# A Method to Construct Continued-Fraction Approximations and Its Applications

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**Abstract**

In this paper, we provide a method to construct a continued-fraction approximation based upon a given asymptotic expansion. As applications of the method developed here, we establish several continued-fraction approximations for the gamma and the digamma (or psi) functions. Finally, some closely-related open problems are also presented.

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## 1 Introduction

The gamma function $\Gamma(x)$ given by

$$\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t}\, \mathrm{d}t \qquad (x > 0)$$

is one of the most important functions in mathematical analysis and has applications in many diverse areas. The logarithmic derivative $\psi(x)$ of the gamma function $\Gamma(x)$ given by

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)} \qquad \text{or} \qquad \ln\Gamma(x)=\int_1^x \psi(x)\,\mathrm{d}t$$

is known as the psi (or digamma) function. The psi function $\psi(x)$ is connected to the Euler-Mascheroni constant $\gamma$ and the harmonic numbers $H_n$ by means of the following well-known relation (see [1, p. 258, Eq. (6.3.2)]):

$$\psi(n+1)=-\gamma+H_n \qquad (n\in\mathbb{N}:=\{1,2,3,\cdots\}), \tag{1.1}$$

where

$$H_n:=\sum_{k=1}^{n}\frac{1}{k} \qquad (n\in\mathbb{N})$$

is the $n$th harmonic number and $\gamma$ is the Euler-Mascheroni constant defined by

$$\gamma=\lim_{n\to\infty}D_n=0.577215664\cdots,$$

where

$$D_n=\sum_{k=1}^{n}\frac{1}{k}-\ln n. \tag{1.2}$$

Various approximations of the psi function $\psi(x)$ are used in the relation (1.1) and interpreted as approximation for the harmonic number $H_n$ or as approximations of the Euler-Mascheroni constant $\gamma$.

There has been significant interest and research on $\gamma$ as evidenced by survey papers (see, for details, [14]) and expository books (see, for example, [19]), which reveal its essential properties and surprising connections with other areas of the mathematical sciences.

The following two-sided inequality for the difference $D_n-\gamma$ was established in [28, 33]:

$$\frac{1}{2(n+1)}<D_n-\gamma<\frac{1}{2n} \qquad (n\in\mathbb{N}).$$

The convergence of the sequence $D_n$ to $\gamma$ is very slow. By changing the logarithmic term in (1.2), DeTemple [15, 16] presented the following inequality:

$$\frac{1}{24(n+1)^2}<R_n-\gamma<\frac{1}{24n^2} \qquad (n\in\mathbb{N}), \tag{1.3}$$

where

$$R_n=H_n-\ln\left(n+\frac{1}{2}\right). \tag{1.4}$$

On the other hand, Negoi [26] proved that the sequence $\{T_n\}_{n\in\mathbb{N}}$ given by

$$T_n = \sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right) \tag{1.5}$$

is strictly increasing and convergent to $\gamma$. Moreover, Negoi [26] proved that

$$\frac{1}{48(n+1)^3} < \gamma - T_n < \frac{1}{48n^3} \qquad (n \in \mathbb{N}). \tag{1.6}$$

The following faster approximation formulas can be found in [10, 12]:

$$X_n := H_n - \ln\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2}\right) = \gamma + O\left(n^{-4}\right) \qquad (n \to \infty) \tag{1.7}$$

and

$$Y_n := H_n - \ln\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3}\right) = \gamma + O\left(n^{-5}\right) \qquad (n \to \infty). \tag{1.8}$$

In view of (1.3), (1.6), (1.7) and (1.8), Chen and Mortici [10] posed the following open problem:

**Open Problem 1.** *For a given $m \in \mathbb{N}$, find the constants $p_j$ $(j = 1, 2, 3, \cdots, m)$ such that*

$$H_n - \ln\left(n\left(1 + \sum_{j=1}^{m} \frac{p_j}{n^j}\right)\right) \tag{1.9}$$

*is the fastest sequence which would converge to $\gamma$.*

Yang [31] first presented the solution of Open Problem 1 by using the Bell polynomials of a logarithmic type. Subsequently, other proofs of Open Problem 1 (1.9) were published by Gavrea and Ivan [17, 18], Lin [20], Chen *et al.* [8], and Chen and Choi [6].

The following familiar Stirling's formula:

$$n! \sim \sqrt{2\pi n}\,\left(\frac{n}{e}\right)^n \qquad (n \to \infty) \tag{1.10}$$

has many applications in statistical physics, probability theory and number theory. Actually, it was first discovered in 1733 by the French mathematician, Abraham de Moivre (1667–1754), in the form given by

$$n! \sim \text{constant} \cdot \sqrt{n}\,\left(\frac{n}{e}\right)^n \qquad (n \to \infty)$$

when he was studying the Gaussian distribution and the central limit theorem. Afterwards, the Scottish mathematician, James Stirling (1692–1770), found the missing constant $\sqrt{2\pi}$ when he was attempting to give the normal approximation of the binomial distribution.

Recently, Sándor and Debnath [29, Theorem 5] proved the following inequality for all positive integers $n \geqq 2$:

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < \Gamma(n+1) < \sqrt{2\pi n}\ \left(\frac{n}{e}\right)^n \ \left(\frac{n}{n-1}\right)^{\frac{1}{2}} \tag{1.11}$$

and proposed the approximation formula given below:

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \ \left(\frac{n}{n-1}\right)^{\frac{1}{2}} \qquad (n \to \infty). \tag{1.12}$$

Motivated by the above-mentioned work by Sándor and Debnath (1.12), Mortici and Srivastava [25] introduced the following class of approximations for all real numbers $a$ and $b$:

$$\Gamma(n+1) \sim \mu_n(a,b) \qquad \left(n \to \infty;\ \mu_n(a,b) := \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1+\frac{1}{n+a}\right)^{-b}\right). \tag{1.13}$$

We note that Stirling's formula (1.10) is obtained from the Mortici-Srivastava result (1.13) in its special case when $b = 0$. Furthermore, the approximation formula (1.12) can be written as follows:

$$\Gamma(n+1) \sim \mu_n\left(-1, -\frac{1}{2}\right) \qquad (n \to \infty).$$

Indeed, in the aforecited work, Mortici and Srivastava [25] proved that

$$\Gamma(n+1) \sim \mu_n\left(-\frac{1}{2}, -\frac{1}{12}\right) \qquad (n \to \infty)$$

is the best approximation among all approximations given by (1.13). We choose to write this best approximation as follows:

$$\Gamma(n+1) \sim \sqrt{2\pi n}\ \left(\frac{n}{e}\right)^n \ \left(1+\frac{1}{n-\frac{1}{2}}\right)^{\frac{1}{12}} \qquad (n \to \infty). \tag{1.14}$$

Mortici and Srivastava [25] also developed the approximation formula (1.14) in order to produce a complete asymptotic expansion (see [25, Theorem 2]).

In this paper, we provide a method to construct a continued-fraction approximation which is based upon a given asymptotic expansion (see Remarks 1 and 2). As applications of our continued-fraction approximation, we develop the approximation formula (1.14) to produce several further continued-fraction approximations (see Theorems 3, 4 and 5). We also establish continued-fraction approximation for the psi function (see Theorem 6). Finally, we present the higher-order estimate for the Euler-Mascheroni constant $\gamma$ (see Theorem 7).

The following lemmas will be useful in our present investigation.

**Lemma 1** (see [8]). *Let*

$$g(x) \sim \sum_{n=1}^{\infty} b_n x^{-n} \qquad (x \to \infty)$$

*be a given asymptotic expansion. Then the composition* $\exp\big(g(x)\big)$ *has asymptotic expansion of the following form*:

$$\exp\big(g(x)\big) \sim \sum_{n=0}^{\infty} a_n x^{-n} \qquad (x \to \infty),$$

*where*

$$a_0 = 1 \qquad \textit{and} \qquad a_n = \frac{1}{n} \sum_{k=1}^{n} k b_k a_{n-k} \qquad (n \in \mathbb{N}). \tag{1.15}$$

**Lemma 2** (see [5, Theorem 9]). *Let* $k \geqq 1$ *and* $n \geqq 0$ *be integers. Then, for all real numbers* $x > 0$:

$$S_k(2n; x) < (-1)^{k+1} \psi^{(k)}(x) < S_k(2n+1; x), \tag{1.16}$$

*where*

$$S_k(p; x) = \frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} + \sum_{i=1}^{p} \left[ B_{2i} \prod_{j=1}^{k-1} (2i+j) \right] \frac{1}{x^{2i+k}},$$

$\{B_n\}_{n \in \mathbb{N}_0}$ $(\mathbb{N}_0 = \mathbb{N} \cup \{0\})$ *are the Bernoulli numbers defined by*

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

It follows from (1.16) that

$$\begin{aligned} &\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} < \psi'(x) \\ &\qquad < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} + \frac{5}{66x^{11}} \qquad (x > 0). \end{aligned} \tag{1.17}$$

By using the recurrence formula:

$$\psi'(x+1) = \psi'(x) - \frac{1}{x^2},$$

we deduce from (1.17) that, for $x > 0$,

$$\begin{aligned}\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} &< \psi'(x+1) \\ &< \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} + \frac{5}{66x^{11}}.\end{aligned} \tag{1.18}$$

The numerical calculations presented in this work were performed by using the Maple software for symbolic computations.

# 2 A Method for the Construction of Continued-Fraction Approximations

In this section, we present a method to construct a continued-fraction approximation based upon a given asymptotic expansion (see Remark 1 and Remark 2 below).

Theorem 1 generalizes an earlier result [13, Lemma 1.1].

**Theorem 1.** *Let* $a_\ell \neq 0 \quad (\ell \in \mathbb{N})$ *and*

$$A(x) \sim \sum_{j=\ell}^{\infty} \frac{a_j}{x^j} \qquad (x \to \infty) \tag{2.1}$$

*be a given asymptotic expansion. Define the function* $B(x)$ *by*

$$A(x) = \frac{a_\ell}{B(x)}.$$

*Then the function* $B(x) = \frac{a_\ell}{A(x)}$ *has the following asymptotic expansion*:

$$B(x) \sim x^\ell + b_{-(\ell-1)}x^{\ell-1} + b_{-(\ell-2)}x^{\ell-2} + \cdots + b_{-1}x + b_0 + \sum_{j=1}^{\infty} \frac{b_j}{x^j} \qquad (x \to \infty),$$

*where*

$$\begin{cases} b_{-(\ell-1)} &= -\dfrac{a_{\ell+1}}{a_\ell} \\ b_{-(\ell-2)} &= -\dfrac{a_{\ell+2}+a_{\ell+1}b_{-(\ell-1)}}{a_\ell} \\ &\vdots \\ b_{-1} &= -\dfrac{a_{2\ell-1}+a_{2\ell-2}b_{-(\ell-1)}+a_{2\ell-3}b_{-(\ell-2)}+\cdots+a_{\ell+1}b_{-2}}{a_\ell} \\ b_0 &= -\dfrac{a_{2\ell}+a_{j+2\ell-1}b_{-(\ell-1)}+a_{2\ell-2}b_{-(\ell-2)}+\cdots+a_{\ell+1}b_{-1}}{a_\ell} \\ b_j &= -\dfrac{1}{a_\ell}\Bigg(a_{j+2\ell}+a_{j+2\ell-1}b_{-(\ell-1)}+a_{j+2\ell-2}b_{-(\ell-2)}+\cdots+a_{j+\ell+1}b_{-1} \\ &\qquad +\sum_{k=1}^{j} a_{k+\ell}b_{j-k}\Bigg) \quad (j\in\mathbb{N}). \end{cases} \tag{2.2}$$

*Proof.* We begin by considering

$$\begin{aligned} \frac{a_\ell}{A(x)} &\sim x^\ell + \sum_{j=-(\ell-1)}^{\infty}\frac{b_j}{x^j} \\ &= x^\ell + b_{-(\ell-1)}x^{\ell-1}+b_{-(\ell-2)}x^{\ell-2}+\cdots+b_{-1}x+\sum_{j=0}^{\infty}\frac{b_j}{x^j} \qquad (x\to\infty), \end{aligned} \tag{2.3}$$

where $b_j$ $(j\in\{-(\ell-1),-(\ell-2),-1,0\}\cup\mathbb{N})$ are real numbers to be determined.

Upon writing (2.3) as follows:

$$\sum_{j=\ell}^{\infty}\frac{a_j}{x^j}\left(x^\ell+b_{-(\ell-1)}x^{\ell-1}+b_{-(\ell-2)}x^{\ell-2}+\cdots+b_{-1}x+\sum_{k=0}^{\infty}\frac{b_k}{x^k}\right)\sim a_\ell,$$

$$\sum_{j=\ell+1}^{\infty}\frac{a_j}{x^{j-\ell}}+\sum_{j=\ell}^{\infty}\frac{a_jb_{-(\ell-1)}}{x^{j-\ell+1}}+\sum_{j=\ell}^{\infty}\frac{a_jb_{-(\ell-2)}}{x^{j-\ell+2}}+\cdots+\sum_{j=\ell}^{\infty}\frac{a_jb_{-1}}{x^{j-1}}\sim-\sum_{j=0}^{\infty}\frac{a_{j+\ell}}{x^{j+\ell}}\sum_{k=0}^{\infty}\frac{b_k}{x^k},$$

$$\begin{aligned} \sum_{j=0}^{\infty}\frac{a_{j+\ell+1}}{x^{j+1}}+\sum_{j=0}^{\infty}\frac{a_{j+\ell}b_{-(\ell-1)}}{x^{j+1}}+\sum_{j=0}^{\infty}\frac{a_{j+\ell}b_{-(\ell-2)}}{x^{j+2}}+\cdots+\sum_{j=0}^{\infty}\frac{a_{j+\ell}b_{-1}}{x^{j+\ell-1}} \\ \sim-\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{a_{k+\ell}b_{j-k}}{x^{j+\ell}}. \end{aligned} \tag{2.4}$$

It is easy to see that

$$\sum_{j=0}^{\infty}\frac{a_{j+\ell+1}}{x^{j+1}}=\frac{a_{\ell+1}}{x}+\frac{a_{\ell+2}}{x^2}+\cdots+\frac{a_{2\ell-1}}{x^{\ell-1}}+\sum_{j=0}^{\infty}\frac{a_{j+2\ell}}{x^{j+\ell}},$$

$$\sum_{j=0}^{\infty}\frac{a_{j+\ell}b_{-(\ell-1)}}{x^{j+1}}=\frac{a_{\ell}b_{-(\ell-1)}}{x}+\frac{a_{\ell+1}b_{-(\ell-1)}}{x^2}+\cdots+\frac{a_{2\ell-2}b_{-(\ell-1)}}{x^{\ell-1}}+\sum_{j=0}^{\infty}\frac{a_{j+2\ell-1}b_{-(\ell-1)}}{x^{j+\ell}},$$

$$\sum_{j=0}^{\infty}\frac{a_{j+\ell}b_{-(\ell-2)}}{x^{j+2}}=\frac{a_{\ell}b_{-(\ell-2)}}{x^2}+\cdots+\frac{a_{2\ell-3}b_{-(\ell-2)}}{x^{\ell-1}}+\sum_{j=0}^{\infty}\frac{a_{j+2\ell-2}b_{-(\ell-2)}}{x^{j+\ell}},$$

$$\vdots$$

$$\sum_{j=0}^{\infty}\frac{a_{j+\ell}b_{-1}}{x^{j+\ell-1}}=\frac{a_{\ell}b_{-1}}{x^{\ell-1}}+\sum_{j=0}^{\infty}\frac{a_{j+\ell+1}b_{-1}}{x^{j+\ell}}.$$

Adding these equations, we see that the left-hand side of (2.4) can be written as follows:

$$\begin{aligned}&\sum_{j=0}^{\infty}\frac{a_{j+\ell+1}}{x^{j+1}}+\sum_{j=0}^{\infty}\frac{a_{j+\ell}b_{-(\ell-1)}}{x^{j+1}}+\sum_{j=0}^{\infty}\frac{a_{j+\ell}b_{-(\ell-2)}}{x^{j+2}}+\cdots+\sum_{j=0}^{\infty}\frac{a_{j+\ell}b_{-1}}{x^{j+\ell-1}}\\&=\frac{a_{\ell+1}+a_{\ell}b_{-(\ell-1)}}{x}+\frac{a_{\ell+2}+a_{\ell+1}b_{-(\ell-1)}+a_{\ell}b_{-(\ell-2)}}{x^2}\\&\quad+\cdots+\frac{a_{2\ell-1}+a_{2\ell-2}b_{-(\ell-1)}+a_{2\ell-3}b_{-(\ell-2)}+\cdots+a_{\ell}b_{-1}}{x^{\ell-1}}\\&\quad+\sum_{j=0}^{\infty}\frac{a_{j+2\ell}+a_{j+2\ell-1}b_{-(\ell-1)}+a_{j+2\ell-2}b_{-(\ell-2)}+\cdots+a_{j+\ell+1}b_{-1}}{x^{j+\ell}}.\end{aligned}\tag{2.5}$$

Equating the coefficients of equal powers of $x$ on the right-hand sides of (2.4) and (2.5), we get

$$\begin{aligned}&a_{\ell+1}+a_{\ell}b_{-(\ell-1)}=0,\\&a_{\ell+2}+a_{\ell+1}b_{-(\ell-1)}+a_{\ell}b_{-(\ell-2)}=0,\\&\vdots\\&a_{2\ell-1}+a_{2\ell-2}b_{-(\ell-1)}+a_{2\ell-3}b_{-(\ell-2)}+\cdots+a_{\ell}b_{-1}=0\end{aligned}\tag{2.6}$$

and

$$a_{j+2\ell} + a_{j+2\ell-1}b_{-(\ell-1)} + a_{j+2\ell-2}b_{-(\ell-2)} + \cdots + a_{j+\ell+1}b_{-1} = -\sum_{k=0}^{j} a_{k+\ell}b_{j-k}. \tag{2.7}$$

We now find from (2.6) that

$$\begin{aligned} b_{-(\ell-1)} &= -\frac{a_{\ell+1}}{a_\ell}, \\ b_{-(\ell-2)} &= -\frac{a_{\ell+2} + a_{\ell+1}b_{-(\ell-1)}}{a_\ell}, \\ &\vdots \\ b_{-1} &= -\frac{a_{2\ell-1} + a_{2\ell-2}b_{-(\ell-1)} + a_{2\ell-3}b_{-(\ell-2)} + \cdots + a_{\ell+1}b_{-2}}{a_\ell}. \end{aligned}$$

By setting $j = 0$, we deduce from (2.7) that

$$b_0 = -\frac{a_{2\ell} + a_{j+2\ell-1}b_{-(\ell-1)} + a_{2\ell-2}b_{-(\ell-2)} + \cdots + a_{\ell+1}b_{-1}}{a_\ell}.$$

Moreover, for $j \in \mathbb{N}$, we have

$$a_{j+2\ell} + a_{j+2\ell-1}b_{-(\ell-1)} + a_{j+2\ell-2}b_{-(\ell-2)} + \cdots + a_{j+\ell+1}b_{-1} = -a_\ell b_j - \sum_{k=1}^{j} a_{k+\ell}b_{j-k},$$

which yields

$$b_j = -\frac{1}{a_\ell}\left(a_{j+2\ell} + a_{j+2\ell-1}b_{-(\ell-1)} + a_{j+2\ell-2}b_{-(\ell-2)} + \cdots + a_{j+\ell+1}b_{-1} + \sum_{k=1}^{j} a_{k+\ell}b_{j-k}\right)$$
$$(j \in \mathbb{N}).$$

The proof of Theorem 1 is thus completed. □

The choice $\ell = 1, 2, 3$ in Theorem 1 yields Corollaries 1, 2 and 3, respectively.

**Corollary 1.** Let $a_1 \neq 0$ and

$$A_1(x) \sim \sum_{j=1}^{\infty} \frac{a_j}{x^j} \qquad (x \to \infty)$$

be a given asymptotic expansion. Define the function $B(x)$ by

$$A_1(x) = \frac{a_1}{B(x)}.$$

Then the function $B(x) = \frac{a_1}{A_1(x)}$ has asymptotic expansion of the following form:

$$B(x) \sim x + \sum_{j=0}^{\infty} \frac{b_j}{x^j} \qquad (x \to \infty),$$

where

$$b_0 = -\frac{a_2}{a_1} \qquad \text{and} \qquad b_j = -\frac{1}{a_1}\left(a_{j+2} + \sum_{k=1}^{j} a_{k+1} b_{j-k}\right) \qquad (j \in \mathbb{N}). \tag{2.8}$$

**Remark 1.** Corollary 1 provides a method to construct a continued-fraction approximation based upon a given asymptotic expansion. The details of this method are given below.

Let $a_1 \neq 0$ and

$$A(x) \sim \sum_{j=1}^{\infty} \frac{a_j}{x^j} \qquad (x \to \infty) \tag{2.9}$$

be a given asymptotic expansion. Then the asymptotic expansion (2.9) can be transformed into the continued-fraction approximation of the form:

$$A(x) \approx \cfrac{a_1}{x + b_0 + \cfrac{b_1}{x + c_0 + \cfrac{c_1}{x + d_0 + \ddots}}} \qquad (x \to \infty), \tag{2.10}$$

wherein the constants are given by the following recurrence relations:

$$\begin{cases} b_0 = -\frac{a_2}{a_1} \quad \text{and} \quad b_j = -\frac{1}{a_1}\left(a_{j+2} + \sum\limits_{k=1}^{j} a_{k+1} b_{j-k}\right); \\ c_0 = -\frac{b_2}{b_1} \quad \text{and} \quad c_j = -\frac{1}{b_1}\left(b_{j+2} + \sum\limits_{k=1}^{j} b_{k+1} c_{j-k}\right) \\ d_0 = -\frac{c_2}{c_1} \quad \text{and} \quad d_j = -\frac{1}{c_1}\left(c_{j+2} + \sum\limits_{k=1}^{j} c_{k+1} d_{j-k}\right) \\ \quad \cdots \quad \cdots \end{cases} \tag{2.11}$$

Clearly, since $a_j \Longrightarrow b_j \Longrightarrow c_j \Longrightarrow d_j \Longrightarrow \cdots$, the asymptotic expansion (2.9) is transformed into the continued-fraction approximation (2.10), and the constants in the right-hand side of (2.10) are determined by the system (2.11).

**Corollary 2.** Let $a_2 \neq 0$ and

$$A_2(x) \sim \sum_{j=2}^{\infty} \frac{a_j}{x^j} \qquad (x \to \infty) \tag{2.12}$$

be a given asymptotic expansion. Define the function $B(x)$ by

$$A_2(x) = \frac{a_2}{B(x)}.$$

Then the function $B(x) = \frac{a_2}{A_2(x)}$ has asymptotic expansion of the following form:

$$B(x) \sim x^2 + b_{-1}x + b_0 + \sum_{j=1}^{\infty} \frac{b_j}{x^j} \qquad (x \to \infty),$$

where

$$b_{-1} = -\frac{a_3}{a_2}, \qquad b_0 = \frac{-a_2^5 a_4 + a_2^4 a_3^2}{a_2^6}$$

$$\text{and} \qquad b_j = -\frac{1}{a_2}\left(a_{j+4} + a_{j+3}b_{-1} + \sum_{k=1}^{j} a_{k+2}a_{j-k}\right) \qquad (j \in \mathbb{N}). \tag{2.13}$$

**Corollary 3.** Let $\mu_3 \neq 0$ and

$$F(x) \sim \sum_{j=3}^{\infty} \frac{\mu_j}{x^j} \qquad (x \to \infty) \tag{2.14}$$

be a given asymptotic expansion. Define the function $G(x)$ by

$$F(x) = \frac{\mu_3}{G(x)}.$$

Then the function $G(x) = \frac{\mu_3}{F(x)}$ has asymptotic expansion of the following form:

$$G(x) \sim x^3 + a_{-2}x^2 + a_{-1}x + a_0 + \sum_{j=1}^{\infty} \frac{a_j}{x^j} \qquad (x \to \infty),$$

where

$$a_{-2} = -\frac{\mu_4}{\mu_3}, \qquad a_{-1} = -\frac{\mu_3\mu_5 - \mu_4^2}{\mu_3^2}, \quad a_0 = -\frac{\mu_3^2\mu_6 - 2\mu_3\mu_4\mu_5 + \mu_4^3}{\mu_3^3}$$

$$\text{and} \qquad a_j = -\frac{1}{\mu_3}\left(\mu_{j+6} + \mu_{j+5}a_{-2} + \mu_{j+4}a_{-1} + \sum_{k=1}^{j} \mu_{k+3}a_{j-k}\right) \qquad (j \in \mathbb{N}). \tag{2.15}$$

We next prove the following result.

**Theorem 2.** *Let $a_\ell \neq 0 \ \ (\ell \in \mathbb{N})$ and*

$$A(x) \sim \sum_{j=\ell}^{\infty} \frac{a_j}{x^{2j-1}} \qquad (x \to \infty) \tag{2.16}$$

*be a given asymptotic expansion. Define the function $B(x)$ by*

$$A(x) = \frac{a_\ell}{B(x)}.$$

*Then the function $B(x) = \frac{a_\ell}{A(x)}$ has asymptotic expansion of the following form:*

$$B(x) \sim x^{2\ell-1} + b_{-(\ell-2)}x^{2\ell-3} + b_{-(\ell-3)}x^{2\ell-5} + \cdots + b_{-1}x^3 + b_0 x + \sum_{j=1}^{\infty} \frac{b_j}{x^{2j-1}} \qquad (x \to \infty),$$

*where*

$$\begin{cases} b_{-(\ell-2)} &= -\dfrac{a_{\ell+1}}{a_\ell} \\ b_{-(\ell-3)} &= -\dfrac{a_{\ell+2} + a_{\ell+1}b_{-(\ell-2)}}{a_\ell} \\ &\ \vdots \\ b_{-1} &= -\dfrac{a_{2\ell-1} + a_{2\ell-2}b_{-(\ell-1)} + a_{2\ell-3}b_{-(\ell-2)} + \cdots + a_{\ell+1}b_{-2}}{a_\ell} \\ b_0 &= -\dfrac{a_{2\ell-2} + a_{2\ell-3}b_{-(\ell-2)} + a_{2\ell-4}b_{-(\ell-3)} + \cdots + a_{\ell+1}b_{-2}}{a_\ell} \\ b_j &= -\dfrac{1}{a_\ell}\Bigg(a_{j+2\ell-1} + a_{j+2\ell-2}b_{-(\ell-2)} + a_{j+2\ell-3}b_{-(\ell-3)} + \cdots + a_{j+\ell+1}b_{-1} \\ &\qquad + \sum\limits_{k=1}^{j} a_{k+\ell}b_{j-k}\Bigg) \qquad (j \in \mathbb{N}). \end{cases} \tag{2.17}$$

*Proof.* We first let

$$\begin{aligned} \frac{a_\ell}{A(x)} &\sim x^{2\ell-1} + \sum_{j=-(\ell-2)}^{\infty} \frac{b_j}{x^{2j-1}} \\ &= x^{2\ell-1} + b_{-(\ell-2)}x^{2\ell-3} + b_{-(\ell-3)}x^{2\ell-5} + \cdots + b_{-1}x^3 + b_0 x + \sum_{j=1}^{\infty} \frac{b_j}{x^{2j-1}} \qquad (x \to \infty), \end{aligned} \tag{2.18}$$

where $b_j \quad (j \in \{-(\ell-2), -(\ell-3), -1, 0\} \cup \mathbb{N})$ are real numbers to be determined. Then, by writing (2.18) as follows:

$$\sum_{j=\ell}^{\infty} \frac{a_j}{x^{2j-1}} \left( x^{2\ell-1} + b_{-(\ell-2)} x^{2\ell-3} + b_{-(\ell-3)} x^{2\ell-5} + \cdots + b_{-1} x^3 + \sum_{k=0}^{\infty} \frac{b_k}{x^{2k-1}} \right) \sim a_\ell,$$

$$\sum_{j=\ell+1}^{\infty} \frac{a_j}{x^{2j-2\ell}} + \sum_{j=\ell}^{\infty} \frac{a_j b_{-(\ell-2)}}{x^{2j-2\ell+2}} + \sum_{j=\ell}^{\infty} \frac{a_j b_{-(\ell-3)}}{x^{2j-2\ell+4}} + \cdots + \sum_{j=\ell}^{\infty} \frac{a_j b_{-1}}{x^{2j-4}} \sim -\sum_{j=0}^{\infty} \frac{a_{j+\ell}}{x^{2(j+\ell)-1}} \sum_{k=0}^{\infty} \frac{b_k}{x^{2k-1}},$$

and

$$\begin{aligned} &\sum_{j=0}^{\infty} \frac{a_{j+\ell+1}}{x^{2j+2}} + \sum_{j=0}^{\infty} \frac{a_{j+\ell} b_{-(\ell-2)}}{x^{2j+2}} + \sum_{j=0}^{\infty} \frac{a_{j+\ell} b_{-(\ell-3)}}{x^{2j+4}} + \cdots + \sum_{j=0}^{\infty} \frac{a_{j+\ell} b_{-1}}{x^{2j+2\ell-4}} \\ &\qquad \sim -\sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{a_{k+\ell} b_{j-k}}{x^{2j+2\ell-2}}. \end{aligned} \tag{2.19}$$

It is easy to see that

$$\sum_{j=0}^{\infty} \frac{a_{j+\ell+1}}{x^{2j+2}} = \frac{a_{\ell+1}}{x^2} + \frac{a_{\ell+2}}{x^4} + \cdots + \frac{a_{2\ell-2}}{x^{2\ell-4}} + \sum_{j=0}^{\infty} \frac{a_{j+2\ell-1}}{x^{2j+2\ell-2}},$$

$$\sum_{j=0}^{\infty} \frac{a_{j+\ell} b_{-(\ell-2)}}{x^{2j+2}} = \frac{a_\ell b_{-(\ell-2)}}{x^2} + \frac{a_{\ell+1} b_{-(\ell-2)}}{x^4} + \cdots + \frac{a_{2\ell-3} b_{-(\ell-2)}}{x^{2\ell-4}} + \sum_{j=0}^{\infty} \frac{a_{j+2\ell-2} b_{-(\ell-2)}}{x^{2j+2\ell-2}},$$

$$\sum_{j=0}^{\infty} \frac{a_{j+\ell} b_{-(\ell-3)}}{x^{2j+4}} = \frac{a_\ell b_{-(\ell-3)}}{x^4} + \cdots + \frac{a_{2\ell-4} b_{-(\ell-3)}}{x^{2\ell-4}} + \sum_{j=0}^{\infty} \frac{a_{j+2\ell-3} b_{-(\ell-3)}}{x^{2j+2\ell-2}},$$

$$\vdots$$

$$\sum_{j=0}^{\infty} \frac{a_{j+\ell} b_{-1}}{x^{2j+2\ell-4}} = \frac{a_\ell b_{-1}}{x^{2\ell-4}} + \sum_{j=0}^{\infty} \frac{a_{j+\ell+1} b_{-1}}{x^{2j+2\ell-2}}.$$

Adding these equations, we see the left-hand side of (2.19) can be written as follows:

$$\sum_{j=0}^{\infty}\frac{a_{j+\ell+1}}{x^{2j+2}}+\sum_{j=0}^{\infty}\frac{a_{j+\ell}b_{-(\ell-2)}}{x^{2j+2}}+\sum_{j=0}^{\infty}\frac{a_{j+\ell}b_{-(\ell-3)}}{x^{2j+4}}+\cdots+\sum_{j=0}^{\infty}\frac{a_{j+\ell}b_{-1}}{x^{2j+2\ell-4}}$$
$$=\frac{a_{\ell+1}+a_\ell b_{-(\ell-2)}}{x^2}+\frac{a_{\ell+2}+a_{\ell+1}b_{-(\ell-2)}+a_\ell b_{-(\ell-3)}}{x^4}$$
$$+\cdots+\frac{a_{2\ell-2}+a_{2\ell-3}b_{-(\ell-2)}+a_{2\ell-4}b_{-(\ell-3)}+\cdots+a_\ell b_{-1}}{x^{2\ell-4}}$$
$$+\sum_{j=0}^{\infty}\frac{a_{j+2\ell-1}+a_{j+2\ell-2}b_{-(\ell-2)}+a_{j+2\ell-3}b_{-(\ell-3)}+\cdots+a_{j+\ell+1}b_{-1}}{x^{2j+2\ell-2}}. \tag{2.20}$$

Equating the coefficients of equal powers of $x$ on the right-hand sides of (2.19) and (2.20), we get

$$\begin{aligned}
&a_{\ell+1}+a_\ell b_{-(\ell-2)}=0,\\
&a_{\ell+2}+a_{\ell+1}b_{-(\ell-2)}+a_\ell b_{-(\ell-3)}=0,\\
&\quad\vdots\\
&a_{2\ell-2}+a_{2\ell-3}b_{-(\ell-2)}+a_{2\ell-4}b_{-(\ell-3)}+\cdots+a_{\ell+1}b_{-2}+a_\ell b_{-1}=0
\end{aligned} \tag{2.21}$$

and

$$a_{j+2\ell-1}+a_{j+2\ell-2}b_{-(\ell-2)}+a_{j+2\ell-3}b_{-(\ell-3)}+\cdots+a_{j+\ell+1}b_{-1}=-\sum_{k=0}^{j}a_{k+\ell}b_{j-k} \qquad (j\in\mathbb{N}_0). \tag{2.22}$$

We now find from (2.21) that

$$\begin{aligned}
b_{-(\ell-2)}&=-\frac{a_{\ell+1}}{a_\ell},\\
b_{-(\ell-3)}&=-\frac{a_{\ell+2}+a_{\ell+1}b_{-(\ell-2)}}{a_\ell},\\
&\ \ \vdots\\
b_{-1}&=-\frac{a_{2\ell-2}+a_{2\ell-3}b_{-(\ell-2)}+a_{2\ell-4}b_{-(\ell-3)}+\cdots+a_{\ell+1}b_{-2}}{a_\ell}.
\end{aligned}$$

For $j = 0$, we obtain from (2.22) that

$$b_0 = -\frac{a_{2\ell-1} + a_{2\ell-2}b_{-(\ell-2)} + a_{2\ell-3}b_{-(\ell-3)} + \cdots + a_{\ell+1}b_{-1}}{a_\ell}$$

and, for $j \in \mathbb{N}$, we have

$$a_{j+2\ell-1} + a_{j+2\ell-2}b_{-(\ell-2)} + a_{j+2\ell-3}b_{-(\ell-3)} + \cdots + a_{j+\ell+1}b_{-1} = -a_\ell b_j - \sum_{k=1}^{j} a_{k+\ell}b_{j-k},$$

which yields

$$b_j = -\frac{1}{a_\ell}\left(a_{j+2\ell-1} + a_{j+2\ell-2}b_{-(\ell-2)} + a_{j+2\ell-3}b_{-(\ell-3)} + \cdots + a_{j+\ell+1}b_{-1} + \sum_{k=1}^{j} a_{k+\ell}b_{j-k}\right)$$

for $j \in \mathbb{N}$. The proof of Theorem 2 is thus completed. $\square$

Theorem 2 implies Corollaries 4 and 5 below.

**Corollary 4.** Let $a_1 \neq 0$ and

$$A_1(x) \sim \sum_{j=1}^{\infty} \frac{a_j}{x^{2j-1}} \qquad (x \to \infty) \tag{2.23}$$

be a given asymptotic expansion. Define the function $B(x)$ by

$$A_1(x) = \frac{a_1}{B(x)}.$$

Then the function $B(x) = \frac{a_1}{A_1(x)}$ has asymptotic expansion of the following form:

$$B(x) \sim x + \sum_{j=1}^{\infty} \frac{b_j}{x^{2j-1}} \qquad (x \to \infty),$$

where

$$b_1 = -\frac{a_2}{a_1} \qquad \text{and} \qquad b_j = -\frac{1}{a_1}\left(a_{j+1} + \sum_{k=1}^{j-1} a_{k+1}b_{j-k}\right) \qquad (j \in \mathbb{N} \setminus \{1\}). \tag{2.24}$$

*Proof.* We write the last line in (2.17) as follows:

$$b_j = -\frac{1}{a_\ell}\Bigg(a_{j+2\ell-1} + a_{j+2\ell-2}b_{-(\ell-2)} + a_{j+2\ell-3}b_{-(\ell-3)} \\ + \cdots + a_{j+\ell+1}b_{-1} + a_{j+\ell}b_0 + \sum_{k=1}^{j-1} a_{k+\ell}b_{j-k}\Bigg)$$

for $j \in \mathbb{N}$, where an empty sum is understood to be zero. Choosing $\ell = 1$ and noticing that

$$b_{-(\ell-2)} = b_{-(\ell-3)} = \cdots = b_{-1} = b_0 = 0,$$

we get

$$b_j = -\frac{1}{a_1}\left(a_{j+1} + \sum_{k=1}^{j-1} a_{k+\ell}b_{j-k}\right) \qquad (j \in \mathbb{N}),$$

which gives the desired formula (2.24) asserted by Corollary 4. □

**Remark 2.** Corollary 4 provides a method to convert the asymptotic expansion (2.23) into a continued fraction of the form:

$$A_1(x) \approx \cfrac{a_1}{x + \cfrac{b_1}{x + \cfrac{c_1}{x + \cfrac{d_1}{x + \ddots}}}} \qquad (x \to \infty), \tag{2.25}$$

where the constants in the right-hand side of (2.25) are given by the following recurrence relations:

$$\begin{cases} b_1 = -\frac{a_2}{a_1} \quad \text{and} \quad b_j = -\frac{1}{a_1}\left(a_{j+1} + \sum\limits_{k=1}^{j-1} a_{k+1}b_{j-k}\right) \\ c_1 = -\frac{b_2}{b_1} \quad \text{and} \quad c_j = -\frac{1}{b_1}\left(b_{j+1} + \sum\limits_{k=1}^{j-1} b_{k+1}c_{j-k}\right) \\ d_1 = -\frac{c_2}{c_1} \quad \text{and} \quad d_j = -\frac{1}{c_1}\left(c_{j+1} + \sum\limits_{k=1}^{j-1} c_{k+1}d_{j-k}\right) \\ \quad \cdots \quad \cdots \end{cases} \tag{2.26}$$

Clearly, since

$$a_j \Longrightarrow b_j \Longrightarrow c_j \Longrightarrow d_j \Longrightarrow \cdots,$$

the asymptotic expansion (2.23) is transformed into the continued-fraction approximation (2.25), and the constants in the right-hand side of (2.25) are determined by the system (2.26).

**Corollary 5.** Let $\lambda_2 \neq 0$ and

$$F(x) \sim \sum_{j=2}^{\infty} \frac{\lambda_j}{x^{2j-1}} \qquad (x \to \infty) \tag{2.27}$$

be a given asymptotic expansion. Define the function $G(x)$ by

$$F(x) = \frac{\lambda_2}{G(x)}.$$

Then the function $G(x) = \frac{\lambda_2}{F(x)}$ has asymptotic expansion of the following form:

$$G(x) \sim x^3 + a_0 x + \sum_{j=1}^{\infty} \frac{a_j}{x^{2j-1}} \qquad (x \to \infty),$$

where

$$a_0 = -\frac{\lambda_3}{\lambda_2} \qquad \text{and} \qquad a_j = -\frac{1}{\lambda_2}\left(\lambda_{j+3} + \sum_{k=1}^{j} \lambda_{k+2} a_{j-k}\right) \qquad (j \in \mathbb{N}). \tag{2.28}$$

# 3 Continued-Fraction Approximations for the Gamma Function

In this section, we develop the approximation formula (1.14) in order to derive various other continued-fraction approximations associate with the gamma function $\Gamma(x)$.

Let $r \neq 0$ be a given real number and $\ell \geqq 0$ be a given integer. Chen and Lin [9] proved that the gamma function $\Gamma(x)$ has the following asymptotic expansion:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \sum_{j=1}^{\infty} \frac{b_j(\ell, r)}{x^j}\right)^{x^\ell / r} \qquad (x \to \infty), \tag{3.1}$$

with the coefficients $b_j(\ell, r)$ $(j \in \mathbb{N})$ given by

$$b_j(\ell, r) = \sum \frac{r^{k_1+k_2+\cdots+k_j}}{k_1! k_2! \cdots k_j!} \left(\frac{B_2}{1\cdot 2}\right)^{k_1} \left(\frac{B_3}{2\cdot 3}\right)^{k_2} \cdots \left(\frac{B_{j+1}}{j(j+1)}\right)^{k_j}, \tag{3.2}$$

where $\{B_n\}_{n\in\mathbb{N}_0}$ are the Bernoulli numbers, summed over all nonnegative integers $k_j$ satisfying the following equation:

$$(1+\ell)k_1 + (2+\ell)k_2 + \cdots + (j+\ell)k_j = j.$$

The choice $(\ell, r) = (0, 12)$ in (3.1) yields

$$\begin{aligned} A(x) &:= \left( \frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x} \right)^{12} - 1 \sim \sum_{j=1}^{\infty} \frac{a_j}{x^j} \\ &= \frac{1}{x} + \frac{1}{2x^2} + \frac{2}{15x^3} + \frac{1}{120x^4} + \frac{1}{840x^5} + \frac{149}{25200x^6} - \frac{19}{6300x^7} - \frac{131}{22400x^8} \\ &\quad + \frac{663799}{99792000x^9} + \frac{12748781}{1397088000x^{10}} - \frac{81764339}{4540536000x^{11}} \\ &\quad - \frac{23598827489}{1089728640000x^{12}} + \cdots \qquad (x \to \infty), \end{aligned} \tag{3.3}$$

where the coefficients $a_j \equiv b_j(0, 12)$ $(j \in \mathbb{N})$ are given by

$$a_j = \sum \frac{12^{k_1+k_2+\cdots+k_j}}{k_1! k_2! \cdots k_j!} \left( \frac{B_2}{1 \cdot 2} \right)^{k_1} \left( \frac{B_3}{2 \cdot 3} \right)^{k_2} \cdots \left( \frac{B_{j+1}}{j(j+1)} \right)^{k_j}, \tag{3.4}$$

summed over all nonnegative integers $k_j$ satisfying the following equation:

$$k_1 + 2k_2 + \cdots + jk_j = j.$$

Based upon the asymptotic expansion (3.3) and by using the system (2.11), we develop the approximation formula (1.14) with a view to deriving a continued-fraction approximation given by Theorem 3.

**Theorem 3.** *It is asserted that*

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left( 1 + \cfrac{1}{x - \frac{1}{2} + \cfrac{\frac{7}{60}}{x + \cfrac{\frac{317}{2940}}{x + \ddots}}} \right)^{\frac{1}{12}} \qquad (x \to \infty). \tag{3.5}$$

*Proof.* By Remark 1, we can convert the asymptotic expansion (3.3) into a continued-fraction approximation of the form:

$$A(x) \approx \cfrac{a_1}{x + b_0 + \cfrac{b_1}{x + c_0 + \cfrac{c_1}{x + d_0 + \ddots}}} \qquad (x \to \infty), \tag{3.6}$$

where the constants in the right-hand side can be determined by using the system (2.11). Moreover, by noting that

$$a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{2}{15}, \quad a_4 = \frac{1}{120}, \quad a_5 = \frac{1}{840}, \quad a_6 = \frac{149}{25200}, \quad \cdots,$$

we find from the first recurrence relation in (2.11) that

$$\begin{aligned}
b_0 &= -\frac{a_2}{a_1} = -\frac{1}{2}, \\
b_1 &= -\frac{a_3 + a_2 b_0}{a_1} = \frac{7}{60}, \\
b_2 &= -\frac{a_4 + a_2 b_1 + a_3 b_0}{a_1} = 0, \\
b_3 &= -\frac{a_5 + a_2 b_2 + a_3 b_1 + a_4 b_0}{a_1} = -\frac{317}{25200}, \\
b_4 &= -\frac{a_6 + a_2 b_3 + a_3 b_2 + a_4 b_1 + a_5 b_0}{a_1} = 0, \quad \cdots.
\end{aligned}$$

From the second recurrence relation in (2.11), we have

$$\begin{aligned}
c_0 &= -\frac{b_2}{b_1} = 0, \\
c_1 &= -\frac{b_3 + b_2 c_0}{b_1} = \frac{317}{2940}, \\
c_2 &= -\frac{b_4 + b_2 c_1 + b_3 c_0}{b_1} = 0, \quad \cdots.
\end{aligned}$$

Continuing the above process, we get

$$d_0 = -\frac{c_2}{c_1} = 0, \quad \cdots.$$

We see that (3.6) can be written as (3.5). The proof of Theorem 3 is thus completed. □

**Remark 3.** Based upon the asymptotic expansion (3.3), following the same method as was used in the proof of Theorem 3, we find that

$$\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x}\right)^{12} \approx 1 + \cfrac{1}{x - \frac{1}{2} + \cfrac{\frac{7}{60}}{x + \cfrac{\frac{317}{2940}}{x + \cfrac{\frac{30397}{62132}}{x + \cfrac{\frac{17752261513}{19078981020}}{x + \cfrac{\frac{2864122300479077017}{1984243256463202020}}{x + \ddots}}}}}} \tag{3.7}$$

as $x \to \infty$. Moreover, based upon the continued-fraction approximation (3.7), we can find new inequalities for the gamma function $\Gamma(x)$. For example, we find for $n \geqq 1$ that

$$1 + \cfrac{1}{n - \frac{1}{2} + \cfrac{\frac{7}{60}}{n + \cfrac{\frac{317}{2940}}{n + \cfrac{\frac{30397}{62132}}{n}}}} < \left( \frac{\Gamma(n+1)}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} \right)^{12} < 1 + \cfrac{1}{n - \frac{1}{2} + \cfrac{\frac{7}{60}}{n + \cfrac{\frac{317}{2940}}{n}}}. \tag{3.8}$$

As $n \to \infty$, the following approximation formulas hold true:

$$\Gamma(n+1) \sim \rho_n := \sqrt{\pi}\left(\frac{n}{e}\right)^n \left(8n^3 + 4n^2 + n + \frac{1}{30}\right)^{1/6} \quad \text{(Ramanujan's formula)}, \tag{3.9}$$

$$\begin{aligned}\Gamma(n+1) \sim \kappa_n := \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{n - \frac{1}{2}}\right)^{\frac{1}{12}} \left(1 + \frac{1}{\left(n - \frac{1}{2}\right)^3}\right)^{-\frac{7}{720}} \\ \cdot \left(1 + \frac{1}{\left(n - \frac{1}{2}\right)^4}\right)^{\frac{7}{480}} \quad \text{(Mortici-Srivastava formula [25])}\end{aligned} \tag{3.10}$$

and

$$\Gamma(n+1) \sim \nu_n := \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \cfrac{1}{n - \frac{1}{2} + \cfrac{\frac{7}{60}}{n + \cfrac{\frac{317}{2940}}{n}}}\right)^{\frac{1}{12}} \quad \text{(New formula)}. \tag{3.11}$$

It is observed from the following Table that, among the approximation formulas (3.9), (3.10) and (3.11), for $n \in \mathbb{N}$, the formula (3.11) is believe to be the best one.

**Table.** Comparison among approximation formulas (3.9), (3.10) and (3.11)

| $n$ | $\frac{\rho_n - n!}{n!}$ | $\frac{\kappa_n - n!}{n!}$ | $\frac{\nu_n - n!}{n!}$ |
|---|---|---|---|
| 1 | $2.833 \times 10^{-4}$ | $3.091 \times 10^{-2}$ | $2.160 \times 10^{-4}$ |
| 10 | $8.587 \times 10^{-8}$ | $1.822 \times 10^{-7}$ | $5.047 \times 10^{-11}$ |
| 100 | $9.451 \times 10^{-12}$ | $1.512 \times 10^{-12}$ | $5.127 \times 10^{-18}$ |
| 1000 | $9.538 \times 10^{-16}$ | $1.486 \times 10^{-17}$ | $5.128 \times 10^{-25}$ |
| 10000 | $9.547 \times 10^{-20}$ | $1.483 \times 10^{-22}$ | $5.128 \times 10^{-32}$ |

Recently, Mortici and Srivastava [25, Theorem 2] proved, as $x \to \infty$, that

$$\begin{aligned}\Gamma(x+1) &\sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1+\frac{1}{x-\frac{1}{2}}\right)^{\frac{1}{12}} \exp\left(\sum_{i=1}^{\infty} \frac{\lambda_i}{x^{2i-1}}\right) \\ &= \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1+\frac{1}{x-\frac{1}{2}}\right)^{\frac{1}{12}} \exp\bigg(-\frac{7}{720x^3}-\frac{1}{4032x^5}-\frac{1}{1280x^7} \\ &\qquad +\frac{245}{304128x^9}-\frac{32287}{16773120x^{11}}+\frac{105}{16384x^{13}}-\frac{7407701}{250675200x^{15}} \\ &\qquad +\frac{169109795}{941359104x^{17}}-\frac{401519531}{288358400x^{19}}+\cdots\bigg),\end{aligned} \tag{3.12}$$

where

$$\lambda_i = \frac{B_{2i}}{2i(2i-1)} - \frac{1}{12(2i-1)} - \frac{1}{12}\sum_{j=1}^{2i-2} \frac{1}{j2^{2i-j-1}} \binom{-j}{2i-j-1} \qquad (i \in \mathbb{N}). \tag{3.13}$$

We convert the asymptotic expansion (3.12) into a continued fraction given by Theorem 4 below.

**Theorem 4.** *For $x \to \infty$, it is asserted that*

$$\begin{aligned}\Gamma(x+1) &\approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1+\frac{1}{x-\frac{1}{2}}\right)^{\frac{1}{12}} \\ &\quad \cdot \exp\left(\cfrac{-\frac{7}{720}}{x^3-\frac{5}{196}x+\cfrac{-\frac{1531}{19208}}{x+\cfrac{\frac{2700395}{2475627}}{x+\cfrac{\frac{16496398810339}{14188933884840}}{x+\cfrac{\frac{408791430136295352 3}{1929557042068438120}}{x+\ddots}}}}}\right).\end{aligned} \tag{3.14}$$

*Proof.* Let us put

$$F(x) = \ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(1+\frac{1}{x-\frac{1}{2}}\right)^{\frac{1}{12}}}\right). \tag{3.15}$$

Then, by noting that $\lambda_1 = 0$, we have

$$F(x) \sim \sum_{i=2}^{\infty} \frac{\lambda_i}{x^{2i-1}} \qquad (x \to \infty), \tag{3.16}$$

where $\lambda_i$ are given in (3.13).

We now define the function $G(x)$ by

$$F(x) = \frac{\lambda_2}{G(x)} \qquad \left(\lambda_2 = -\frac{7}{720}\right). \tag{3.17}$$

By Corollary 5, we find for $x \to \infty$ that

$$G(x) = \frac{\lambda_2}{F(x)} \sim x^3 - \frac{5}{196}x + A_1(x), \tag{3.18}$$

where

$$\begin{aligned} A_1(x) = \sum_{j=1}^{\infty} \frac{a_j}{x^{2j-1}} = &-\frac{1531}{19208x} + \frac{2700395}{31059336x^3} - \frac{31009745857}{158278376256x^5} + \frac{6779851492025}{10340853915392x^7} \\ &- \frac{517524935589060758 39}{17055583996812641280x^9} + \frac{7830963178548566644339 9}{4234332986942018408448x^{11}} \\ &- \frac{7742687957195958707976251459}{5394540225364131452362752 0x^{13}} + \cdots, \end{aligned} \tag{3.19}$$

and the coefficients $a_j$ in (3.19) can be calculated by the following recurrence relation:

$$a_0 = -\frac{\lambda_3}{\lambda_2}, \quad a_j = -\frac{1}{\lambda_2}\left(\lambda_{j+3} + \sum_{k=1}^{j} \lambda_{k+2} a_{j-k}\right) \qquad (j \in \mathbb{N}).$$

By Remark 2, the asymptotic expansion (3.19) can be transformed into the following continued-fraction approximation:

$$A_1(x) \approx \cfrac{a_1}{x + \cfrac{b_1}{x + \cfrac{c_1}{x + \cfrac{d_1}{x + \ddots}}}} \qquad (x \to \infty), \tag{3.20}$$

where the constants in the right-hand side can be determined by using the system (2.26).

We see from (3.19) that

$$\begin{aligned}
&a_1 = -\frac{1531}{19208}, \quad a_2 = \frac{2700395}{31059336}, \quad a_3 = -\frac{31009745857}{158278376256}, \\
&a_4 = \frac{6779851492025}{10340853915392}, \quad a_5 = -\frac{5175249355890607 5839}{17055583996812641280}, \ \cdots .
\end{aligned}$$

From the first recurrence relation in (2.26), we have

$$\begin{aligned}
b_1 &= -\frac{a_2}{a_1} = \frac{2700395}{2475627}, \\
b_2 &= -\frac{a_3 + a_2 b_1}{a_1} = -\frac{336661200211}{265467647016}, \\
b_3 &= -\frac{a_4 + a_2 b_2 + a_3 b_1}{a_1} = \frac{223241534903487835}{53648887720757472}, \\
b_4 &= -\frac{a_5 + a_2 b_3 + a_3 b_2 + a_4 b_1}{a_1} = -\frac{2568646334805333128611964 23}{119804221740759675297235 20}, \ \cdots .
\end{aligned}$$

Also, from the second recurrence relation in (2.26), we get

$$\begin{aligned}
c_1 &= -\frac{b_2}{b_1} = \frac{16496398810339}{14188933884840}, \\
c_2 &= -\frac{b_3 + b_2 c_1}{b_1} = -\frac{194269584893463401}{78871712215566400}, \ \cdots .
\end{aligned}$$

Continuing the above process, it is seen that

$$d_1 = -\frac{c_2}{c_1} = \frac{4087914301362953523}{1929557042068438120}, \quad \cdots .$$

We thus find for $x \to \infty$ that

$$A_1(x) \approx \cfrac{-\frac{1531}{19208}}{x + \cfrac{\frac{2700395}{2475627}}{x + \cfrac{\frac{16496398810339}{14188933884840}}{x + \cfrac{\frac{4087914301362953523}{1929557042068438120}}{x + \ddots}}}}. \tag{3.21}$$

From (3.17), (3.18) and (3.21), we obtain the desired result (3.14). The proof of Theorem 4 is thus completed. □

**Remark 4.** By applying a lemma of Mortici [23, 24], You [32, Theorem 1] proved for $n \to \infty$ that

$$\Gamma(n+1) \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1+\frac{1}{n-\frac{1}{2}}\right)^{\frac{1}{12}} \cdot \exp\left(\frac{1}{n}\cfrac{-\frac{7}{720}}{n^2-\frac{5}{196}+\cfrac{-\frac{1531}{19208}}{n^2+\frac{2700395}{2475627}+\cfrac{-\frac{336661200211}{265467647016}}{n+\frac{4555949691907915}{1388500719857988}+\ddots}}}\right). \tag{3.22}$$

We find that

$$\cfrac{-\frac{7}{720}}{n^3-\frac{5}{196}n+\cfrac{-\frac{1531}{19208}}{n+\cfrac{\frac{2700395}{2475627}}{n+\cfrac{\frac{16496398810339}{14188933884840}}{n+\cfrac{\frac{4087914301362953523}{1929557042068438120}}{n}}}}} = \frac{1}{n}\cfrac{-\frac{7}{720}}{n^2-\frac{5}{196}+\cfrac{-\frac{1531}{19208}}{n^2+\frac{2700395}{2475627}+\cfrac{-\frac{336661200211}{265467647016}}{n+\frac{4555949691907915}{1388500719857988}}}}. \tag{3.23}$$

This development seems to indicate that the formula (3.14) is equivalent to the formula (3.22).

By Lemma 1, we obtain from (3.12) for $x \to \infty$ that

$$\begin{aligned} F(x) &:= \frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(1+\frac{1}{x-\frac{1}{2}}\right)^{\frac{1}{12}}} - 1 \\ &\sim -\frac{7}{720x^3} - \frac{1}{4032x^5} + \frac{49}{1036800x^6} - \frac{1}{1280x^7} + \frac{1}{414720x^8} + \frac{19841227}{24634368000x^9} \\ &\quad + \frac{6199}{812851200x^{10}} - \frac{104610517}{54344908800x^{11}} - \frac{3793207123}{496628858880000x^{12}} + \cdots. \end{aligned} \tag{3.24}$$

Let us now define the function $G(x)$ by

$$F(x) = \frac{-\frac{7}{720}}{G(x)}. \tag{3.25}$$

Then, by Corollary 3, we find for $x \to \infty$ that

$$G(x) = \frac{-\frac{7}{720}}{F(x)} \sim x^3 - \frac{5}{196}x + \frac{7}{1440} + A_2(x), \tag{3.26}$$

where

$$\begin{aligned} A_2(x) = &-\frac{1531}{19208x} + \frac{700005796811}{8050579891200x^3} - \frac{803771788246897}{4102575512555520x^5} \\ &+ \frac{292889867213204249}{446724889144934400x^7} - \frac{2173108372968318747066599103411}{716170794259761532403712000000x^9} + \cdots . \end{aligned} \tag{3.27}$$

The asymptotic expansion (3.27) can be transformed into the continued-fraction approximation of the form:

$$A_2(x) \approx \cfrac{-\frac{1531}{19208}}{x + \cfrac{\frac{700005796811}{641682518400}}{x + \cfrac{\frac{13851424906662663998852 3}{119170597422441942748800}}{x + \ddots}}} \qquad (x \to \infty), \tag{3.28}$$

where the constants in the right-hand side are determined by using the system (2.26).

From (3.25), (3.26) and (3.28), we obtain Theorem 5 below, which converts the asymptotic expansion (3.24) into a continued fraction.

**Theorem 5.** *For $x \to \infty$, the following asymptotic formula holds true*:

$$\begin{aligned} \Gamma(x+1) \sim &\sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{x - \frac{1}{2}}\right)^{\frac{1}{12}} \\ &\cdot \left(1 + \cfrac{-\frac{7}{720}}{x^3 - \frac{5}{196}x + \frac{7}{1440} + \cfrac{-\frac{1531}{19208}}{x + \cfrac{\frac{700005796811}{641682518400}}{x + \cfrac{\frac{138514249066626639988523}{119170597422441942748800}}{x + \ddots}}}}\right). \end{aligned} \tag{3.29}$$

# 4 Psi (or Digamma) Function and the Euler-Mascheroni Constant

In this section, we first establish a continued-fraction approximation for the psi (or digamma) function $\psi(x)$. Based upon the obtained result, we the present the higher-order estimates for the Euler-Mascheroni constant $\gamma$.

The function $\psi\left(x+\frac{1}{2}\right)$ is known to have the following asymptotic formula (see [21, p. 33]):

$$\psi\left(x+\frac{1}{2}\right) \sim \ln x - \sum_{k=0}^{\infty} \frac{B_{2k+2}\left(\frac{1}{2}\right)}{(2k+2)x^{2k+2}} \qquad (x \to \infty), \tag{4.1}$$

where $\{B_n(x)\}_{n\in\mathbb{N}_0}$ denotes the Bernoulli polynomials defined by the following generating function:

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\, \frac{t^n}{n!} \qquad (|t| < 2\pi). \tag{4.2}$$

In terms of the Bernoulli numbers $\{B_n\}_{n\in\mathbb{N}_0}$, that is, $\{B_n(0)\}_{n\in\mathbb{N}_0}$, it is known that (see, for example, [1, p. 805])

$$B_n\left(\frac{1}{2}\right) = -\left(1-\frac{1}{2^{n-1}}\right) B_n \qquad (n \in \mathbb{N}_0), \tag{4.3}$$

the expansion formula (4.1) can be written as follows:

$$\psi\left(x+\frac{1}{2}\right) \sim \ln x + \sum_{k=1}^{\infty} \left(1-\frac{1}{2^{2k-1}}\right)\left(\frac{B_{2k}}{2k}\right)\, x^{2k} \qquad (x \to \infty). \tag{4.4}$$

By Lemma 1, we have

$$\begin{aligned} e^{\psi\left(x+\frac{1}{2}\right)} &\sim x \exp\left[\sum_{k=1}^{\infty}\left(1-\frac{1}{2^{2k-1}}\right)\, \left(\frac{B_{2k}}{2k}\right)\, x^{2k}\right] \\ &\sim x\sum_{n=0}^{\infty}\frac{a_n}{x^{2n}} = x + \sum_{n=1}^{\infty}\frac{a_n}{x^{2n-1}} \qquad (x \to \infty), \end{aligned} \tag{4.5}$$

where

$$a_0 = 1 \qquad \text{and} \qquad a_n = \frac{1}{n}\sum_{k=1}^{n} k\left(1-\frac{1}{2^{2k-1}}\right)\, \left(\frac{B_{2k}}{2k}\right)\, a_{n-k} \qquad (n \in \mathbb{N}). \tag{4.6}$$

We are thus led to the following asymptotic formula:

$$
\begin{aligned}
e^{\psi\left(x+\frac{1}{2}\right)} - x \sim \sum_{n=1}^{\infty} \frac{a_n}{x^{2n-1}} \\
&= \frac{1}{24x} - \frac{37}{5760x^3} + \frac{10313}{2903040x^5} - \frac{5509121}{1393459200x^7} \\
&\quad + \frac{2709398569}{367873228800x^9} - \frac{499769010050743}{24103053950976000x^{11}} + \cdots \qquad (4.7)
\end{aligned}
$$

as $x \to \infty$.

Theorem 6 transforms the asymptotic expansion (4.7) into a continued fraction of the form (4.8).

**Theorem 6.** *For $x \to \infty$, it is asserted that*

$$
e^{\psi\left(x+\frac{1}{2}\right)} - x \approx \cfrac{a_1}{x + \cfrac{b_1}{x + \cfrac{c_1}{x + \cfrac{d_1}{x + \ddots}}}}, \tag{4.8}
$$

*where*

$$
a_1 = \frac{1}{24}, \quad b_1 = \frac{37}{240}, \quad c_1 = \frac{74381}{186480}, \quad d_1 = \frac{2153427637}{2774113776}, \quad \cdots .
$$

*Proof.* Let the function $A(x)$ be given by

$$
A(x) = e^{\psi\left(x+\frac{1}{2}\right)} - x.
$$

It follows from (4.4) that

$$
A(x) \sim \sum_{j=1}^{\infty} \frac{a_j}{x^{2j-1}} \qquad (x \to \infty), \tag{4.9}
$$

where the coefficients $a_j \quad (j \in \mathbb{N})$ are given in (4.6). By Remark 2, the asymptotic expansion (4.9) can be transformed into the continued-fraction approximation of the form

$$
A(x) \approx \cfrac{a_1}{x + \cfrac{b_1}{x + \cfrac{c_1}{x + \cfrac{d_1}{x + \ddots}}}} \qquad x \to \infty, \tag{4.10}
$$

where the constants in the right-hand side can be determined using the system (2.26).

We see from (4.7) that

$$a_1 = \frac{1}{24}, \quad a_2 = -\frac{37}{5760x}, \quad a_3 = \frac{10313}{2903040}, \quad a_4 = -\frac{5509121}{1393459200}, \quad a_5 = \frac{2709398569}{367873228800}, \quad \cdots .$$

We obtain from the first recurrence relation in (2.26) that

$$\begin{aligned}
b_1 &= -\frac{a_2}{a_1} = \frac{37}{240}, \\
b_2 &= -\frac{a_3 + a_2 b_1}{a_1} = -\frac{74381}{1209600}, \\
b_3 &= -\frac{a_4 + a_2 b_2 + a_3 b_1}{a_1} = \frac{499469}{6912000}, \\
b_4 &= -\frac{a_5 + a_2 b_3 + a_3 b_2 + a_4 b_1}{a_1} = -\frac{2345759788879}{16094453760000}, \quad \cdots .
\end{aligned}$$

Moreover, from the second recurrence relation in (2.26), we get

$$\begin{aligned}
c_1 &= -\frac{b_2}{b_1} = \frac{74381}{186480}, \\
c_2 &= -\frac{b_3 + b_2 c_1}{b_1} = -\frac{2153427637}{6954958080}, \quad \cdots .
\end{aligned}$$

Continuing the above process, we have

$$d_1 = -\frac{c_2}{c_1} = \frac{2153427637}{2774113776}, \quad \cdots .$$

We thus have completed the proof of Theorem 6. $\square$

As $x \to \infty$, the equation (4.8) can be written as follows:

$$\psi\left(x + \frac{1}{2}\right) \approx \ln\left( x + \cfrac{\frac{1}{24}}{x + \cfrac{\frac{37}{240}}{x + \cfrac{\frac{74381}{186480}}{x + \cfrac{\frac{2153427637}{2774113776}}{x + \ddots}}}} \right). \tag{4.11}$$

Now, upon setting $x = n + \frac{1}{2}$ in (4.11), we find for $n \to \infty$ that

$$\gamma \approx H_n - \ln\left(n + \frac{1}{2} + \cfrac{\frac{1}{24}}{n + \frac{1}{2} + \cfrac{\frac{37}{240}}{n + \frac{1}{2} + \cfrac{\frac{74381}{186480}}{n + \frac{1}{2} + \cfrac{\frac{2153427637}{2774113776}}{n + \frac{1}{2} + \ddots}}}}\right). \tag{4.12}$$

By changing the logarithmic term in (1.2), we are going now to derive a higher-order estimate for the Euler-Mascheroni constant $\gamma$. Indeed, if we let

$$U_n = H_n - \ln\left(n + \frac{1}{2} + \cfrac{\frac{1}{24}}{n + \frac{1}{2} + \cfrac{\frac{37}{240}}{n + \frac{1}{2}}}\right), \tag{4.13}$$

by using the Maple software, we obtain

$$U_n - \gamma = \frac{74381}{29030400 n^6} + O\left(\frac{1}{n^7}\right) \qquad (n \to \infty). \tag{4.14}$$

Motivated by (4.14), we establish Theorem 7 below, which provides the higher-order estimate for the Euler-Mascheroni constant $\gamma$. Remarkably, the convergence of the sequence $U_n$ to $\gamma$ is faster than that of the sequence $Y_n$ defined by (1.8).

**Theorem 7.** *For $n \geqq 1$, we have*

$$\frac{74381}{29030400(n + \frac{63}{100})^6} < U_n - \gamma < \frac{74381}{29030400(n + \frac{1}{2})^6}. \tag{4.15}$$

*Proof.* In order to prove (4.15), it suffices to show that

$$f(n) > 0 \qquad \text{and} \qquad g(n) < 0 \qquad (n \in \mathbb{N}),$$

where

$$f(x) = \psi(x+1) - \ln\left(x + \frac{1}{2} + \cfrac{\frac{1}{24}}{x + \frac{1}{2} + \cfrac{\frac{37}{240}}{x + \frac{1}{2}}}\right) - \frac{74381}{29030400(n + \frac{63}{100})^6}$$

and

$$g(x) = \psi(x+1) - \ln\left(x + \frac{1}{2} + \cfrac{\frac{1}{24}}{x + \frac{1}{2} + \cfrac{\frac{37}{240}}{x + \frac{1}{2}}}\right) - \frac{74381}{29030400(n+\frac{1}{2})^6}.$$

Differentiating $f(x)$ and using the right-hand side of (1.18), we have

$$\begin{aligned}
f'(x) &= \psi'(x+1) - \frac{2(57600x^4 + 115200x^3 + 101760x^2 + 44160x + 9179)}{(240x^2 + 240x + 97)(480x^3 + 720x^2 + 454x + 107)} + \frac{74381}{4838400(x + \frac{63}{100})^7} \\
&< \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} + \frac{5}{66x^{11}} \\
&\quad - \frac{2(57600x^4 + 115200x^3 + 101760x^2 + 44160x + 9179)}{(240x^2 + 240x + 97)(480x^3 + 720x^2 + 454x + 107)} + \frac{74381}{4838400(x + \frac{63}{100})^7} \\
&= -\frac{f_1(x-3)}{2970x^{11}(100x + 63)^7(240x^2 + 240x + 97)(480x^3 + 720x^2 + 454x + 107)},
\end{aligned}$$

where

$$\begin{aligned}
f_1(x) &= 2595290749847364886436366082 + 20099909043227097376167706083x \\
&\quad + 57640582879801692383134513530x^2 + 93299887267720070931978816744x^3 \\
&\quad + 99888564914785863082055344320x^4 + 76518511193633369306361441372x^5 \\
&\quad + 43795552166475427284343650801x^6 + 19181471436764520289478294094x^7 \\
&\quad + 6501543576796264038230012670x^8 + 1708061605083425665456126695x^9 \\
&\quad + 345203867549102161486735500x^{10} + 52716866090761185950050000x^{11} \\
&\quad + 5888228135878079600000000x^{12} + 454068335914132500000000x^{13} \\
&\quad + 21613250671650000000000x^{14} + 478641735000000000000x^{15}.
\end{aligned}$$

Hence, clearly, $f'(x) < 0$ for $x \geqq 3$, and we have

$$f(x) > \lim_{t\to\infty} f(t) = 0 \qquad (x \geqq 3).$$

Direct computations yield

$$f(1) = 0.00000006339\cdots \qquad \text{and} \qquad f(2) = 0.0000007538\cdots.$$

Consequently, the inequality $f(n) > 0$ holds true for all $n \in \mathbb{N}$.

Next, upon differentiating $g(x)$ and using the left-hand side of (1.18), we have

$$\begin{aligned} g'(x) &= \psi'(x+1) - \frac{2(57600x^4+115200x^3+101760x^2+44160x+9179)}{(240x^2+240x+97)(480x^3+720x^2+454x+107)} + \frac{74381}{4838400(x+\frac{1}{2})^7} \\ &> \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} \\ &\quad - \frac{2(57600x^4+115200x^3+101760x^2+44160x+9179)}{(240x^2+240x+97)(480x^3+720x^2+454x+107)} + \frac{74381}{4838400(x+\frac{1}{2})^7} \\ &= \frac{g_1(x-3)}{37800x^9(2x+1)^6(480x^3+720x^2+454x+107)(240x^2+240x+97)}, \end{aligned}$$

where

$$\begin{aligned} g_1(x) &= 83188275652737 + 2081335933362051x + 5696424067987728x^2 \\ &\quad + 7331238869573304x^3 + 5659377310564002x^4 + 2882512211350014x^5 \\ &\quad + 1009870638085332x^6 + 246216780083736x^7 + 41230462609413x^8 \\ &\quad + 4537072471519x^9 + 296215637760x^{10} + 8712224640x^{11}. \end{aligned}$$

Hence, the inequality $g'(x) > 0$ for $x \geqq 3$, and we have

$$g(x) < \lim_{t\to\infty} g(t) = 0 \qquad (x \geqq 3).$$

Direct computations would yield

$$g(1) = -0.0000882\cdots \qquad \text{and} \qquad g(2) = -0.000001998\cdots.$$

Hence, clearly, the inequality $g(n) < 0$ holds true for all $n \in \mathbb{N}$. The proof of Theorem 7 is thus completed. □

**Remark 5.** For $n \in \mathbb{N}$, the following higher-order approximation holds true:

$$\frac{2913008718640511}{1149236702517657600(n+\frac{4}{5})^{10}} < I_n - \gamma < \frac{2913008718640511}{1149236702517657600(n+\frac{1}{2})^{10}}, \tag{4.16}$$

where

$$I_n = H_n - \ln\left( n + \frac{1}{2} + \cfrac{\frac{1}{24}}{n+\frac{1}{2} + \cfrac{\frac{37}{240}}{n+\frac{1}{2} + \cfrac{\frac{74381}{186480}}{n+\frac{1}{2} + \cfrac{\frac{2153427637}{2774113776}}{n+\frac{1}{2}}}}} \right). \tag{4.17}$$

Following the same method as was used in the proof of Theorem 7, we can prove (4.16). Here we omit the proof.

# 5 Concluding Remarks and Open Problems

In our present investigation, we have provided a potentially useful method in order to construct a continued-fraction approximation based upon a given asymptotic expansion. As applications of the method which we have developed here, we have successfully established a number of continued-fraction approximations for the gamma and the digamma (or psi) functions.

We choose to conclude our paper by presenting some closely-related open problems.

**I. The Alzer-Martins Inequalities.** It is known, for $r > 0$ and $n \in \mathbb{N}$, that

$$\frac{n}{n+1} < \left( \frac{\frac{1}{n} \sum_{i=1}^{n} i^r}{\frac{1}{n+1} \sum_{i=1}^{n+1} i^r} \right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{5.1}$$

(I.1) In the year 1988, while investigating a problem on Lorentz sequence spaces, Martins [22] published the right-hand inequality in (5.1), namely,

$$\left( \frac{\frac{1}{n} \sum_{i=1}^{n} i^r}{\frac{1}{n+1} \sum_{i=1}^{n+1} i^r} \right)^{\frac{1}{r}} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \quad (r > 0) \qquad \text{(Martins inequality)}.$$

(I.2) The left-hand inequality in (5.1) was proved in 1993 by Alzer [3], namely,

$$\frac{n}{n+1} < \left( \frac{\frac{1}{n} \sum_{i=1}^{n} i^r}{\frac{1}{n+1} \sum_{i=1}^{n+1} i^r} \right)^{\frac{1}{r}} \quad (r > 0) \qquad \text{(Alzer inequality)}.$$

(I.3) In the year 1994, Alzer [4] showed that, if $r < 0$, the Martins inequality is reversed, that is,

$$\left( \frac{\frac{1}{n} \sum_{i=1}^{n} i^r}{\frac{1}{n+1} \sum_{i=1}^{n+1} i^r} \right)^{\frac{1}{r}} > \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \quad (r < 0) \qquad \text{(Reversed Martins inequality)}.$$

(I.4) In the year 2005, Chen and Qi [11] proved that the Alzer inequality is valid for all real numbers $r$, that is,

$$\frac{n}{n+1} < \left( \frac{\frac{1}{n}\sum\limits_{i=1}^{n} i^r}{\frac{1}{n+1}\sum\limits_{i=1}^{n+1} i^r} \right)^{\frac{1}{r}} \qquad (r \in \mathbb{R} \setminus \{0\}) \qquad \text{(Extended Alzer inequality)}.$$

We note here that

$$\lim_{r \to 0} \left( \frac{\frac{1}{n}\sum\limits_{i=1}^{n} i^r}{\frac{1}{n+1}\sum\limits_{i=1}^{n+1} i^r} \right)^{\frac{1}{r}} = \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}.$$

The inequality (5.1) has indeed attracted much interest of from many mathematicians and has motivated a large number of research papers concerning its new proofs as well as its various extensions, generalizations and improvements. See also [2] for some historical notes.

**The Chen-Qi Conjecture.** Chen and Qi [11] posed the following conjecture:

*For any given natural number* $n$*, the function* $f(r)$ *given by*

$$f(r) = \begin{cases} \left( \dfrac{\frac{1}{n}\sum\limits_{i=1}^{n} i^r}{\frac{1}{n+1}\sum\limits_{i=1}^{n+1} i^r} \right)^{\frac{1}{r}} & (r \neq 0) \\[2em] \dfrac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} & (r = 0) \end{cases}$$

*is strictly decreasing on* $(-\infty, \infty)$.

**Remark 6.** If the Chen-Qi conjecture can be proved, then we obtain a unified treatment of the results (I.1) to (I.4).

Upon differentiation, we get

$$r^2\, \frac{f'(r)}{f(r)} = x_{n+1} - x_n,$$

where

$$x_n = \ln\left( \frac{1}{n}\sum_{j=1}^{n} j^r \right) - \frac{\sum\limits_{j=1}^{n} j^r\ \ln(j^r)}{\sum\limits_{j=1}^{n} j^r}. \tag{5.2}$$

Thus, in order to prove the Chen-Qi conjecture (that is, $f'(r) < 0$), it suffices to show the following *Open Problem*.

**Open Problem 2.** *Prove that, for any given $r \in \mathbb{R}$, the sequence $(x_n)$, defined by* (5.2), *is strictly decreasing.*

**II. Infinite Product Formulas.** We begin by recalling, among several useful equivalent forms (see [30, Section 1.1]), the following familiar Weierstrass canonical product form of the Gamma function $\Gamma(z)$ (see, for example, [1, p. 255, Entry (6.1.3)]; see also [30, p. 1, Eq. (2)]):

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left\{ e^{-\frac{z}{n}} \left(1 + \frac{z}{n}\right) \right\}, \tag{5.3}$$

where $\gamma$ denotes the Euler-Mascheroni constant.

In the year 2013, Chen and Choi [7] proved the following theorem.

**Theorem 8** (see [7])**.** *Let*

$$\mathcal{A}(p, q) = \prod_{j=1}^{\infty} \left\{ e^{-\frac{p}{j}} \left(1 + \frac{p}{j} + \frac{q}{j^2}\right) \right\} \qquad (p, q \in \mathbb{C};\ \Re(p) > 0)\,. \tag{5.4}$$

*Then*

$$\mathcal{A}(p, q) = \frac{e^{-p\gamma}}{\Gamma\left(1 + \frac{1}{2}p - \frac{1}{2}\sqrt{p^2 - 4q}\right) \Gamma\left(1 + \frac{1}{2}p + \frac{1}{2}\sqrt{p^2 - 4q}\right)} \qquad (p, q \in \mathbb{C})\,. \tag{5.5}$$

**Remark 7.** Upon setting $q = 0$ and replacing $p$ by $z$ in (5.5), we get

$$\mathcal{A}(z, 0) = \prod_{j=1}^{\infty} \left\{ e^{-\frac{z}{j}} \left(1 + \frac{z}{j}\right) \right\} = \frac{e^{-z\gamma}}{\Gamma(z+1)}. \tag{5.6}$$

which, in view of the following recurrence relation:

$$\Gamma(z + 1) = z\Gamma(z),$$

is seen to be equivalent to the Weierstrass canonical product form (5.3) of the Gamma function. Obviously, therefore, the Choi-Srivastava product formula (5.5) can be looked upon as a generalization of the Weierstrass canonical product form (5.3) of the Gamma function.

In light of the well-known $\Gamma$-function integral given by

$$\Gamma(z) = \int_0^{\infty} t^{z-1}\, e^{-t}\, \mathrm{d}t \qquad \big(\Re(z) > 0\big),$$

we propose the following *Open Problem*.

**Open Problem 3.** *Find an explicit integral expression for $\mathcal{A}(p, q)$ or $\frac{1}{\mathcal{A}(p,q)}$, where $\mathcal{A}(p, q)$ is given by* (5.4).

## References

[1] M. Abramowitz and I. A. Stegun (Editors), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Applied Mathematics Series, Vol. **55**, National Bureau of Standards, Washington, D.C., 1964; Reprinted by Dover Publications, New York, 1965 (see also [27]).

[2] S. Abramovich, J. Barić, M. Matić and J. Pečarić, On Van de Lune-Alzer's inequality, J. Math. Inequal. **1** (2007), 563–587.

[3] H. Alzer, On an inequality of H. Minc and L. Sathre, J. Math. Anal. Appl. **179** (1993), 396–402.

[4] H. Alzer, Refinement of an inequality of G. Bennett, Discrete Math. **135** (1994), 39–46.

[5] H. Alzer, On some inequalities for the gamma and psi functions, Math. Comput. **66** (1997), 373–389.

[6] C.-P. Chen and J. Choi, Unified treatment of several asymptotic expansions concerning some mathematical constants, Appl. Math. Comput. **305** (2017), 348–363.

[7] C.-P. Chen and J. Choi, Two infinite product formulas with two parameters, Integral Transforms Spec. Funct. **24** (2013), 357–363.

[8] C.-P. Chen, N. Elezović and L. Vukšić, Asymptotic formulae associated with the Wallis power function and digamma function, J. Class. Anal. **2** (2013), 151–166.

[9] C.-P. Chen and L. Lin, Remarks on asymptotic expansions for the gamma function, Appl. Math. Lett. **25** (2012), 2322–2326.

[10] C.-P. Chen and C. Mortici, New sequence converging towards the Euler-Mascheroni constant, Comput. Math. Appl. **64** (2012), 391–398.

[11] C.-P. Chen and F. Qi, Extension of an inequality of H. Alzer for negative powers, Tamkang J. Math. **36** (2005), 69–72.

[12] C.-P. Chen, H. M. Srivastava, L. Li and S. Manyama, Inequalities and monotonicity properties for the psi (or digamma) function and estimates for the Euler-Mascheroni constant, Integral Transforms Spec. Funct. **22** (2011), 681–693.

[13] C.-P. Chen and Q. Wang, Asymptotic expansions and continued fraction approximations for the harmonic number, RGMIA Res. Rep. Coll. **22** (2019), Article ID 6, 1–18 [`https://rgmia.org/v22.php`].

[14] T. P. Dence and J. B. Dence, A survey of Euler's constant, Math. Mag. **82** (2009), 255–265.

[15] D. W. DeTemple, A quicker convergence to Euler's constant, Amer. Math. Monthly **100** (1993), 468–470.

[16] D. W. DeTemple, The non-integer property of sums of reciprocals of consecutive integers, Math. Gaz. **75** (1991), 193–194.

[17] I. Gavrea and M. Ivan, A solution to an open problem on the Euler-Mascheroni constant, Appl. Math. Comput. **224** (2013), 54–57.

[18] I. Gavrea and M. Ivan, Optimal rate of convergence for sequences of a prescribed form, J. Math. Anal. Appl. **402** (2013), 35–43.

[19] J. Havil, Gamma: Exploring Euler's Constant, Princeton University Press, Princeton, New Jersey, 2003.

[20] L. Lin, Asymptotic formulas associated with psi function with applications, J. Math. Anal. Appl. **405** (2013), 52–56.

[21] Y. L. Luke, The Special Functions and Their Approximations, Vol. **I**, Mathematics in Science and Engineering, Vols. **53-I** and **53-II**, A Series of Monographs and Textbooks, Academic Press, New York and London, 1969.

[22] J. S. Martins, Arithmetic and geometric means: An application to Lorentz sequence spaces, Math. Nachr. **139** (1988), 281–288.

[23] C. Mortici, Product approximations via asymptotic integration, Amer. Math. Monthly **117** (2010), 434–441.

[24] C. Mortici, New improvements of the Stirling formula, Appl. Math. Comput. **217** (2010), 699–704.

[25] C. Mortici and H. M. Srivastava, A product approximation of the gamma function, Numer. Algorithms **69** (2015), 595–610.

[26] T. Negoi, A faster convergence to the constant of Euler, Gazeta Matematică Ser. A **15** (1997), 111–113 (in Romanian).

[27] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark (Editors), NIST Handbook of Mathematical Functions [With 1 CD-ROM (Windows, Macintosh and UNIX)], U. S. Department of Commerce, National Institute of Standards and Technology, Washington, D. C., 2010; Cambridge University Press, Cambridge, London and New York, 2010 (see also [1]).

[28] P. J. Rippon, Convergence with pictures, Amer. Math. Monthly **93** (1986), 476–478.

[29] J. Sándor and L. Debnath, On certain inequalities involving the constant $e$ and their applications, J. Math. Anal. Appl. 249 (2000), 569–582.

[30] H. M. Srivastava and J. Choi, Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.

[31] S.-J. Yang, On an open problem of Chen and Mortici concerning the Euler-Mascheroni constant, J. Math. Anal. Appl. **396** (2012), 689–693.

[32] X. You, Continued fraction approximation and inequality of the gamma function, *Results Math.* **73** (2018), Article ID 20, 1–8 [`https://doi.org/10.1007/s00025-018-0797-6`].

[33] R. M. Young, Euler's constant, Math. Gaz. **75** (1991), 187–190.