ON SOME ψ CAPUTO FRACTIONAL OSTROWSKI LIKE INEQUALITIES

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Abstract. In this paper we obtain some ψ Caputo fractional Ostrowski like inequalities. Ostrowski type inequalities involving one, two and three functions using ψ Caputo fractional derivatives definition are obtained.

1. Introduction

In the year 1938 Ostrowski [17] has obtained the useful result on inequality which is as follows: Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b) i.e $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty .$$

In past few decades the inequalities have attracted the interest from large number of mathematicians from various branches of mathematics see [2, 3, 16, 7].

Fractional Calculus is the study of non integer order derivative and integration which has applications in many areas of science and technology. During last few decades theory of fractional calculus has been developed rapidly due to its applications in various fields.

Recently many authors have studied the various type of Ostrowski like inequalities using various fractional order derivative and integral definitions see [4, 5, 6, 12, 13, 14, 18, 19, 20]. In [10, 15] the authors have obtained results on Ostrowski inequalities in functions of two and three variables using the right Caputo fractional derivatives. In [11] authors have studied Gruss and Cebysev type inequalities on local fractional integral. Results on Ostrowski type inequalities on conformable fractional calculus are obtained by authors in [9].

Motivated by the results in the above papers in this paper we obtain Some Ostrowski inequalities involving one, two and three functions using the ψ Caputo fractional derivatives of a function with respect to another functions.

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2. Preliminaries

Now in this section we give some definitions and lemmas for ψ Caputo fractional derivative and integral used in our subsequent discussions As given in [8] the fractional integral and derivative of a function f with respect to another function ψ are defined as follows:

Definition 2.1. [1, 8]Let $I=[a,b]$ be an interval, $\alpha>0$, f is an integrable function defined on I and $\psi\in C^1(I)$ an increasing function such that $\psi'(x)\neq 0$ for all $x\in I$ then fractional derivative and integral of f is given by

$$I_{a+}^{\alpha,\psi}f(x)=\frac{1}{\Gamma(\alpha)}\int\limits_a^x\psi'(t)\left(\psi(x)-\psi(t)\right)^{\alpha-1}f(t)\,dt,$$

and

$$\begin{aligned}D_{a+}^{\alpha,\psi}f(x)&=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{a+}^{n-\alpha,\psi}f(x)\\&=\frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\int\limits_a^x\psi'(t)\left(\psi(x)-\psi(t)\right)^{n-\alpha-1}f(t)\,dt,\end{aligned}$$

respectively. Similarly right fractional integral and right fractional derivative are

$$I_{b-}^{\alpha,\psi}f(x)=\frac{1}{\Gamma(\alpha)}\int\limits_a^x\psi'(t)\left(\psi(t)-\psi(x)\right)^{\alpha-1}f(t)\,dt,$$

and

$$\begin{aligned}D_{b-}^{\alpha,\psi}f(x)&=\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{b-}^{n-\alpha,\psi}f(x)\\&=\frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\int\limits_a^x\psi'(t)\left(\psi(t)-\psi(x)\right)^{n-\alpha-1}f(t)\,dt.\end{aligned}$$

In [1] the authors have reformulated the above definition and given the following definition of left ψ Caputo type fractional derivative

Definition 2.2. Let $\alpha>0$, $n\in\mathbb{N}$, I is the interval $-\infty\leq a<b\leq\infty$, $f,\psi\in C^n(I)$ two functions such that ψ is increasing and $\psi'(x)\neq 0$ for all $x\in I$. The left ψ-Caputo fractional derivative of f of order α is given by

$$D_{a+}^{\alpha,\psi}f(x)=I_{a+}^{n-\alpha,\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n f(x),$$

and the right ψ-Caputo fractional derivative of f is given by

$$D_{b-}^{\alpha,\psi}f(x)=I_{b-}^{n-\alpha,\psi}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n f(x).$$

For given $\alpha = m \in \mathbb{N}$

$$D_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{n-\alpha-1} f_\psi^{[n]}(t)\, dt,$$

and

$$D_{b-}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \psi'(t) (\psi(t) - \psi(x))^{n-\alpha-1} (-1)^n f_\psi^{[n]}(t)\, dt.$$

In particular when $\alpha \in (0,1)$ then

$$D_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (\psi(x) - \psi(t))^{-\alpha} f'(t)\, dt,$$

and

$$D_{b-}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (\psi(t) - \psi(x))^{-\alpha} f'(t)\, dt.$$

In [Theorem 4 [1]] the author has given the relation between fractional derivative and integral with respect to the same function as

Lemma. [1] Given a function $f \in C^n[a,b]$ and $\alpha > 0$, we have

$$I_{a+}^{\alpha,\psi C} D_{a+}^{\alpha,\psi} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(a)}{k!} (\psi(x) - \psi(a))^k.$$

and

$$I_{b-}^{\alpha,\psi C} D_{b-}^{\alpha,\psi} f(x) = f(x) - \sum_{k=0}^{n-1} (-1)^k \frac{f_\psi^{[k]}(b)}{k!} (\psi(b) - \psi(x))^k.$$

In particular given $\alpha \in (0,1)$, we have as given by author in [1]

$$I_{a+}^{\alpha,\psi C} D_{a+}^{\alpha,\psi} f(x) = f(x) - f(a).$$

and

$$I_{b-}^{\alpha,\psi C} D_{b-}^{\alpha,\psi} f(x) = f(x) - f(b).$$

The ψ fractional Taylor's formula as given in [1] is

$$f(x) = \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(a)}{k!} (\psi(x) - \psi(a))^k + I_{a+}^{\alpha,\psi C} D_{a+}^{\alpha,\psi} f(x). \tag{2.3}$$

If $f_\psi^k(a) = 0$ for $k = 1, ..., n-1$. in (2.3)then we have

$$f(x) - f(a) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1}\, {}^C D_{a+}^{\alpha,\psi} f(t)\, dt. \tag{2.4}$$

3. Main Results

Now in our result we give the ψ Caputo fractional Ostrowski inequality involving one function as follows:

Theorem 3.1. Let $\alpha \in (0,1),\ f : [a,b] \to \mathbb{R}$ be a continuous functions on [a,b] and ${}^{C}D^{\alpha,\psi}_{a+}f : (a,b) \to \mathbb{R}$ are bounded i.e $\left\| {}^{C}D^{\alpha,\psi}_{a+}f \right\| = Sup \left| {}^{C}D^{\alpha,\psi}_{a+}f(t) \right| < \infty$ then

$$\left| \frac{1}{(\psi(x)-\psi(a))} \int_a^b f(x)dx - f(a) \right| \leq \frac{\left\| {}^{C}D^{\alpha,\psi}_{a+}f \right\|_{\infty}}{\Gamma(\alpha+2)} (\psi(x)-\psi(a))^{\alpha}. \tag{3.1}$$

Proof. We have from (2.4)

$$f(x) - f(a) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x)-\psi(t))^{\alpha-1} \, {}^{C}D^{\alpha,\psi}_{a+} f(t)\, dt. \tag{3.2}$$

Now using the properties of modulus we have

$$\begin{aligned} |f(x) - f(a)| &\leq \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x)-\psi(t))^{\alpha-1} \, {}^{C}D^{\alpha,\psi}_{a+} f(t)\, dt \\ &\leq \frac{1}{\Gamma(\alpha)} \left(\int_a^x \psi'(t) (\psi(x)-\psi(t))^{\alpha-1} \right) \left\| {}^{C}D^{\alpha,\psi}_{a+}f \right\|_{\infty} \\ &= \frac{\left\| {}^{C}D^{\alpha,\psi}_{a+}f \right\|_{\infty}}{\Gamma(\alpha+1)} (\psi(x)-\psi(t))^{\alpha}. \end{aligned} \tag{3.3}$$

Thus we have

$$\begin{aligned} \left| \frac{1}{(\psi(x)-\psi(a))} \int_a^b f(x)dx - f(a) \right| &= \left| \frac{1}{(\psi(x)-\psi(a))} \int_a^b (f(x)-f(a))\, dx \right| \\ &\leq \frac{1}{(\psi(x)-\psi(a))} \int_a^b |f(x)-f(a)|\, dx \\ &\leq \frac{\left\| {}^{C}D^{\alpha,\psi}_{a+}f \right\|_{\infty}}{\Gamma(\alpha+2)} (\psi(b)-\psi(a))^{\alpha}. \end{aligned} \tag{3.4}$$

Which is required inequality.

□

Now in our next theorem we give the ψ Caputo fractional Ostrowski inequality involving two functions

Theorem 3.2. Let $\alpha \in (0,1)$, $f,g : [a,b] \to \mathbb{R}$ be a continuous functions on [a,b] and ${}^{C}D_{a+}^{\alpha,\psi}f$, ${}^{C}D_{a+}^{\alpha,\psi}g$: $(a,b) \to \mathbb{R}$ are bounded i.e $\left\|{}^{C}D_{a+}^{\alpha,\psi}f\right\| = Sup\left|{}^{C}D_{a+}^{\alpha,\psi}f(t)\right| < \infty$ and $\left\|{}^{C}D_{a+}^{\alpha,\psi}g\right\| = Sup\left|{}^{C}D_{a+}^{\alpha,\psi}g(t)\right| < \infty$ then

$$\begin{aligned} &\left|2\int_a^b f(x)\,g(x)\,dx - \int_a^b [f(a)\,g(x) - f(x)\,g(a)]dx\right| \\ &\leq \left\|{}^{C}D_{a+}^{\alpha,\psi}f\right\|_{\infty} I_{a+}^{\alpha+1,\psi}\,|g(x)| + \left\|{}^{C}D_{a+}^{\alpha,\psi}g\right\|_{\infty} I_{a+}^{\alpha+1,\psi}\,|f(x)|\,. \end{aligned} \tag{3.5}$$

Proof. From the hypotheses we have

$$f(x) - f(a) = \frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)\,(\psi(x) - \psi(t))^{\alpha-1}\,{}^{C}D_{a+}^{\alpha,\psi}f(t)\,dt, \tag{3.6}$$

and

$$g(x) - g(a) = \frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)\,(\psi(x) - \psi(t))^{\alpha-1}\,{}^{C}D_{a+}^{\alpha,\psi}g(t)\,dt. \tag{3.7}$$

Multiplying (3.6) by $g(x)$ and (3.7) by $f(x)$ we have

$$f(x)\,g(x) - f(a)\,g(x) = \frac{g(x)}{\Gamma(\alpha)}\int_a^x \psi'(t)\,(\psi(x) - \psi(t))^{\alpha-1}\,{}^{C}D_{a+}^{\alpha,\psi}f(t)\,dt, \tag{3.8}$$

$$f(x)\,g(x) - f(x)\,g(a) = \frac{f(x)}{\Gamma(\alpha)}\int_a^x \psi'(t)\,(\psi(x) - \psi(t))^{\alpha-1}\,{}^{C}D_{a+}^{\alpha,\psi}g(t)\,dt. \tag{3.9}$$

Adding (3.8) and (3.9) we get

$$\begin{aligned} &2f(x)\,g(x) - f(a)\,g(x) - f(x)\,g(a) \\ &= \frac{g(x)}{\Gamma(\alpha)}\int_a^x \psi'(t)\,(\psi(x) - \psi(t))^{\alpha-1}\,{}^{C}D_{a+}^{\alpha,\psi}f(t)\,dt \\ &+ \frac{f(x)}{\Gamma(\alpha)}\int_a^x \psi'(t)\,(\psi(x) - \psi(t))^{\alpha-1}\,{}^{C}D_{a+}^{\alpha,\psi}g(t)\,dt. \end{aligned} \tag{3.10}$$

Now integrating both sides of (3.10) with respect to x over $[a,b]$ we have

$$\begin{aligned} &2\int_a^b f(x)\,g(x) - \int_a^b [f(a)\,g(x) - f(x)\,g(a)]\,dx \\ &= \int_a^b \frac{g(x)}{\Gamma(\alpha)}\left(\int_a^x \psi'(t)\,(\psi(x) - \psi(t))^{\alpha-1}\,{}^{C}D_{a+}^{\alpha,\psi}f(t)\,dt\right)dx \end{aligned}$$

$$+\int_a^b \frac{f(x)}{\Gamma(\alpha)}\left(\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\, {}^C D_{a+}^{\alpha,\psi} g(t)\,dt\right)dx. \tag{3.11}$$

From (3.11) and using the properties of modulus we have

$$\left|2\int_a^b f(x)\,g(x) - \int_a^b [f(a)\,g(x) - f(x)\,g(a)]\,dx\right|$$

$$\leq \int_a^b \frac{|g(x)|}{\Gamma(\alpha)}\left(\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\left|{}^C D_{a+}^{\alpha,\psi} f(t)\right| dt\right)dx$$

$$+\int_a^b \frac{|f(x)|}{\Gamma(\alpha)}\left(\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\left|{}^C D_{a+}^{\alpha,\psi} g(t)\right| dt\right)dx. \tag{3.12}$$

From above equation it is easy to see that

$$\left|2\int_a^b f(x)\,g(x) - \int_a^b [f(a)\,g(x) - f(x)\,g(a)]\,dx\right|$$

$$\leq \frac{\left\|{}^C D_{a+}^{\alpha,\psi} f\right\|_\infty}{\Gamma(\alpha)}\int_a^b |g(x)|\left(\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\,dt\right)dx$$

$$+\frac{\left\|{}^C D_{a+}^{\alpha,\psi} g\right\|_\infty}{\Gamma(\alpha)}\int_a^b |f(x)|\left(\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\,dt\right)dx$$

$$=\frac{\left\|{}^C D_{a+}^{\alpha,\psi} f\right\|_\infty}{\Gamma(\alpha)}\int_a^b (\psi(x)-\psi(a))^{\alpha}\,|g(x)|\,dx$$

$$+\frac{\left\|{}^C D_{a+}^{\alpha,\psi} g\right\|_\infty}{\Gamma(\alpha)}\int_a^b (\psi(x)-\psi(a))^{\alpha}\,|f(x)|\,dx$$

$$=\left\|{}^C D_{a+}^{\alpha,\psi} f\right\|_\infty I_{a+}^{\alpha+1,\psi} g(x) + \left\|{}^C D_{a+}^{\alpha,\psi} g\right\|_\infty I_{a+}^{\alpha+1,\psi} f(x). \tag{3.12}$$

Which is required inequality.

□

Now in our next theorem we give the ψ Caputo fractional Ostrowski inequality involving three functions.

Theorem 3.3. Let $\alpha \in (0,1)$, $f, g, h : [a,b] \to \mathbb{R}$ be a continuous functions on [a,b] and ${}^C D_{a+}^{\alpha,\psi} f$, ${}^C D_{a+}^{\alpha,\psi} g$, ${}^C D_{a+}^{\alpha,\psi} h$: $(a,b) \to \mathbb{R}$ are bounded i.e $\left\|{}^C D_{a+}^{\alpha,\psi} f\right\| =$

$sup\left|{}^{C}D_{a+}^{\alpha,\psi}f(t)\right|<\infty$, $\left\|{}^{C}D_{a+}^{\alpha,\psi}g\right\|=sup\left|{}^{C}D_{a+}^{\alpha,\psi}g(t)\right|<\infty$ and $\left\|{}^{C}D_{a+}^{\alpha,\psi}h\right\|=sup\left|{}^{C}D_{a+}^{\alpha,\psi}h(t)\right|<\infty$ then

$$\left|3\int_a^b f(x)\,g(x)\,h(x)-\int_a^b\left[f(a)\,g(x)\,h(x)-f(x)\,g(a)\,h(x)-h(a)\,f(x)\,g(x)\right]dx\right|$$
$$\leq\left\|{}^{C}D_{a+}^{\alpha,\psi}f\right\|_\infty I_{a+}^{\alpha+1,\psi}\left|g(x)\,h(x)\right|+\left\|{}^{C}D_{a+}^{\alpha,\psi}g\right\|_\infty I_{a+}^{\alpha+1,\psi}\left|f(x)\,h(x)\right|$$
$$+\left\|{}^{C}D_{a+}^{\alpha,\psi}h\right\|_\infty I_{a+}^{\alpha+1,\psi}\left|f(x)\,g(x)\right|. \tag{3.13}$$

Proof. For any x and a from hypotheses we have

$$f(x)-f(a)=\frac{1}{\Gamma(\alpha)}\int_a^x\psi'(t)\,(\psi(x)-\psi(t))^{\alpha-1}\,{}^{C}D_{a+}^{\alpha,\psi}f(t)\,dt, \tag{3.14}$$

$$g(x)-g(a)=\frac{1}{\Gamma(\alpha)}\int_a^x\psi'(t)\,(\psi(x)-\psi(t))^{\alpha-1}\,{}^{C}D_{a+}^{\alpha,\psi}h(t)\,dt, \tag{3.15}$$

$$h(x)-h(a)=\frac{1}{\Gamma(\alpha)}\int_a^x\psi'(t)\,(\psi(x)-\psi(t))^{\alpha-1}\,{}^{C}D_{a+}^{\alpha,\psi}h(t)\,dt, \tag{3.16}$$

Multiplying both sides of (3.14),(3.15) and (3.16) by $g(x)h(x)$, $f(x)h(x)$ and $f(x)g(x)$ respectively we have

$$f(x)\,g(x)\,h(x)-f(a)\,g(x)\,h(x)=\frac{g(x)\,h(x)}{\Gamma(\alpha)}\int_a^x\psi'(t)\,(\psi(x)-\psi(t))^{\alpha-1}\,{}^{C}D_{a+}^{\alpha,\psi}f(t)\,dt, \tag{3.17}$$

$$f(x)\,h(x)g(x)-f(x)\,h(x)g(a)=\frac{f(x)\,h(x)}{\Gamma(\alpha)}\int_a^x\psi'(t)\,(\psi(x)-\psi(t))^{\alpha-1}\,{}^{C}D_{a+}^{\alpha,\psi}g(t)\,dt, \tag{3.18}$$

$$f(x)\,g(x)\,h(x)-f(x)\,g(x)\,h(a)=\frac{f(x)\,g(x)}{\Gamma(\alpha)}\int_a^x\psi'(t)\,(\psi(x)-\psi(t))^{\alpha-1}\,{}^{C}D_{a+}^{\alpha,\psi}h(t)\,dt. \tag{3.19}$$

Adding (3.17),(3.18) and (3.19) we have

$$3f(x)\,g(x)\,h(x)-f(a)\,g(x)\,h(x)-f(x)\,h(x)g(a)-f(x)\,g(x)\,h(a)$$
$$=\frac{g(x)\,h(x)}{\Gamma(\alpha)}\int_a^x\psi'(t)\,(\psi(x)-\psi(t))^{\alpha-1}\,{}^{C}D_{a+}^{\alpha,\psi}f(t)\,dt$$

$$+\frac{f(x)\,h(x)}{\Gamma(\alpha)}\int_a^x \psi'(t)\,(\psi(x)-\psi(t))^{\alpha-1}\,{}^{C}D_{a+}^{\alpha,\psi}g(t)\,dt$$
$$+\frac{f(x)\,g(x)}{\Gamma(\alpha)}\int_a^x \psi'(t)\,(\psi(x)-\psi(t))^{\alpha-1}\,{}^{C}D_{a+}^{\alpha,\psi}h(t)\,dt. \tag{3.20}$$

Integrating both sides of (3.20) with respect to x over $[a,b]$ we have

$$3\int_a^b f(x)\,g(x)\,h(x)\,dx-\int_a^b [f(a)\,g(x)\,h(x)-f(x)\,g(a)\,h(x)-f(x)\,g(x)\,h(a)]\,dx$$
$$=\int_a^b \frac{g(x)\,h(x)}{\Gamma(\alpha)}\left(\int_a^x \psi'(t)\,(\psi(x)-\psi(t))^{\alpha-1}\,{}^{C}D_{a+}^{\alpha,\psi}f(t)\,dt\right)dx$$
$$+\int_a^b \frac{f(x)\,h(x)}{\Gamma(\alpha)}\left(\int_a^x \psi'(t)\,(\psi(x)-\psi(t))^{\alpha-1}\,{}^{C}D_{a+}^{\alpha,\psi}g(t)\,dt\right)dx$$
$$+\int_a^b \frac{f(x)\,g(x)}{\Gamma(\alpha)}\left(\int_a^x \psi'(t)\,(\psi(x)-\psi(t))^{\alpha-1}\,{}^{C}D_{a+}^{\alpha,\psi}h(t)\,dt\right)dx. \tag{3.21}$$

From (3.21) and using the properties of modulus we have

$$\left|3\int_a^b f(x)\,g(x)\,h(x)\,dx-\right.$$
$$\left.-\int_a^b [f(a)\,g(x)\,h(x)-f(x)\,g(a)\,h(x)-f(x)\,g(x)\,h(a)]\,dx\right|$$
$$=\int_a^b \frac{|g(x)\,h(x)|}{\Gamma(\alpha)}\left(\int_a^x \psi'(t)\,(\psi(x)-\psi(t))^{\alpha-1}\left|{}^{C}D_{a+}^{\alpha,\psi}f(t)\right|dt\right)dx$$
$$+\int_a^b \frac{|f(x)\,h(x)|}{\Gamma(\alpha)}\left(\int_a^x \psi'(t)\,(\psi(x)-\psi(t))^{\alpha-1}\left|{}^{C}D_{a+}^{\alpha,\psi}g(t)\right|dt\right)dx$$
$$+\int_a^b \frac{|f(x)\,g(x)|}{\Gamma(\alpha)}\left(\int_a^x \psi'(t)\,(\psi(x)-\psi(t))^{\alpha-1}\left|{}^{C}D_{a+}^{\alpha,\psi}h(t)\right|dt\right)dx$$

$$\leq \frac{\left\| {}^{C}D_{a+}^{\alpha,\psi} f \right\|_{\infty}}{\Gamma(\alpha)} \int_{a}^{b} |g(x)\,h(x)| \left(\int_{a}^{x} \psi'(t) \left(\psi(x) - \psi(t)\right)^{\alpha-1} dt \right) dx$$

$$+ \frac{\left\| {}^{C}D_{a+}^{\alpha,\psi} g \right\|_{\infty}}{\Gamma(\alpha)} \int_{a}^{b} |f(x)\,h(x)| \left(\int_{a}^{x} \psi'(t) \left(\psi(x) - \psi(t)\right)^{\alpha-1} dt \right) dx$$

$$+ \frac{\left\| {}^{C}D_{a+}^{\alpha,\psi} h \right\|_{\infty}}{\Gamma(\alpha)} \int_{a}^{b} |f(x)\,g(x)| \left(\int_{a}^{x} \psi'(t) \left(\psi(x) - \psi(t)\right)^{\alpha-1} dt \right) dx$$

$$= \frac{\left\| {}^{C}D_{a+}^{\alpha,\psi} f \right\|_{\infty}}{\Gamma(\alpha)} \int_{a}^{b} \left(\psi(x) - \psi(a)\right)^{\alpha} |g(x)\,h(x)|\, dx$$

$$+ \frac{\left\| {}^{C}D_{a+}^{\alpha,\psi} g \right\|_{\infty}}{\Gamma(\alpha)} \int_{a}^{b} \left(\psi(x) - \psi(a)\right)^{\alpha} |f(x)\,h(x)|\, dx$$

$$+ \frac{\left\| {}^{C}D_{a+}^{\alpha,\psi} h \right\|_{\infty}}{\Gamma(\alpha)} \int_{a}^{b} \left(\psi(x) - \psi(a)\right)^{\alpha} |f(x)\,g(x)|\, dx$$

$$= \left\| {}^{C}D_{a+}^{\alpha,\psi} f \right\|_{\infty} I_{a+}^{\alpha+1,\psi} |g(x)\,h(x)| + \left\| {}^{C}D_{a+}^{\alpha,\psi} g \right\|_{\infty} I_{a+}^{\alpha+1,\psi} |f(x)\,h(x)|$$

$$+ \left\| {}^{C}D_{a+}^{\alpha,\psi} h \right\|_{\infty} I_{a+}^{\alpha+1,\psi} |f(x)\,g(x)|. \tag{3.22}$$

which proves our result.

□

4. Some Other Ostrowski Tye Inequalities

Now in our next theorem we give the Ostrowski type inequality of two functions on ψ fractional derivative

Theorem 4.1. Let f, g and ${}^{C}D_{a+}^{\alpha,\psi} f, {}^{C}D_{a+}^{\alpha,\psi} g$ be as in Theorem 3.2. . Then

$$\left| \int_{a}^{b} f(x)\,g(x)\,dx - g(a) \int_{a}^{b} f(x)\,dx - f(a) \int_{a}^{b} g(x)\,dx + (b-a)\,f(a)\,g(a) \right|$$

$$\leq \frac{\left\| {}^{C}D_{a+}^{\alpha,\psi} f \right\|_{\infty} \left\| {}^{C}D_{a+}^{\alpha,\psi} g \right\|_{\infty}}{\left[\Gamma(\alpha+1)\right]^{2}} \left(\psi(x) - \psi(a)\right)^{2\alpha}. \tag{4.1}$$

Proof. Multiplying right hand side and left hand side of (3.6) and (3.7)

$$f(x)\,g(x) - f(x)\,g(a) - f(a)\,g(x) + f(a)\,g(a)$$

$$= \left\{ \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} \, {}^C D_{a+}^{\alpha,\psi} f(t) \, dt \right\}$$
$$\times \left\{ \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} \, {}^C D_{a+}^{\alpha,\psi} g(t) \, dt \right\}. \tag{4.2}$$

Integrating (4.2) with respect to x over $[a, b]$ we have

$$\int_a^b f(x) g(x) \, dx - g(a) \int_a^b f(x) \, dx - f(a) \int_a^b g(x) \, dx + (b-a) f(a) g(a)$$
$$= \int_a^b \left[\left\{ \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} \, {}^C D_{a+}^{\alpha,\psi} f(t) \, dt \right\} \right.$$
$$\left. \times \left\{ \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} \, {}^C D_{a+}^{\alpha,\psi} g(t) \, dt \right\} \right]. \tag{4.3}$$

From (4.3) and using properties of modulus we have

$$\left| \int_a^b f(x) g(x) \, dx - g(a) \int_a^b f(x) \, dx - f(a) \int_a^b g(x) \, dx + (b-a) f(a) g(a) \right|$$
$$\leq \frac{\left\| {}^C D_{a+}^{\alpha,\psi} f \right\|_\infty \left\| {}^C D_{a+}^{\alpha,\psi} g \right\|_\infty}{[\Gamma(\alpha+1)]^2} (\psi(x) - \psi(a))^{2\alpha}. \tag{4.3}$$

Now in our next theorem we give the Ostrowski type inequality of three functions on ψ fractional derivative

Theorem 4.2. Let f, g, h and ${}^C D_{a+}^{\alpha,\psi} f$, ${}^C D_{a+}^{\alpha,\psi} g$, ${}^C D_{a+}^{\alpha,\psi} h$ be as in Theorem 3.3. . Then

$$\int_a^b f(x) g(x) h(x) \, dx - g(a) \int_a^b f(x) h(x) dx - f(a) \int_a^b g(x) h(x) \, dx$$
$$+ f(a) g(a) \int_a^b h(x) \, dx - h(a) \int_a^b f(x) g(x) \, dx + g(a) h(a) \int_a^b f(x) dx$$
$$+ f(a) h(a) \int_a^b g(x) dx - (b-a) f(a) g(a) h(a)$$

$$\leq \frac{\left\| D_{a+}^{\alpha,\psi} f \right\|_{\infty} \left\| D_{a+}^{\alpha,\psi} g \right\|_{\infty} \left\| D_{a+}^{\alpha,\psi} h \right\|_{\infty}}{\left(\Gamma\left(\alpha+1\right)\right)^{3}} \left(\psi\left(x\right)-\psi\left(\alpha\right)\right)^{3\alpha}. \tag{4.4}$$

Proof. Multiplying right hand side and left hand side of (3.14), (3.15) and (3.16) equations we have

$$\begin{aligned}
&\left(f(x)-f(a)\right)\left(g(x)-g(a)\right)\left(h(x)-h(a)\right) \\
&= \left\{ \frac{1}{\Gamma\left(\alpha\right)} \int_{a}^{x} \psi'\left(t\right)\left(\psi\left(x\right)-\psi\left(t\right)\right)^{\alpha-1} {}^{C}D_{a+}^{\alpha,\psi} f\left(t\right) dt \right\} \\
&\times \left\{ \frac{1}{\Gamma\left(\alpha\right)} \int_{a}^{x} \psi'\left(t\right)\left(\psi\left(x\right)-\psi\left(t\right)\right)^{\alpha-1} {}^{C}D_{a+}^{\alpha,\psi} g\left(t\right) dt \right\} \\
&\times \left\{ \frac{1}{\Gamma\left(\alpha\right)} \int_{a}^{x} \psi'\left(t\right)\left(\psi\left(x\right)-\psi\left(t\right)\right)^{\alpha-1} {}^{C}D_{a+}^{\alpha,\psi} h\left(t\right) dt \right\}.
\end{aligned} \tag{4.5}$$

From (4.5) we have

$$\begin{aligned}
&f\left(x\right)g\left(x\right)h\left(x\right)-f(x)g(a)h(x)-f(a)g(x)h(x)+f(a)g(a)h(x) \\
&-f\left(x\right)g\left(x\right)h\left(a\right)+f\left(x\right)g\left(a\right)h\left(a\right)+f\left(a\right)g\left(x\right)h\left(a\right)-f\left(a\right)g\left(a\right)h\left(a\right) \\
&= \left\{ \frac{1}{\Gamma\left(\alpha\right)} \int_{a}^{x} \psi'\left(t\right)\left(\psi\left(x\right)-\psi\left(t\right)\right)^{\alpha-1} {}^{C}D_{a+}^{\alpha,\psi} f\left(t\right) dt \right\} \\
&\times \left\{ \frac{1}{\Gamma\left(\alpha\right)} \int_{a}^{x} \psi'\left(t\right)\left(\psi\left(x\right)-\psi\left(t\right)\right)^{\alpha-1} {}^{C}D_{a+}^{\alpha,\psi} g\left(t\right) dt \right\} \\
&\times \left\{ \frac{1}{\Gamma\left(\alpha\right)} \int_{a}^{x} \psi'\left(t\right)\left(\psi\left(x\right)-\psi\left(t\right)\right)^{\alpha-1} {}^{C}D_{a+}^{\alpha,\psi} h\left(t\right) dt \right\}.
\end{aligned} \tag{4.6}$$

Now integrating with respect to x over $[a, b]$ and using the properties of modulus we get the required inequality (4.4).

□

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