

where $x,\ y\in C$, and provided the above integrals exist.

3. h-Convexity of Integral Means

Assume that the function $f : C \subseteq X \to [0, \infty)$ is an h-convex function with $h \in L[0,1]$. Let $y, x \in C$. We say that the function is f is *hemi-Lebesgue integrable* on C if $[0,1] \ni t \mapsto f[(1-t)x + ty]$ is Lebesgue integrable on $[0,1]$ for all $(x,y) \in C^2 := C \times C$.

If $p : [0,1] \to [0,\infty)$ is Lebesgue integrable, then we can define the function $F_p : C \times C \to \mathbb{R}$ by

$$F_p(x,y) = \int_0^1 f((1-t)x + ty)\, p(t)\, dt. \tag{3.1}$$

For $p \equiv 1$ we can consider the function

$$F(x,y) = \int_0^1 f((1-t)x + ty)\, dt \tag{3.2}$$

for all $(x,y) \in C^2$.

Theorem 3. *Assume that the function $f : C \subseteq X \to [0,\infty)$ is an h-convex function with $h \in L[0,1]$. If f is hemi-Lebesgue integrable on C, then the function F_p defined by (3.1) is h-convex on C^2. Moreover, if p is symmetric on $[0,1]$ then $F_p(y,x) = F_p(x,y)$ for all $(x,y) \in C^2$, namely F_p is symmetric on C^2.*

Proof. Let $(x,y), (u,v) \in C^2$ and $\alpha \in [0,1]$. Then

$$\begin{aligned}
&F_p(\alpha(x,y) + (1-\alpha)(u,v)) \\
&= F_p(\alpha x + (1-\alpha)u, \alpha y + (1-\alpha)v) \\
&= \int_0^1 f((1-t)(\alpha x + (1-\alpha)u) + t(\alpha y + (1-\alpha)v))\, p(t)\, dt \\
&= \int_0^1 f(\alpha((1-t)x + ty) + (1-\alpha)((1-t)u + tv))\, p(t)\, dt \\
&\le \int_0^1 \{h(\alpha) f((1-t)x + ty) + h(1-\alpha) f((1-t)u + tv)\}\, p(t)\, dt \\
&\text{(by the } h\text{-convexity of } f) \\
&= h(\alpha)\int_0^1 f((1-t)x + ty)\, p(t)\, dt + h(1-\alpha)\int_0^1 f((1-t)u + tv)\, p(t)\, dt \\
&= h(\alpha) F_p(x,y) + h(1-\alpha) F_p(u,v),
\end{aligned}$$

which proves the convexity of F_p on C^2.

For $(x,y) \in C^2$, we have, by changing the variable $s = 1-t$, $t \in [0,1]$, that

$$\begin{aligned}
F_p(y,x) &= \int_0^1 f((1-t)y + tx)\, p(t)\, dt = \int_0^1 f(sy + (1-s)x)\, p(1-s)\, ds \\
&= \int_0^1 f((1-s)x + sy)\, p(s)\, ds = F_p(x,y)
\end{aligned}$$

and the theorem is proved. □

Corollary 2. *Assume that the function* $f : C \subseteq X \to [0, \infty)$ *is an* h*-convex function with* $h \in L[0,1]$ *and* f *is hemi-Lebesgue integrable on* C*. Then the functions*

$$F(x,y) := \int_0^1 f((1-t)x + ty)\,dt,$$

$$F_{\left|\cdot-\frac{1}{2}\right|}(x,y) := \int_0^1 f((1-t)x + ty)\left|t - \frac{1}{2}\right| dt$$

and

$$F_{\cdot(1-\cdot)}(x,y) := \int_0^1 f((1-t)x + ty)\,t(1-t)\,dt$$

are h*-convex and symmetric on* C^2*.*

We have:

Theorem 4. *Assume that the function* $f : C \subseteq X \to [0, \infty)$ *is an* h*-convex function with* $h \in L[0,1]$ *and* f *is hemi-Lebesgue integrable on* C*. If* p *is Lebesgue integrable and symmetric on* $[0,1]$*, then*

$$\begin{aligned}(3.3)\quad & \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{x+y}{2}\right) \int_0^1 p(t)\,dt \\ & \leq \int_0^1 \left(\int_0^1 f\left([(1-t)(1-\alpha) + \alpha t]\,x + [(1-t)\alpha + t(1-\alpha)]\,y \right) p(t)\,dt \right) d\alpha \\ & \leq 2 \int_0^1 f((1-t)x + ty)\,p(t)\,dt \int_0^1 h(\alpha)\,d\alpha \\ & \leq 2\,[f(x) + f(y)] \int_0^1 h(t)\,p(t)\,dt \int_0^1 h(\alpha)\,d\alpha\end{aligned}$$

for all $(x,y) \in C^2$*.*

Proof. From the inequality (2.6) for the h-convex F_p we have

$$\begin{aligned}(3.4)\quad & \frac{1}{2h\left(\frac{1}{2}\right)} F_p\left(\frac{(x,y)+(u,v)}{2}\right) \\ & \leq \int_0^1 \left[F_p((1-\alpha)(x,y) + \alpha(u,v)) \right] d\alpha \leq \left[F_p(x,y) + F_p(u,v) \right] \int_0^1 h(\alpha)\,d\alpha,\end{aligned}$$

for all (x,y), $(u,v) \in C^2$.

If we take $(u,v) = (y,x)$ in (3.4), then we get

$$\begin{aligned}(3.5)\quad & \frac{1}{2h\left(\frac{1}{2}\right)} F_p\left(\frac{(x,y)+(y,x)}{2}\right) \\ & \leq \int_0^1 \left[F_p((1-\alpha)(x,y) + \alpha(y,x)) \right] d\alpha \leq \left[F_p(x,y) + F_p(y,x) \right] \int_0^1 h(\alpha)\,d\alpha,\end{aligned}$$

for all $(x,y) \in C^2$.

Observe that

$$F_p\left(\frac{(x,y)+(y,x)}{2}\right) = F_p\left(\frac{x+y}{2}, \frac{x+y}{2}\right) = f\left(\frac{x+y}{2}\right) \int_0^1 p(t)\,dt,$$

$$
\begin{aligned}
&\int_0^1 [F_p((1-\alpha)(x,y)+\alpha(y,x))]\,d\alpha \\
&= \int_0^1 [F_p((1-\alpha)x+\alpha y,(1-\alpha)y+\alpha x)]\,d\alpha \\
&= \int_0^1 \left(\int_0^1 f((1-t)((1-\alpha)x+\alpha y)+t((1-\alpha)y+\alpha x))\,p(t)\,dt\right)d\alpha \\
&= \int_0^1 \left(\int_0^1 f([(1-t)(1-\alpha)+\alpha t]x+[(1-t)\alpha+t(1-\alpha)]y)\,p(t)\,dt\right)d\alpha
\end{aligned}
$$

and

$$
[F_p(x,y)+F_p(y,x)]\int_0^1 h(t)\,dt = 2\int_0^1 f((1-t)x+ty)\,p(t)\,dt\int_0^1 h(\alpha)\,d\alpha.
$$

Then by (3.5) we get the first part of (3.3).

Since

$$
\int_0^1 f((1-t)x+ty)\,p(t)\,dt \le [f(x)+f(y)]\int_0^1 h(t)\,p(t)\,dt,
$$

hence the last part of (3.3) also holds. □

Remark 3. *If the function $f : C \subseteq X \to [0,\infty)$ is a convex function, namely $h(t)=t$, $t\in[0,1]$ then from (3.3) we get*

$$
\begin{aligned}
(3.6)\quad & f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt \\
&\le \int_0^1 \left(\int_0^1 f([(1-t)(1-\alpha)+\alpha t]x+[(1-t)\alpha+t(1-\alpha)]y)\,p(t)\,dt\right)d\alpha \\
&\le \int_0^1 f((1-t)x+ty)\,p(t)\,dt
\end{aligned}
$$

provided that p is Lebesgue integrable and symmetric on $[0,1]$ and $(x,y)\in C^2$.

If we write (3.3) for the case of P-type functions $f : C \to [0,\infty)$, i.e., $h(t)=1$, $t\in[0,1]$, then we get the inequality

$$
\begin{aligned}
(3.7)\quad & \frac{1}{2}f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt \\
&\le \int_0^1 \left(\int_0^1 f([(1-t)(1-\alpha)+\alpha t]x+[(1-t)\alpha+t(1-\alpha)]y)\,p(t)\,dt\right)d\alpha \\
&\le 2\int_0^1 f((1-t)x+ty)\,p(t)\,dt \le 2[f(x)+f(y)]\int_0^1 p(t)\,dt,
\end{aligned}
$$

provided that p is Lebesgue integrable and symmetric on $[0,1]$, where $(x,y)\in C^2$.

If f is Breckner s-convex on C, for $s \in (0,1)$, then by taking $h(t) = t^s$ in (2.18) we get

$$
\begin{aligned}
(3.8)\quad & 2^{s-1} f\left(\frac{x+y}{2}\right) \int_0^1 p(t)\,dt \\
& \leq \int_0^1 \left(\int_0^1 f\left(\left[(1-t)(1-\alpha) + \alpha t\right] x + \left[(1-t)\alpha + t(1-\alpha)\right] y\right) p(t)\,dt \right) d\alpha \\
& \leq \frac{2}{s+1} \int_0^1 f\left((1-t)x + ty\right) p(t)\,dt \leq \frac{2}{s+1} \left[f(x) + f(y)\right] \int_0^1 t^s p(t)\,dt
\end{aligned}
$$

provided that p is Lebesgue integrable and symmetric on $[0,1]$, where $(x,y) \in C^2$.

If $(X, \|\cdot\|)$ is a normed linear space, $s \in (0,1)$, then for any $x, y \in X$ we have

$$
\begin{aligned}
(3.9)\quad & 2^{s-1} \left\|\frac{x+y}{2}\right\|^s \int_0^1 p(t)\,dt \\
& \leq \int_0^1 \left(\int_0^1 \left\|\left[(1-t)(1-\alpha) + \alpha t\right] x + \left[(1-t)\alpha + t(1-\alpha)\right] y\right\|^s p(t)\,dt \right) d\alpha \\
& \leq \frac{2}{s+1} \int_0^1 \left\|(1-t)x + ty\right\|^s p(t)\,dt \leq \frac{2}{s+1} \left[\|x\|^s + \|y\|^s\right] \int_0^1 t^s p(t)\,dt,
\end{aligned}
$$

provided that p is Lebesgue integrable and symmetric on $[0,1]$.

If $f : C \to [0,\infty)$ is of s-Godunova-Levin type, with $s \in [0,1)$, then by taking $h(t) = \frac{1}{t^s}$ in (3.3) we get

$$
\begin{aligned}
(3.10)\quad & \\
& \frac{1}{2^{s+1}} f\left(\frac{x+y}{2}\right) \int_0^1 p(t)\,dt \\
& \leq \int_0^1 \left(\int_0^1 f\left(\left[(1-t)(1-\alpha) + \alpha t\right] x + \left[(1-t)\alpha + t(1-\alpha)\right] y\right) p(t)\,dt \right) d\alpha \\
& \leq \frac{2}{1-s} \int_0^1 f\left((1-t)x + ty\right) p(t)\,dt \\
& \leq \frac{2}{1-s} \left[f(x) + f(y)\right] \int_0^1 \frac{p(t)}{t^s}\,dt,
\end{aligned}
$$

provided that p is Lebesgue integrable and symmetric on $[0,1]$, where $(x,y) \in C^2$.

4. Some Examples for Functions of a Real Variable

Let $g : I \to \mathbb{R}$ an integrable function on the interval I and $p : [0,1] \to [0,\infty)$ a symmetric and integrable function on $[0,1]$. For $(a,b) \in I^2$ we consider the function

$$
G_p(a,b) = \int_0^1 g\left((1-t)a + tb\right) p(t)\,dt. \tag{4.1}
$$

If $b = a$, then

$$
G_p(a,a) = g(a) \int_0^1 p(t)\,dt.
$$

If $a \neq b$, then by making the change of variable $u = (1-t)a + tb$, $t \in [0,1]$, we have $du = (b-a)\,dt$, $t = \frac{u-a}{b-a}$ and from (4.1) we obtain

$$G_p(a,b) = \begin{cases} \int_a^b g(u)\, p\left(\frac{u-a}{b-a}\right) du, & (a,b) \in I^2,\ a \neq b \\ \\ g(a) \int_0^1 p(t)\, dt, & (a,b) \in I^2,\ a = b. \end{cases} \tag{4.2}$$

In particular, we can consider the functions G, $G_{\left|\cdot - \frac{1}{2}\right|}$, $G_{\cdot(1-\cdot)} : I^2 \to \mathbb{R}$ defined by

$$G(a,b) := \begin{cases} \frac{1}{b-a} \int_a^b g(u)\, du, & (a,b) \in I^2,\ a \neq b, \\ \\ g(a), & (a,b) \in I^2,\ a \neq b, \end{cases}$$

$$G_{\left|\cdot - \frac{1}{2}\right|}(a,b) := \begin{cases} \frac{1}{(b-a)^2} \int_a^b g(u) \left|u - \frac{a+b}{2}\right| du, & (a,b) \in I^2,\ a \neq b, \\ \\ \frac{1}{4} g(a), & (a,b) \in I^2,\ a \neq b, \end{cases}$$

and

$$G_{\cdot(1-\cdot)}(a,b) := \begin{cases} \frac{1}{(b-a)^3} \int_a^b g(u)(u-a)(b-u)\, du, & (a,b) \in I^2,\ a \neq b, \\ \\ \frac{1}{6} g(a), & (a,b) \in I^2,\ a \neq b. \end{cases}$$

By utilising the general Theorem 3, we can state the following result concerning the h-convexity of the weighted integral mean (4.2):

Proposition 1. *Assume that the function $g : I \subseteq \mathbb{R} \to [0,\infty)$ is an h-convex function with $h \in L[0,1]$. If g is Lebesgue integrable on I, then the function G_p defined by (4.2) is h-convex on I^2. Moreover, if p is symmetric on $[0,1]$ then $G_p(b,a) = G_p(a,b)$ for all $(a,b) \in I^2$.*

We observe that, if g is a convex function on I, then G_p is convex on I^2. In the particular case when $p \equiv 1$, we recapture Wulbert's result from 2003, [53], who showed that the integral mean of a convex function is globally convex as a function of two variables.

The above proposition can be used as a simple tool to build h-convex functions (P-type functions, Breckner s-convex functions, s-Godunova-Levin type functions etc...) on $I^2 \subset \mathbb{R}^2$ starting with the same kind of function defined on I. The details are omitted.

References

[1] M. Alomari and M. Darus, The Hadamard's inequality for s-convex function. *Int. J. Math. Anal.* (Ruse) **2** (2008), no. 13-16, 639–646.

[2] M. Alomari and M. Darus, Hadamard-type inequalities for s-convex functions. *Int. Math. Forum* **3** (2008), no. 37-40, 1965–1975.

[3] G. A. Anastassiou, Univariate Ostrowski inequalities, revisited. *Monatsh. Math.,* **135** (2002), no. 3, 175–189.

[4] N. S. Barnett, P. Cerone, S. S. Dragomir, M. R. Pinheiro,and A. Sofo, Ostrowski type inequalities for functions whose modulus of the derivatives are convex and applications. *Inequality Theory and Applications*, **Vol. 2** (Chinju/Masan, 2001), 19–32, Nova Sci. Publ., Hauppauge, NY, 2003. Preprint: *RGMIA Res. Rep. Coll.* **5** (2002), No. 2, Art. 1 [Online http://rgmia.org/papers/v5n2/Paperwapp2q.pdf].

[5] E. F. Beckenbach, Convex functions, *Bull. Amer. Math. Soc.* **54**(1948), 439–460.

[6] M. Bombardelli and S. Varošanec, Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities. *Comput. Math. Appl.* **58** (2009), no. 9, 1869–1877.

[7] W. W. Breckner, Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen. (German) *Publ. Inst. Math.* (Beograd) (N.S.) **23(37)** (1978), 13–20.

[8] W. W. Breckner and G. Orbán, Continuity properties of rationally s-convex mappings with values in an ordered topological linear space. Universitatea "Babeş-Bolyai", Facultatea de Matematica, Cluj-Napoca, 1978. viii+92 pp.

[9] P. Cerone and S. S. Dragomir, Midpoint-type rules from an inequalities point of view, Ed. G. A. Anastassiou, *Handbook of Analytic-Computational Methods in Applied Mathematics*, CRC Press, New York. 135-200.

[10] P. Cerone and S. S. Dragomir, New bounds for the three-point rule involving the Riemann-Stieltjes integrals, in *Advances in Statistics Combinatorics and Related Areas*, C. Gulati, et al. (Eds.), World Science Publishing, 2002, 53-62.

[11] P. Cerone, S. S. Dragomir and J. Roumeliotis, Some Ostrowski type inequalities for n-time differentiable mappings and applications, *Demonstratio Mathematica*, **32**(2) (1999), 697—712.

[12] G. Cristescu, Hadamard type inequalities for convolution of h-convex functions. *Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity* **8** (2010), 3–11.

[13] S. S. Dragomir, Ostrowski's inequality for monotonous mappings and applications, *J. KSIAM,* **3**(1) (1999), 127-135.

[14] S. S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Comp. Math. Appl.,* **38** (1999), 33-37.

[15] S. S. Dragomir, On the Ostrowski's inequality for Riemann-Stieltjes integral, *Korean J. Appl. Math.,* **7** (2000), 477-485.

[16] S. S. Dragomir, On the Ostrowski's inequality for mappings of bounded variation and applications, *Math. Ineq. & Appl.,* **4**(1) (2001), 33-40.

[17] S. S. Dragomir, On the Ostrowski inequality for Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ where f is of Hölder type and u is of bounded variation and applications, *J. KSIAM,* **5**(1) (2001), 35-45.

[18] S. S. Dragomir, Ostrowski type inequalities for isotonic linear functionals, *J. Inequal. Pure & Appl. Math.,* **3**(5) (2002), Art. 68.

[19] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 2, Article 31, 8 pp.

[20] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.* **3** (2002), No. 2, Article 31.

[21] S. S. Dragomir, An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.* **3** (2002), No.3, Article 35.

[22] S. S. Dragomir, An Ostrowski like inequality for convex functions and applications, *Revista Math. Complutense*, **16**(2) (2003), 373-382.

[23] S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type.* Springer Briefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1

[24] S. S. Dragomir, Inequalities of Hermite-Hadamard type for h-convex functions on linear spaces, *Proyecciones Journal of Mathematics*, **34** (2015), No 4, pp. 323-341.

[25] S. S. Dragomir, P. Cerone, J. Roumeliotis and S. Wang, A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis, *Bull. Math. Soc. Sci. Math. Romanie,* **42(90)** (4) (1999), 301-314.

[26] S.S. Dragomir and S. Fitzpatrick, The Hadamard inequalities for s-convex functions in the second sense. *Demonstratio Math.* **32** (1999), no. 4, 687–696.

[27] S.S. Dragomir and S. Fitzpatrick,The Jensen inequality for s-Breckner convex functions in linear spaces. *Demonstratio Math.* **33** (2000), no. 1, 43–49.

[28] S. S. Dragomir and B. Mond, On Hadamard's inequality for a class of functions of Godunova and Levin. *Indian J. Math.* **39** (1997), no. 1, 1–9.

[29] S. S. Dragomir and C. E. M. Pearce, On Jensen's inequality for a class of functions of Godunova and Levin. *Period. Math. Hungar.* **33** (1996), no. 2, 93–100.

[30] S. S. Dragomir and C. E. M. Pearce, Quasi-convex functions and Hadamard's inequality, *Bull. Austral. Math. Soc.* **57** (1998), 377-385.
[31] S. S. Dragomir, J. Pečarić and L. Persson, Some inequalities of Hadamard type. *Soochow J. Math.* **21** (1995), no. 3, 335–341.
[32] S. S. Dragomir and Th. M. Rassias (Eds), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publisher, 2002.
[33] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28** (1997), 239-244.
[34] S. S. Dragomir and S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11** (1998), 105-109.
[35] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules, *Indian J. of Math.,* **40**(3) (1998), 245-304.
[36] A. El Farissi, Simple proof and refinement of Hermite-Hadamard inequality, *J. Math. Ineq.* **4** (2010), No. 3, 365–369.
[37] E. K. Godunova and V. I. Levin, Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions. (Russian) *Numerical mathematics and mathematical physics* (Russian), 138–142, 166, Moskov. Gos. Ped. Inst., Moscow, 1985
[38] H. Hudzik and L. Maligranda, Some remarks on s-convex functions. *Aequationes Math.* **48** (1994), no. 1, 100–111.
[39] E. Kikianty and S. S. Dragomir, Hermite-Hadamard's inequality and the p-HH-norm on the Cartesian product of two copies of a normed space, *Math. Inequal. Appl.* (in press)
[40] U. S. Kirmaci, M. Klaričić Bakula, M. E Özdemir and J. Pečarić, Hadamard-type inequalities for s-convex functions. *Appl. Math. Comput.* **193** (2007), no. 1, 26–35.
[41] M. A. Latif, On some inequalities for h-convex functions. *Int. J. Math. Anal.* (Ruse) **4** (2010), no. 29-32, 1473–1482.
[42] D. S. Mitrinović and I. B. Lacković, Hermite and convexity, *Aequationes Math.* **28** (1985), 229–232.
[43] D. S. Mitrinović and J. E. Pečarić, Note on a class of functions of Godunova and Levin. *C. R. Math. Rep. Acad. Sci. Canada* **12** (1990), no. 1, 33–36.
[44] C. E. M. Pearce and A. M. Rubinov, P-functions, quasi-convex functions, and Hadamard-type inequalities. *J. Math. Anal. Appl.* **240** (1999), no. 1, 92–104.
[45] J. E. Pečarić and S. S. Dragomir, On an inequality of Godunova-Levin and some refinements of Jensen integral inequality. *Itinerant Seminar on Functional Equations, Approximation and Convexity* (Cluj-Napoca, 1989), 263–268, Preprint, 89-6, Univ. "Babeş-Bolyai", Cluj-Napoca, 1989.
[46] J. Pečarić and S. S. Dragomir, A generalization of Hadamard's inequality for isotonic linear functionals, *Radovi Mat.* (Sarajevo) **7** (1991), 103–107.
[47] M. Radulescu, S. Radulescu and P. Alexandrescu, On the Godunova-Levin-Schur class of functions. *Math. Inequal. Appl.* **12** (2009), no. 4, 853–862.
[48] M. Z. Sarikaya, A. Saglam, and H. Yildirim, On some Hadamard-type inequalities for h-convex functions. *J. Math. Inequal.* **2** (2008), no. 3, 335–341.
[49] E. Set, M. E. Özdemir and M. Z. Sarıkaya, New inequalities of Ostrowski's type for s-convex functions in the second sense with applications. *Facta Univ. Ser. Math. Inform.* **27** (2012), no. 1, 67–82.
[50] M. Z. Sarikaya, E. Set and M. E. Özdemir, On some new inequalities of Hadamard type involving h-convex functions. *Acta Math. Univ. Comenian.* (N.S.) **79** (2010), no. 2, 265–272.
[51] M. Tunç, Ostrowski-type inequalities via h-convex functions with applications to special means. *J. Inequal. Appl.* **2013**, 2013:326.
[52] S. Varošanec, On h-convexity. *J. Math. Anal. Appl.* **326** (2007), no. 1, 303–311.
[53] D. E. Wulbert, Favard's inequality on average values of convex functions, *Mathematical and Computer Modelling*, Vol. **37** (2003), no. 12-13, pp. 1383–1391.

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