SCHUR CONVEXITY OF INTEGRAL MEANS

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Abstract. For a Lebesgue integrable function $p:[0,1]\to[0,\infty)$ we consider the function $S_{f,p}$, $M_{f,p}:D\to\mathbb{R}$ defined by

$$S_{f,p}\left(x,y\right)=\int_0^1 f\left(tx+\left(1-t\right)y,ty+\left(1-t\right)x\right)p\left(t\right)dt$$

and

$$\begin{aligned}M_{f,p}\left(x,y\right)&=\int_0^1 f\left(tx+\left(1-t\right)y,ty+\left(1-t\right)x\right)p\left(t\right)dt\\&\quad-f\left(\frac{x+y}{2},\frac{x+y}{2}\right)\int_0^1 p\left(t\right)dt,\end{aligned}$$

where $f:D\to\mathbb{R}$ is Schur convex on the symmetric convex subset D of a X^2, where X is a linear space. In this paper we show among others that $S_{f,p}$ and $M_{f,p}$ preserve the Schur convexity of f. We also provide some applications for norms and Schur convex functions of two real variable.

1. Introduction

For any $x=(x_1,...,x_n)\in\mathbb{R}^n$, let $x_{[1]}\geq...\geq x_{[n]}$ denote the components of x in decreasing order, and let $x_{\downarrow}=\left(x_{[1]},...,x_{[n]}\right)$ denote the decreasing rearrangement of x. For x, $y\in\mathbb{R}^n$, $x\prec y$ if, by definition,

$$\begin{cases}\sum_{i=1}^k x_{[i]}\leq\sum_{i=1}^k y_{[i]},\ k=1,...,n-1;\\ \\ \sum_{i=1}^n x_{[i]}=\sum_{i=1}^n y_{[i]}.\end{cases}$$

When $x\prec y$, x is said to be *majorized* by y (y *majorizes* x). This notation and terminology was introduced by Hardy, Littlewood and Pólya in 1934.

Functions that preserve the ordering of majorization are said to be Schur-convex. Perhaps "Schur-increasing" would be more appropriate, but the term "Schur-convex" is by now well entrenched in the literature, as mentioned in [8, p.80].

A real-valued function ϕ defined on a set $\mathcal{A}\subset\mathbb{R}^n$ is said to be *Schur-convex* on $\mathcal{A}$ if

$$x\prec y\text{ on }\mathcal{A}\Rightarrow\phi\left(x\right)\leq\phi\left(y\right). \tag{1.1}$$

If, in addition, $\phi\left(x\right)<\phi\left(y\right)$ whenever $x\prec y$ but x is not a permutation of y, then ϕ is said to be *strictly Schur-convex* on $\mathcal{A}$. If $\mathcal{A}=\mathbb{R}^n$, then ϕ is simply said to be Schur-convex or strictly Schur-convex.

For fundamental properties of Schur convexity see the monograph [8] and the references therein. For some recent results, see [3]-[6] and [9]-[11].

1991 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Convex functions, Schur convex functions, Integral inequalities, Hermite-Hadamard inequality.

RGMIA Res. Rep. Coll. 22 (2019), Art. 78, 13 pp. Received 07/08/19

Let $\mathcal{A} \subset \mathbb{R}^n$ be a set with the following properties:

(i) $\mathcal{A}$ is *symmetric* in the sense that $x \in \mathcal{A} \Rightarrow x\Pi \in \mathcal{A}$ for all permutations Π of the coordinates.

(ii) $\mathcal{A}$ is convex and has a nonempty interior.

We have the following result, [8, p. 85].

Theorem 1. *If ϕ is continuously differentiable on the interior of $\mathcal{A}$ and continuous on $\mathcal{A}$, then necessary and sufficient conditions for ϕ to be Schur-convex on $\mathcal{A}$ are*

$$\phi \text{ is symmetric on } \mathcal{A} \tag{1.2}$$

and

$$(z_1 - z_2)\left[\frac{\partial \phi(z)}{\partial x_1} - \frac{\partial \phi(z)}{\partial x_2}\right] \geq 0 \text{ for all } z \in \mathcal{A}. \tag{1.3}$$

Another interesting characterization of Schur convex functions ϕ on $\mathcal{A}$ was obtained by C. Stępniak in [12]:

Theorem 2. *Let ϕ be any function defined on a symmetric convex set $\mathcal{A}$ in $\mathbb{R}^n$. Then the function ϕ is Schur convex on $\mathcal{A}$ if and only if*

$$\phi(x_1, ..., x_i, ..., x_j, ..., x_n) = \phi(x_1, ..., x_j, ..., x_i, ..., x_n) \tag{1.4}$$

for all $(x_1, ..., x_n) \in \mathcal{A}$ and $1 \leq i < j \leq n$ and

$$\phi(\lambda x_1 + (1-\lambda) x_2, \lambda x_2 + (1-\lambda) x_1, x_3, ..., x_n) \leq \phi(x_1, ..., x_n) \tag{1.5}$$

for all $(x_1, ..., x_n) \in \mathcal{A}$ and for all $\lambda \in (0,1)$,

It is well known that any symmetric convex function defined on a symmetric convex set $\mathcal{A}$ is Schur convex, [8, p. 97]. If the function $\phi : \mathcal{A} \to \mathbb{R}$ is symmetric and quasi-convex, namely

$$\phi(\alpha u + (1-\alpha) v) \leq \max\{\phi(u), \phi(v)\}$$

for all $\alpha \in [0,1]$ and u, $v \in \mathcal{A}$, a symmetric convex set, then ϕ is Schur convex on $\mathcal{A}$ [8, p. 98].

Let X be a linear space and $G \subset X^2 := X \times X$ a convex set. We say that G is *symmetric* if $(x, y) \in G$ implies that $(y, x) \in G$. If $D \subset X$ is a convex subset of X, then the Cartesian product $G := D^2 := D \times D$ is convex and symmetric in X^2.

Motivated by the characterization result of Stępniak above, we say that a function $f : G \to \mathbb{R}$ will be called *Schur convex* on the convex and symmetric set $G \subset X^2$ if

$$f(t(x, y) + (1-t)(y, x)) \leq f(x, y) \tag{1.6}$$

for all $(x, y) \in G$ and for all $t \in [0,1]$.

If $G = D^2$ then we recapture the general concept of Schur convexity introduced by Burai and Makó in 2016, [1].

We say that the function $f : G \to \mathbb{R}$ is symmetric on G if $f(x, y) = f(y, x)$ for all $(x, y) \in G$. If the function f is symmetric on G and the inequality holds for a given $t \in (0,1)$ and for all $(x, y) \in G$, then we say that f is *t-Schur convex* on G.

The following fact follows from the definition of Schur convex functions:

Proposition 1. *If $f : G \to \mathbb{R}$ is Schur convex on the convex and symmetric set $G \subset X^2$, then f is symmetric on G.*

For $(x,y) \in G$, as in [1], let us define the following auxiliary function $\varphi_{(x,y)}$: $[0,1] \to R$ by

$$\varphi_{f,(x,y)}(t) = f(t(x,y) + (1-t)(y,x)) = f(tx + (1-t)y, ty + (1-t)x). \tag{1.7}$$

The properties of this function are as follows [4]:

Lemma 1. *Let $G \subset X^2$ be a convex and symmetric set and $f : G \to \mathbb{R}$ a symmetric function on G. Then f is Schur convex on G if and only if for all arbitrarily fixed $(x,y) \in G$ the function $\varphi_{f,(x,y)}$ is monotone decreasing on $[0,1/2)$, monotone increasing on $(1/2,1]$, and $\varphi_{f,(x,y)}$ has a global minimum at $1/2$.*

The proof of this result in the case of $G = D^2$ was given in [1].

We have the following weighted double integral inequality [4]:

Theorem 3. *Assume that the function $f : G \to \mathbb{R}$ is Schur convex on the convex and symmetric set $G \subset X^2$. Then for any Lebesgue integrable function $w : [0,1] \to [0,\infty)$ we have*

$$\begin{aligned} & f\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \int_0^1 w(s)\,ds \\ & \leq \int_0^1 f(sx + (1-s)y, sy + (1-s)x)\, w(s)\,ds \\ & \leq f(x,y) \int_0^1 w(s)\,ds \end{aligned} \tag{1.8}$$

for all $(x,y) \in G$.

In particular, we have

$$f\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \leq \int_0^1 f(sx + (1-s)y, sy + (1-s)x)\,ds \leq f(x,y) \tag{1.9}$$

for all $(x,y) \in G$.

For a Lebesgue integrable function $p : [0,1] \to [0,\infty)$ we consider the function $S_{f,p}$, $M_{f,p} : D \to \mathbb{R}$ defined by

$$S_{f,p}(x,y) = \int_0^1 f(tx + (1-t)y, ty + (1-t)x)\, p(t)\,dt$$

and

$$\begin{aligned} M_{f,p}(x,y) = & \int_0^1 f(tx + (1-t)y, ty + (1-t)x)\, p(t)\,dt \\ & - f\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \int_0^1 p(t)\,dt, \end{aligned}$$

where $f : D \to \mathbb{R}$ is Schur convex on the symmetric convex subset D of a X^2, where X is a linear space.

Motivated by the above results, in this paper we show among others that $S_{f,p}$ and $M_{f,p}$ preserve the Schur convexity of f. We also provide some applications for Schur convex and convex functions of two real variable.

2. Schur Convexity for Functions of Composite Arguments

Assume that the function $f : G \to \mathbb{R}$ is *Schur convex* on the convex and symmetric set $G \subset X^2$. For $t \in [0,1]$, we define the function $S_{f,t} : G \to \mathbb{R}$ defined by

$$S_{f,t}(x,y) := f(t(x,y) + (1-t)(y,x)) = f(tx + (1-t)y, ty + (1-t)x). \tag{2.1}$$

In the case when $t = 0$ or $t = 1$ the definition (2.1) becomes, by the symmetry of f in G, that

$$S_{f,0}(x,y) = S_{f,1}(x,y) = f(x,y), \ \ (x,y) \in G.$$

We have:

Theorem 4. *Assume that the function $f : G \to \mathbb{R}$ is Schur convex on G then $S_{f,t}$ is Schur convex on G for all $t \in (0,1)$.*

Proof. Let $(x,y) \in G$ and $s \in [0,1]$, $t \in (0,1)$. Observe that

$$\begin{aligned}
& t(sx + (1-s)y, sy + (1-s)x) + (1-t)(sy + (1-s)x, sx + (1-s)y) \\
& = t(s(x,y) + (1-s)(y,x)) + (1-t)(s(y,x) + (1-s)(x,y)) \\
& = s[t(x,y) + (1-t)(y,x)] + (1-s)[t(y,x) + (1-t)(x,y)] \\
& = s(tx + (1-t)y, ty + (1-t)x) + (1-s)[(ty + (1-t)x, tx + (1-t)y)] \\
& = s(u,v) + (1-s)(v,u),
\end{aligned}$$

where $u := tx + (1-t)y$ and $v := ty + (1-t)x$ for all $(x,y) \in G$ and $s,\, t \in [0,1]$.

By Schur convexity of f on G we get

$$f(s(u,v) + (1-s)(v,u)) \le f(u,v)$$

for all $s \in [0,1]$.

Therefore

$$\begin{aligned}
& S_{f,t}(s(x,y) + (1-s)(y,x)) \\
& = f[t(sx + (1-s)y, sy + (1-s)x) + (1-t)(sy + (1-s)x, sx + (1-s)y)] \\
& \qquad \le f(tx + (1-t)y, ty + (1-t)x) = S_{f,t}(x,y)
\end{aligned} \tag{2.2}$$

for $(x,y) \in G$ and $s,\, t \in [0,1]$.

This proves the Schur convexity of $S_{f,t}$ on G. □

We define for $t \in [0,1]$, $t \neq \frac{1}{2}$ the function $M_{f,t}$ on G by

$$\begin{aligned}
M_{f,t}(x,y) &:= f(t(x,y) + (1-t)(y,x)) - f\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \\
&= f(tx + (1-t)y, ty + (1-t)x) - f\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \\
&= S_{f,t}(x,y) - f\left(\frac{x+y}{2}, \frac{x+y}{2}\right),
\end{aligned}$$

where $f : G \to \mathbb{R}$ is Schur convex on the convex and symmetric subset $G \subset X^2$.

We have the following result.

Corollary 1. *Let f be a Schur convex function on D and $t \in [0,1]$, $t \neq \frac{1}{2}$. Then the function $M_{f,t}$ is Schur convex on D.*

Proof. Let $s \in [0,1]$ and $(x,y) \in G$. Then

$$\begin{aligned}
&M_{f,t}\left(s\left(x,y\right)+\left(1-s\right)\left(y,x\right)\right)\\
&=S_{f,t}\left(s\left(x,y\right)+\left(1-s\right)\left(y,x\right)\right)\\
&-f\left(\frac{sx+\left(1-s\right)y+sy+\left(1-s\right)x}{2},\frac{sx+\left(1-s\right)y+sy+\left(1-s\right)x}{2}\right)\\
&=M_{f,t}\left(s\left(x,y\right)+\left(1-s\right)\left(y,x\right)\right)-f\left(\frac{x+y}{2},\frac{x+y}{2}\right)\\
&\le S_{f,t}\left(x,y\right)-f\left(\frac{x+y}{2},\frac{x+y}{2}\right)=M_{f,t}\left(x,y\right),
\end{aligned}$$

which proves the Schur convexity of $M_{f,t}$ on D. □

Let $(X,\|\cdot\|)$ be a normed space. The function $f(x)=\|x\|^r$, $r\ge 1$ is convex on X. Assume that $q:[0,1]\to[0,\infty)$ is a Lebesgue integrable symmetric function on $[0,1]$. If we define

$$N_{r,q}\left(x,y\right):=\int_0^1\left\|\left(1-\tau\right)x+\tau y\right\|^r q\left(\tau\right)d\tau, \tag{2.3}$$

then we know that $N_{r,p}$ is globally convex on X^2, see [5].

For $t\in[0,1]$, we define

$$\begin{aligned}
S_{r,q,t}\left(x,y\right)&:=N_{r,q}\left(tx+\left(1-t\right)y,ty+\left(1-t\right)x\right)\\
&=\int_0^1\left\|\left(1-\tau\right)\left(tx+\left(1-t\right)y\right)+\tau\left(ty+\left(1-t\right)x\right)\right\|^r q\left(\tau\right)d\tau\\
&=\int_0^1\left\|\left[\left(1-\tau\right)t+\tau\left(1-t\right)\right]x+\left[\left(1-\tau\right)\left(1-t\right)+\tau t\right]y\right\|^r q\left(\tau\right)d\tau
\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
&M_{r,q,t}\left(x,y\right)\\
&:=N_{r,q}\left(tx+\left(1-t\right)y,ty+\left(1-t\right)x\right)-\left\|\frac{x+y}{2}\right\|^r\int_0^1 q\left(\tau\right)d\tau\\
&=\int_0^1\left\|\left[\left(1-\tau\right)t+\tau\left(1-t\right)\right]x+\left[\left(1-\tau\right)\left(1-t\right)+\tau t\right]y\right\|^r q\left(\tau\right)d\tau\\
&-\left\|\frac{x+y}{2}\right\|^r\int_0^1 q\left(\tau\right)d\tau,
\end{aligned} \tag{2.5}$$

where $r\ge 1$ and $(x,y)\in X^2$.

By utilising Theorem 4, we can state the following result:

Proposition 2. *Assume that $q:[0,1]\to[0,\infty)$ is a Lebesgue integrable symmetric function on $[0,1]$, $t\in[0,1]$ and $r\ge 1$. Then the functions $S_{r,q,t}$ and $M_{r,q,t}$ are Schur convex on X^2.*

Let C be a convex subset in X and $f:C^2:=C\times C\to\mathbb{R}$. For $(t,s)\in[0,1]^2$ we consider the function $P_{f,(t,s)}:C^2\to\mathbb{R}$ defined by

$$\begin{aligned}
&P_{f,(t,s)}\left(x,y\right)\\
&:=\frac{1}{2}\left[f\left(tx+\left(1-t\right)y,sx+\left(1-s\right)y\right)+f\left(\left(1-t\right)x+ty,sy+\left(1-s\right)x\right)\right],
\end{aligned}$$

where $(x,y)\in C^2$.

Theorem 5. *Assume that $f:C^2\to R$ is convex on C^2 and $(t,s)\in[0,1]^2$. Then the function $P_{f,(t,s)}$ is Schur convex on C^2.*

Proof. Let $\alpha,\ \beta\geq 0$ with $\alpha+\beta=1$, $(x,y)\in C^2$ and consider

$$\begin{aligned}
&2P_{(t,s)}\left(\alpha\left(x,y\right)+\beta\left(y,x\right)\right)\\
&=P_{(t,s)}\left(\alpha x+\beta y,\alpha y+\beta x\right)\\
&=f\left(t\left(\alpha x+\beta y\right)+\left(1-t\right)\left(\alpha y+\beta x\right),s\left(\alpha x+\beta y\right)+\left(1-s\right)\left(\alpha y+\beta x\right)\right)\\
&+f\left(\left(1-t\right)\left(\alpha x+\beta y\right)+t\left(\alpha y+\beta x\right),s\left(\alpha y+\beta x\right)+\left(1-s\right)\left(\alpha x+\beta y\right)\right).
\end{aligned}$$

Observe that

$$\begin{aligned}
&\left(t\left(\alpha x+\beta y\right)+\left(1-t\right)\left(\alpha y+\beta x\right),s\left(\alpha x+\beta y\right)+\left(1-s\right)\left(\alpha y+\beta x\right)\right)\\
&=\alpha\left(tx+\left(1-t\right)y,sx+\left(1-s\right)y\right)+\beta\left(ty+\left(1-t\right)x,sy+\left(1-s\right)x\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\left(1-t\right)\left(\alpha x+\beta y\right)+t\left(\alpha y+\beta x\right),s\left(\alpha y+\beta x\right)+\left(1-s\right)\left(\alpha x+\beta y\right)\right)\\
&=\alpha\left(\left(1-t\right)x+ty,sy+\left(1-s\right)x\right)+\beta\left(\left(1-t\right)y+tx,sx+\left(1-s\right)y\right).
\end{aligned}$$

Since f is convex on D, hence

$$\begin{aligned}
&f\left[\alpha\left(tx+\left(1-t\right)y,sx+\left(1-s\right)y\right)+\beta\left(ty+\left(1-t\right)x,sy+\left(1-s\right)x\right)\right]\\
&\leq\alpha f\left(tx+\left(1-t\right)y,sx+\left(1-s\right)y\right)+\beta f\left(ty+\left(1-t\right)x,sy+\left(1-s\right)x\right)
\end{aligned}$$

and

$$\begin{aligned}
&f\left[\alpha\left(\left(1-t\right)x+ty,sy+\left(1-s\right)x\right)+\beta\left(\left(1-t\right)y+tx,sx+\left(1-s\right)y\right)\right]\\
&\leq\alpha f\left(\left(1-t\right)x+ty,sy+\left(1-s\right)x\right)+\beta f\left(\left(1-t\right)y+tx,sx+\left(1-s\right)y\right).
\end{aligned}$$

If we add these two inequalities, we get

$$\begin{aligned}
2P_{(t,s)}\left(\alpha\left(x,y\right)+\beta\left(y,x\right)\right)&\leq\alpha f\left(tx+\left(1-t\right)y,sx+\left(1-s\right)y\right)\\
&+\beta f\left(\left(1-t\right)y+tx,sx+\left(1-s\right)y\right)\\
&+\beta f\left(ty+\left(1-t\right)x,sy+\left(1-s\right)x\right)\\
&+\alpha f\left(\left(1-t\right)x+ty,sy+\left(1-s\right)x\right)\\
&=f\left(tx+\left(1-t\right)y,sx+\left(1-s\right)y\right)\\
&+f\left(ty+\left(1-t\right)x,sy+\left(1-s\right)x\right)=2P_{(t,s)}\left(x,y\right),
\end{aligned}$$

which shows that $P_{(t,s)}$ is Schur convex on C^2. □

For $(t,s)\in[0,1]^2$ we also consider the function $Q_{f,(t,s)}:C^2\to\mathbb{R}$ defined by

$$\begin{aligned}
&Q_{f,(t,s)}\left(x,y\right)\\
&:=P_{f,(t,s)}\left(x,y\right)-P_{f,(t,s)}\left(\frac{x+y}{2},\frac{x+y}{2}\right)\\
&=\frac{1}{2}\left[f\left(tx+\left(1-t\right)y,sx+\left(1-s\right)y\right)+f\left(\left(1-t\right)x+ty,sy+\left(1-s\right)x\right)\right]\\
&-f\left(\frac{x+y}{2},\frac{x+y}{2}\right).
\end{aligned}$$

Corollary 2. *Assume that* $f : C^2 \to R$ *is convex on* C^2 *and* $(t,s) \in [0,1]^2$. *Then the function* $Q_{(t,s)}$ *is Schur convex on* C^2.

3. Schur Convexity of Integral Mean

For a Lebesgue integrable function $p : [0,1] \to [0,\infty)$ and a Schur convex function $f : G \to \mathbb{R}$ on the convex and symmetric set $G \subset X^2$ we define the functions $S_{f,p}$ and $M_{f,p}$ on G by

$$\begin{aligned} S_{f,p}(x,y) &:= \int_0^1 S_{f,t}(x,y)\, p(t)\, dt \\ &= \int_0^1 f(tx + (1-t)y, ty + (1-t)x)\, p(t)\, dt \end{aligned}$$

and

$$\begin{aligned} M_{f,p}(x,y) &:= \int_0^1 f(tx + (1-t)y, ty + (1-t)x)\, p(t)\, dt \\ &\quad - f\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \int_0^1 p(t)\, dt. \end{aligned}$$

In particular, if $p \equiv 1$, then we also consider the functions

$$S_f(x,y) := \int_0^1 f(tx + (1-t)y, ty + (1-t)x)\, dt$$

and

$$M_f(x,y) := \int_0^1 f(tx + (1-t)y, ty + (1-t)x)\, dt - f\left(\frac{x+y}{2}, \frac{x+y}{2}\right).$$

We have:

Theorem 6. *Assume that the function* $f : G \to \mathbb{R}$ *is Schur convex on* G *and* $p : [0,1] \to [0,\infty)$ *is a Lebesgue integrable function on* $[0,1]$, *then the functions* $S_{f,p}$ *and* $M_{f,p}$ *are Schur convex on* G.

Proof. Let $s \in [0,1]$ and $(x,y) \in G$. Then, by the Schur convexity of $S_{f,t}$ for $t \in [0,1]$, we have

$$\begin{aligned} S_{f,p}(s(x,y) + (1-s)(y,x)) &= \int_0^1 S_{f,t}(s(x,y) + (1-s)(y,x))\, p(t)\, dt \\ &\le \int_0^1 S_{f,t}(x,y)\, p(t)\, dt = S_{f,p}(x,y), \end{aligned}$$

which proves the Schur convexity of $S_{f,p}$.

The proof for $M_{f,p}$ is similar. □

Corollary 3. *Assume that the function* $f : G \to \mathbb{R}$ *is Schur convex on* G, *then the functions* S_f *and* M_f *are Schur convex on* G.

We also have the following double integral inequalities:

Corollary 4. *Assume that the function $f : G \to \mathbb{R}$ is Schur convex on the convex and symmetric set $G \subset X^2$. Then for any Lebesgue integrable functions w, $p : [0,1] \to [0,\infty)$ we have*

$$\begin{aligned}
(3.1)\qquad & f\left(\frac{x+y}{2},\frac{x+y}{2}\right)\int_0^1 p(t)\,dt\int_0^1 w(s)\,ds \\
&\le \int_0^1\int_0^1 f\left[t(sx+(1-s)y)+(1-t)(sy+(1-s)x),\right. \\
&\left. t(sy+(1-s)x)+(1-t)(sx+(1-s)y)\right]p(t)\,w(s)\,dtds \\
&\le \int_0^1 f(tx+(1-t)y,ty+(1-t)x)\,p(t)\,dt\int_0^1 w(s)\,ds \\
&\left(\le f(x,y)\int_0^1 p(t)\,dt\int_0^1 w(s)\,ds\right)
\end{aligned}$$

for all $(x,y) \in G$.

The proof follows by Theorem 3 applied for the function $S_{f,p}$. This is a refinement of the inequality (1.8) from Introduction.

For p, $w \equiv 1$ we get for $(x,y) \in G$ that

$$\begin{aligned}
(3.2)\qquad & f\left(\frac{x+y}{2},\frac{x+y}{2}\right) \\
&\le \int_0^1\int_0^1 f\left[t(sx+(1-s)y)+(1-t)(sy+(1-s)x),\right. \\
&\left. t(sy+(1-s)x)+(1-t)(sx+(1-s)y)\right]dtds \\
&\le \int_0^1 f(tx+(1-t)y,ty+(1-t)x)\,dt\ \ (\le f(x,y)),
\end{aligned}$$

where $f : G \to \mathbb{R}$ is *Schur convex* on the convex and symmetric set $G \subset X^2$. This is a refinement of the inequality (1.9) from Introduction.

Let $(X,\|\cdot\|)$ be a normed space. The function $f(x) = \|x\|^r$, $r \ge 1$ is convex on X. Assume that $q : [0,1] \to [0,\infty)$ is a Lebesgue integrable *symmetric* function on $[0,1]$. For a Lebesgue integrable function $p : [0,1] \to [0,\infty)$, we can define the norm related functions

$$\begin{aligned}
& S_{r,q,p}(x,y) \\
&:= \int_0^1 N_{r,q}(tx+(1-t)y,ty+(1-t)x)\,p(t)\,dt \\
&= \int_0^1\int_0^1 \|[(1-\tau)t+\tau(1-t)]x+[(1-\tau)(1-t)+\tau t]y\|^r q(\tau)\,p(t)\,d\tau dt
\end{aligned}$$

and

$$\begin{aligned}
& M_{r,q,p}(x,y) \\
&:= \int_0^1\int_0^1 \|[(1-\tau)t+\tau(1-t)]x+[(1-\tau)(1-t)+\tau t]y\|^r q(\tau)\,p(t)\,d\tau dt \\
&- \left\|\frac{x+y}{2}\right\|^r \int_0^1 q(\tau)\,d\tau\int_0^1 p(t)\,dt.
\end{aligned}$$

By making use of Theorem 7 we have the following result:

Proposition 3. *Assume that* $q : [0,1] \to [0,\infty)$ *is a Lebesgue integrable symmetric function on* $[0,1]$, $p : [0,1] \to [0,\infty)$ *is Lebesgue integrable on* $[0,1]$ *and* $r \geq 1$. *Then the functions* $S_{r,q,p}$ *and* $M_{r,q,p}$ *are Schur convex on* X^2.

Consider the two variable weight $W : [0,1]^2 \to [0,\infty)$ that is Lebesgue integrable on $[0,1]^2$ and define

$$\begin{aligned}P_{f,W}(x,y) &:= \int_0^1 \int_0^1 P_{f,(t,s)}(x,y)\, W(t,s)\, dtds \\ &= \frac{1}{2}\int_0^1 \int_0^1 f(tx+(1-t)y, sx+(1-s)y)\, W(t,s)\, dtds \\ &+ \frac{1}{2}\int_0^1 \int_0^1 f((1-t)x+ty, sy+(1-s)x)\, W(t,s)\, dtds.\end{aligned}$$

If W is symmetric on $[0,1]^2$ in the sense that $W(t,s) = W(s,t)$ for all $(t,s) \in [0,1]^2$, then

$$P_{f,W}(x,y) = \int_0^1 \int_0^1 f(tx+(1-t)y, sx+(1-s)y)\, W(t,s)\, dtds.$$

In particular, if $w : [0,1] \to [0,\infty)$ is Lebesgue integrable on $[0,1]$, then by taking $W(t,s) = w(t)\, w(s)$, $(t,s) \in [0,1]^2$ we can also consider the function

$$P_{f,w}(x,y) = \int_0^1 \int_0^1 f(tx+(1-t)y, sx+(1-s)y)\, w(t)\, w(s)\, dtds$$

and the unweighted function

$$P_f(x,y) = \int_0^1 \int_0^1 f(tx+(1-t)y, sx+(1-s)y)\, dtds.$$

In a similar way, we can consider

$$Q_{f,W}(x,y) := P_{f,W}(x,y) - f\left(\frac{x+y}{2}, \frac{x+y}{2}\right)\int_0^1 \int_0^1 W(t,s)\, dtds,$$

$$Q_{f,w}(x,y) := P_{f,w}(x,y) - f\left(\frac{x+y}{2}, \frac{x+y}{2}\right)\left(\int_0^1 w(t)\, dt\right)^2,$$

and

$$Q_f(x,y) := P_f(x,y) - f\left(\frac{x+y}{2}, \frac{x+y}{2}\right).$$

Theorem 7. *Assume that* $f : C^2 \to R$ *is convex on* C^2 *and* $W : [0,1]^2 \to [0,\infty)$ *is Lebesgue integrable on* $[0,1]^2$, *then* $P_{f,W}$ *and* $Q_{f,W}$ *are Schur convex on* C^2.

Proof. Let $\alpha \in [0,1]$ and $(x,y) \in G$. Then, by the Schur convexity of $P_{f,(t,s)}$ for $(t,s) \in [0,1]^2$, we have

$$\begin{aligned}&P_{f,W}(\alpha(x,y) + (1-\alpha)(y,x)) \\ &= \int_0^1 \int_0^1 P_{f,(t,s)}(\alpha(x,y) + (1-\alpha)(y,x))\, W(t,s)\, dtds \\ &\leq \int_0^1 \int_0^1 P_{f,(t,s)}(x,y)\, W(t,s)\, dtds = P_{f,W}(x,y),\end{aligned}$$

which proves the Schur convexity of $P_{f,W}$.

The Schur convexity of $Q_{f,W}$ goes in a similar way. □

Corollary 5. *Assume that $f : C^2 \to R$ is convex on C^2 and $w : [0,1] \to [0,\infty)$ is Lebesgue integrable on $[0,1]$, then $P_{f,w}$ and $Q_{f,w}$ are Schur convex on C^2. In particular, P_f and Q_f are Schur convex on C^2.*

4. Examples for Functions of Two Real Variables

For a Lebesgue integrable function $p : [0,1] \to [0,\infty)$ and a Schur convex function $f : I^2 \to \mathbb{R}$ where I is an interval of real numbers, by changing the variable

$$u = (1-t)\,a + tb,\ t \in [0,1] \text{ with } (a,b) \in I^2 \text{ and } a \neq b$$

we can express the functions $S_{f,p}$ and $M_{f,p}$ on I^2 by

$$\begin{aligned} S_{f,p}(a,b) &= \int_0^1 f\left(ta + (1-t)\,b, tb + (1-t)\,a\right) p(t)\,dt \\ &= \frac{1}{b-a}\int_a^b f(u, a+b-u)\,p\left(\frac{u-a}{b-a}\right) du \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} M_{f,p}(a,b) &= \int_0^1 f\left(ta + (1-t)\,b, tb + (1-t)\,a\right) p(t)\,dt \\ &\quad - f\left(\frac{a+b}{2}, \frac{a+b}{2}\right)\int_0^1 p(t)\,dt \\ &= \frac{1}{b-a}\int_a^b f(u, a+b-u)\,p\left(\frac{u-a}{b-a}\right) du \\ &\quad - f\left(\frac{a+b}{2}, \frac{a+b}{2}\right)\int_0^1 p(t)\,dt. \end{aligned} \tag{4.2}$$

For $(a,b) \in I^2$ with $a = b$ we have

$$S_{f,p}(a,a) = f(a,a)\int_0^1 p(t)\,dt \text{ and } M_{f,p}(a,a) = 0. \tag{4.3}$$

In particular, if $p \equiv 1$, then we also consider the functions

$$S_f(a,b) := \begin{cases} \frac{1}{b-a}\int_a^b f(u, a+b-u)\,du \text{ for } (a,b) \in I^2 \text{ with } a \neq b, \\ \\ f(a,a) \text{ for } (a,b) \in I^2 \text{ with } a = b \end{cases} \tag{4.4}$$

and

$$M_f(a,b) = \begin{cases} \frac{1}{b-a}\int_a^b f(u, a+b-u)\,du - f\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \\ \text{for } (a,b) \in I^2 \text{ with } a \neq b, \\ \\ 0 \text{ for } (a,b) \in I^2 \text{ with } a = b. \end{cases} \tag{4.5}$$

Proposition 4. *Assume that $f : I^2 \to \mathbb{R}$ is Schur convex on I^2 and $p : [0,1] \to [0,\infty)$ is Lebesgue integrable on $[0,1]$, then $S_{f,p}$ and $M_{f,p}$ defined by (4.1)-(4.3) are Schur convex on I^2. In particular, the functions S_f and M_f defined by (4.4) and (4.5) are Schur convex on I^2.*

If $w : [0,1] \to [0,\infty)$ is Lebesgue integrable on $[0,1]$ and $f : I^2 \to \mathbb{R}$ is convex on I^2, then by changing the variables $tb + (1-t)a = u$ and $sb + (1-s)a = v$ and we can also consider the function

$$P_{f,w}(a,b) := \frac{1}{(b-a)^2} \int_a^b \int_a^b f(u,v)\, w\left(\frac{u-a}{b-a}\right) w\left(\frac{v-a}{b-a}\right) dudv \tag{4.6}$$

if $(a,b) \in I^2$ with $a \neq b$ and

$$P_{f,w}(a,a) := f(a,a) \left(\int_0^1 w(t)\, dt\right)^2. \tag{4.7}$$

We also can consider

$$\begin{aligned} Q_{f,w}(a,b) := &\frac{1}{(b-a)^2} \int_a^b \int_a^b f(u,v)\, w\left(\frac{u-a}{b-a}\right) w\left(\frac{v-a}{b-a}\right) dudv \\ &- f\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \left(\int_0^1 w(t)\, dt\right)^2 \end{aligned} \tag{4.8}$$

if $(a,b) \in I^2$ with $a \neq b$ and

$$Q_{f,w}(a,a) := 0. \tag{4.9}$$

In particular, we have

$$P_f(a,b) := \begin{cases} \frac{1}{(b-a)^2} \int_a^b \int_a^b f(u,v)\, dudv \text{ if } (a,b) \in I^2 \text{ with } a \neq b, \\ \\ f(a,a) \text{ if } (a,b) \in I^2 \text{ with } a \neq b \end{cases} \tag{4.10}$$

and

$$Q_f(a,b) := \begin{cases} \frac{1}{(b-a)^2} \int_a^b \int_a^b f(u,v)\, dudv - f\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \\ \text{if } (a,b) \in I^2 \text{ with } a \neq b, \\ \\ 0 \text{ if } (a,b) \in I^2 \text{ with } a \neq b. \end{cases} \tag{4.11}$$

Proposition 5. *Assume that $f : I^2 \to \mathbb{R}$ is convex on I^2 and $w : [0,1] \to [0,\infty)$ is Lebesgue integrable on $[0,1]$, then $P_{f,w}$ and $Q_{f,w}$ defined by (4.6)-(4.9) are Schur convex on I^2. In particular, the functions S_f and M_f defined by (4.10) and (4.11) are Schur convex on I^2.*

In [2] Chu et al. obtained the following results:

Lemma 2. *Suppose $h : I \to \mathbb{R}$ is a continuous function. Function*

$$M_h(x,y) := \begin{cases} \frac{1}{y-x} \int_x^y h(t)\, dt - h\left(\frac{x+y}{2}\right), & (x,y) \in I^2,\ x \neq y \\ \\ 0, & (x,y) \in I^2,\ x = y \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if and only if h is convex (concave) on I. Furthermore, function

$$T_h(x,y) := \begin{cases} \frac{h(x)+h(y)}{2} - \frac{1}{y-x} \int_x^y h(t)\, dt, & (x,y) \in I^2,\ x \neq y \\ \\ 0, (x,y) \in I^2,\ x = y \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if and only if h is convex (concave) on I.

If we take $f_h : I^2 \to \mathbb{R}$ defined as $f_h(x,y) = M_h(x,y)$ then $S_{f_h,p}$ defined by (4.1) becomes

$$
\begin{aligned}
& S_{f_h,p}(a,b) \\
&= \frac{1}{b-a} \int_a^b f_h(u, a+b-u)\, p\left(\frac{u-a}{b-a}\right) du \\
&= \frac{1}{b-a} \int_a^b \left[\frac{1}{a+b-2u} \int_u^{a+b-u} h(t)\, dt - h\left(\frac{a+b}{2}\right)\right] p\left(\frac{u-a}{b-a}\right) du \\
&= \frac{1}{b-a} \int_a^b \frac{1}{a+b-2u} \left(\int_u^{a+b-u} h(t)\, dt\right) p\left(\frac{u-a}{b-a}\right) du \\
&- \frac{1}{b-a} h\left(\frac{a+b}{2}\right) \int_a^b p\left(\frac{u-a}{b-a}\right) du
\end{aligned}
$$

for $a \neq b$ and $S_{f_h,p}(a,a) = 0$ with $a,\ b \in I$.

Therefore, by Proposition 4.1 we conclude that $S_{f_h,p}$ *is Schur convex on* I^2 provided that h is continuous convex on I and $p : [0,1] \to [0,\infty)$ is Lebesgue integrable on $[0,1]$.

In particular

$$
S_{f_h}(a,b) = \begin{cases} \frac{1}{b-a} \int_a^b \frac{1}{a+b-2u} \left(\int_u^{a+b-u} h(t)\, dt\right) du - h\left(\frac{a+b}{2}\right), & a \neq b \\ \\ 0, \quad a = b \end{cases}
$$

is Schur convex on I^2.

If we take now $g_h : I^2 \to \mathbb{R}$ defined as $g_h(x,y) = T_h(x,y)$ then $S_{g_h,p}$ defined by (4.1) becomes

$$
\begin{aligned}
& S_{g_h,p}(a,b) \\
&= \frac{1}{b-a} \int_a^b g_h(u, a+b-u)\, p\left(\frac{u-a}{b-a}\right) du \\
&= \frac{1}{b-a} \int_a^b \left(\frac{h(u) + h(a+b-u)}{2} - \frac{1}{a+b-2u} \int_u^{a+b-u} h(t)\, dt\right) \\
&\times p\left(\frac{u-a}{b-a}\right) du \\
&= \frac{1}{b-a} \int_a^b \frac{h(u) + h(a+b-u)}{2} p\left(\frac{u-a}{b-a}\right) du \\
&- \frac{1}{b-a} \int_a^b \frac{1}{a+b-2u} \left(\int_u^{a+b-u} h(t)\, dt\right) p\left(\frac{u-a}{b-a}\right) du
\end{aligned}
$$

for $a \neq b$ and $S_{f_h,p}(a,a) = 0$ with $a,\ b \in I$.

By Proposition 4.1 we conclude that $S_{g_h,p}$ is Schur convex on I^2 provided that h is continuous convex on I and $p : [0,1] \to [0,\infty)$ is Lebesgue integrable on $[0,1]$.

If p is symmetric on $[0,1]$, namely $p(1-t) = p(t)$ for all $t \in [0,1]$, then

$$
\frac{1}{b-a} \int_a^b \frac{h(u) + h(a+b-u)}{2} p\left(\frac{u-a}{b-a}\right) du = \frac{1}{b-a} \int_a^b h(u)\, p\left(\frac{u-a}{b-a}\right) du
$$

and in this case

$$S_{g_h,p}(a,b) = \frac{1}{b-a}\int_a^b h(u)\,p\left(\frac{u-a}{b-a}\right)du - \frac{1}{b-a}\int_a^b \frac{1}{a+b-2u}\left(\int_u^{a+b-u} h(t)\,dt\right)p\left(\frac{u-a}{b-a}\right)du,$$

which is *Schur convex on* I^2 if h is continuous convex on I.

In particular, we get that

$$S_{g_h}(a,b) = \frac{1}{b-a}\int_a^b h(u)\,du - \frac{1}{b-a}\int_a^b \frac{1}{a+b-2u}\left(\int_u^{a+b-u} h(t)\,dt\right)du,$$

is *Schur convex on* I^2 when h is continuous convex on I.

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