SOME NEW PROPERTIES OF LOG-CONVEX FUNCTIONS DEFINED ON CONVEX SUBSETS IN LINEAR SPACES

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ABSTRACT. For a Lebesgue integrable function $w : [0,1] \to [0,\infty)$ we consider the symmetric functions

$$J_{f,w,r}(x,y) := \frac{\int_0^1 f^r((1-t)x+ty) f^r(tx+(1-t)y) w(t) dt}{f^r(x) f^r(y)}$$

and

$$M_{f,w,r}(x,y) := \frac{\int_0^1 f^r((1-t)x+ty) f^r(tx+(1-t)y) w(t) dt}{f^{2r}\left(\frac{x+y}{2}\right)}$$

where $f : C \to (0,\infty)$ is a log-convex function defined on the convex subset C of a linear space X and $r > 0$.

In this paper we show among others that $J_{f,w,r}$ is Schur concave and $M_{f,w,r}$ is Schur convex on $C \times C$. Some examples for log-convex functions of a real variable are also given.

1. Introduction

A function $f : I \to (0,\infty)$ is said to be *log-convex* or *multiplicatively convex* if $\log f$ is convex, or, equivalently, if for all x, $y \in I$ and $t \in [0,1]$ one has the inequality:

$$f(tx+(1-t)y) \le [f(x)]^t [f(y)]^{1-t}. \tag{1.1}$$

We note that if f and g are convex and g is increasing, then $g \circ f$ is convex; moreover, since $f = \exp(\log f)$, it follows that a log-convex function is convex, but the converse may not necessarily be true. This follows directly from (1.1) because, by the *arithmetic-geometric mean inequality*, we have

$$[f(x)]^t [f(y)]^{1-t} \le tf(x) + (1-t) f(y)$$

for all $x, y \in I$ and $t \in [0,1]$.

Let us recall the *Hermite-Hadamard inequality*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}, \tag{1.2}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I, $a, b \in I$ and $a < b$.

For related results, see [13] and [9].

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Note that if we apply the above inequality for the log-convex functions $f : I \to (0, \infty)$, we have that

$$\ln \left[f\left(\frac{a+b}{2}\right)\right] \le \frac{1}{b-a}\int_a^b \ln f(x)\,dx \le \frac{\ln f(a) + \ln f(b)}{2}, \tag{1.3}$$

from which we get

$$f\left(\frac{a+b}{2}\right) \le \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \le \sqrt{f(a)\,f(b)} \tag{1.4}$$

that is an inequality of Hermite-Hadamard's type for log-convex functions.

By using simple properties of log-convex functions Dragomir and Mond proved in 1998 the following result [11].

Theorem 1. *Let $f : I \to (0, \infty)$ be a log-convex mapping on I and $a, b \in I$ with $a < b$. Then one has the inequality:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \sqrt{f(x)\,f(a+b-x)}dx \le \sqrt{f(a)\,f(b)}. \tag{1.5}$$

The inequality between the first and second term in (1.5) may be improved as follows [11]. A different upper bound for the middle term in (1.5) can be also provided.

Theorem 2. *Let $f : I \to (0, \infty)$ be a log-convex mapping on I and a, $b \in I$ with $a < b$. Then one has the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \\ &\le \frac{1}{b-a}\int_a^b \sqrt{f(x)\,f(a+b-x)}dx \\ &\le \frac{1}{b-a}\int_a^b f(x)\,dx \le L(f(a), f(b)), \end{aligned} \tag{1.6}$$

where $L(p, q)$ is the logarithmic mean of the strictly positive real numbers p, q, i.e.,

$$L(p,q) := \frac{p-q}{\ln p - \ln q} \text{ if } p \ne q \text{ and } L(p,p) := p.$$

The last inequality in (1.6) was obtained in a different context in [14].

As shown in [15], the following result also holds:

Theorem 3. *Let $f : I \to (0, \infty)$ be a log-convex mapping on I and a, $b \in I$ with $a < b$. Then one has the inequalities:*

$$f\left(\frac{a+b}{2}\right) \le \left(\frac{1}{b-a}\int_a^b \sqrt{f(x)}dx\right)^2 \le \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.7}$$

We define the *p-logarithmic mean* as

$$L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, \text{with } a \ne b \\ \\ a, \text{ if } a = b \end{cases}$$

for $p \ne 0, -1$ and a, $b > 0$.

In the recent work [8] we generalized the inequality (1.6) as follows:

Theorem 4. *Let $f : [a,b] \to (0,\infty)$ be a log-convex function on $[a,b]$. Then for any $p > 0$ we have the inequality*

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \\
&\leq \left(\frac{1}{b-a}\int_a^b f^p(x)\,f^p(a+b-x)\,dx\right)^{\frac{1}{2p}} \\
&\leq \left(\frac{1}{b-a}\int_a^b f^{2p}(x)\,dx\right)^{\frac{1}{2p}} \\
&\leq \begin{cases} \left[L_{2p-1}(f(a),f(b))\right]^{1-\frac{1}{2p}}\left[L(f(a),f(b))\right]^{\frac{1}{2p}}, & p \neq \frac{1}{2}; \\ L(f(a),f(b)), & p = \frac{1}{2}. \end{cases}
\end{aligned} \tag{1.8}
$$

If $p \in \left(0, \frac{1}{2}\right)$, then we have

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \\
&\leq \left(\frac{1}{b-a}\int_a^b f^p(x)\,f^p(a+b-x)\,dx\right)^{\frac{1}{2p}} \\
&\leq \left(\frac{1}{b-a}\int_a^b f^{2p}(x)\,dx\right)^{\frac{1}{2p}} \leq \frac{1}{b-a}\int_a^b f(x)\,dx.
\end{aligned} \tag{1.9}
$$

Remark 1. *If we take in (1.8) $p = 1$, then we get*

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \\
&\leq \left(\frac{1}{b-a}\int_a^b f(x)\,f(a+b-x)\,dx\right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{b-a}\int_a^b f^2(x)\,dx\right)^{\frac{1}{2}} \\
&\leq \left[A(f(a),f(b))\right]^{\frac{1}{2}}\left[L(f(a),f(b))\right]^{\frac{1}{2}}.
\end{aligned} \tag{1.10}
$$

If we take $p = \frac{1}{4}$ in (1.9), then we get

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \\
&\leq \left(\frac{1}{b-a}\int_a^b \sqrt[4]{f(x)\,f(a+b-x)\,dx}\right)^{2} \\
&\leq \left(\frac{1}{b-a}\int_a^b \sqrt{f(x)}dx\right)^{2} \leq \frac{1}{b-a}\int_a^b f(x)\,dx.
\end{aligned} \tag{1.11}
$$

This improves the inequality (1.7).

Motivated by the above results, in this paper we study among others the Schur convexity of some functions associated to a log-convex function on C. Some examples for log-convex functions of a real variable are also given.

2. Log Convex Functions on Convex Sets in Linear Spaces

We consider the function $f : C \to \mathbb{R}$ defined on the convex subset C of the linear space X and for each $(x, y) \in C^2 := C \times C$ we introduce the auxiliary function $\varphi_{(x,y)} : [0, 1] \to \mathbb{R}$ defined by

$$\varphi_{(x,y)}(t) := f((1-t)x + ty).$$

It is well known that the function f is convex on C if and only if for each $(x, y) \in C^2$ the auxiliary function $\varphi_{(x,y)}$ is convex on $[0, 1]$.

By utilising the classical Hermite-Hadamard inequality for the convex function $\varphi_{(x,y)}$ on $[0, 1]$ we then have

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x + ty)\,dt \leq \frac{f(x) + f(y)}{2} \tag{2.1}$$

for all $(x, y) \in C^2$.

We say that the function $f : C \to (0, \infty)$ is log-convex on C if

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t} \tag{2.2}$$

for all vectors $x, y \in C$ and $t \in [0, 1]$. By taking the log in (2.2) we deduce that f is log-convex on C if $\ln f$ is convex on C.

Lemma 1. *Consider the function $f : C \to (0, \infty)$. The function f is log-convex on C if and only if for all $(x, y) \in C^2$ the auxiliary function $\varphi_{(x,y)}$ is log-convex on $[0, 1]$.*

Proof. Assume that f is log-convex on C and $(x, y) \in C^2$. Let $\alpha, \beta > 0$ with $\alpha + \beta = 1$ and $t_1, t_2 \in [0, 1]$ then

$$\begin{aligned}
\varphi_{(x,y)}(\alpha t_1 + \beta t_2) &= f((\alpha t_1 + \beta t_2)x + (1 - \alpha t_1 - \beta t_2)y) \\
&= f((\alpha t_1 + \beta t_2)x + (\alpha + \beta - \alpha t_1 - \beta t_2)y) \\
&= f(\alpha[t_1 x + (1-t_1)y] + \beta[t_2 x + (1-t_2)y]) \\
&\leq [f(t_1 x + (1-t_1)y)]^\alpha [f(t_2 x + (1-t_2)y)]^\beta \\
&= \left[\varphi_{(x,y)}(t_1)\right]^\alpha \left[\varphi_{(x,y)}(t_2)\right]^\beta,
\end{aligned}$$

which shows that $\varphi_{(x,y)}$ is log-convex on $[0, 1]$.

Let $(x, y) \in C^2$ and $t \in [0, 1]$, then by the log-convexity of $\varphi_{(x,y)}$ we have

$$\begin{aligned}
f(tx + (1-t)y) &= \varphi_{(x,y)}(t) = \varphi_{(x,y)}(t \cdot 1 + (1-t) \cdot 0) \\
&\leq \left[\varphi_{(x,y)}(1)\right]^t \left[\varphi_{(x,y)}(0)\right]^{1-t} = [f(x)]^t [f(y)]^{1-t},
\end{aligned}$$

which proves the log-convexity of f on C. □

By utilising Theorem 2 and 4 for the auxiliary function $\varphi_{(x,y)}$ we can state the following result for log-convex functions defined on the convex set C of the linear space X.

Theorem 5. *Let $f : C \to (0,\infty)$ be a log-convex function on C and $(x,y) \in C^2$, then*

$$f\left(\frac{x+y}{2}\right) \leq \exp\left[\int_0^1 \ln f(tx+(1-t)y)\,dt\right] \leq \int_0^1 \sqrt{f(tx+(1-t)y)\,f((1-t)x+ty)}dt \leq \int_0^1 f(tx+(1-t)y) \leq L(f(x),f(y)), \tag{2.3}$$

where $L(\cdot,\cdot)$ is the logarithmic mean.

For any $p>0$ we have the inequality

$$\begin{aligned} f\left(\frac{x+y}{2}\right) &\leq \exp\left[\int_0^1 \ln f(tx+(1-t)y)\,dt\right] \\ &\leq \left(\int_0^1 f^p(tx+(1-t)y)\,f^p((1-t)x+ty)\,dt\right)^{\frac{1}{2p}} \\ &\leq \left(\int_0^1 f^{2p}(tx+(1-t)y)\,dt\right)^{\frac{1}{2p}} \\ &\leq \begin{cases} [L_{2p-1}(f(x),f(y))]^{1-\frac{1}{2p}}\,[L(f(x),f(y))]^{\frac{1}{2p}}, & p \neq \frac{1}{2}, \\ L(f(x),f(y)), & p=\frac{1}{2}, \end{cases} \end{aligned} \tag{2.4}$$

where $L_r(\cdot,\cdot)$ is the r-logarithmic mean.

If $p \in \left(0,\frac{1}{2}\right)$, then we have

$$\begin{aligned} f\left(\frac{x+y}{2}\right) &\leq \exp\left[\int_0^1 \ln f(tx+(1-t)y)\,dt\right] \\ &\leq \left(\int_0^1 f^{2p}(tx+(1-t)y)\,dt\right)^{\frac{1}{2p}} \\ &\leq \left(\int_0^1 f^{2p}(tx+(1-t)y)\,dx\right)^{\frac{1}{2p}} \leq \int_0^1 f(tx+(1-t)y)\,dt. \end{aligned} \tag{2.5}$$

Now, for $t \in [0,1]$ we define the function $S_t : C^2 \to (0,\infty)$ by

$$S_{f,t}(x,y) = f(tx+(1-t)y).$$

Lemma 2. *If the function $f : C \to (0,\infty)$ is a log-convex function on C and $t \in (0,1)$, then $S_{f,t}$ is log-convex on C^2.*

Proof. Let α, $\beta > 0$ with $\alpha+\beta=1$ and (x,y), $(u,v) \in C^2$. Then

$$\begin{aligned} S_{f,t}(\alpha(x,y)+\beta(u,v)) &= S_{f,t}(\alpha x+\beta u, \alpha y+\beta v) \\ &= f(t(\alpha x+\beta u)+(1-t)(\alpha y+\beta v)) \\ &= f(\alpha[tx+(1-t)y]+\beta[tu+(1-t)v]) \\ &\leq [f(tx+(1-t)y)]^{\alpha}\,[f(tu+(1-t)v)]^{\beta} \\ &= [S_{f,t}(x,y)]^{\alpha}\,[S_{f,t}(u,v)]^{\beta}, \end{aligned}$$

which shows that $S_{f,t}$ is log-convex on C^2. $\square$

For for $t \in [0,1]$ we define the function $T_{f,t} : C^2 \to (0,\infty)$ by

$$T_{f,t}(x,y) = \frac{S_{f,t}(x,y) + S_{f,1-t}(x,y)}{2} = \frac{f(tx + (1-t)y) + f((1-t)x + ty)}{2}.$$

We observe that $T_{f,t}$ is symmetric on C^2, namely $T_{f,t}(x,y) = T_{f,t}(y,x)$ for all $(x,y) \in C^2$.

Theorem 6. *If the function $f : C \to (0,\infty)$ is a log-convex function on C and $t \in (0,1)$, then $T_{f,t}$ is log-convex on C^2.*

Proof. Let α, $\beta > 0$ with $\alpha + \beta = 1$ and $(x,y), (u,v) \in C^2$. Then by Lemma 2 we have for $t \in (0,1)$ that

$$S_{f,t}(\alpha(x,y) + \beta(u,v)) \le [S_{f,t}(x,y)]^{\alpha} [S_{f,t}(u,v)]^{\beta}$$

and

$$S_{f,1-t}(\alpha(x,y) + \beta(u,v)) \le [S_{f,1-t}(x,y)]^{\alpha} [S_{f,1-t}(u,v)]^{\beta}.$$

If we add these two inequalities we get

$$\begin{aligned} &S_{f,t}(\alpha(x,y) + \beta(u,v)) + S_{f,1-t}(\alpha(x,y) + \beta(u,v)) \\ &\le [S_{f,t}(x,y)]^{\alpha} [S_{f,t}(u,v)]^{\beta} + [S_{f,1-t}(x,y)]^{\alpha} [S_{f,1-t}(u,v)]^{\beta}. \end{aligned} \tag{2.6}$$

If we use the Hölder's type inequality

$$ab + cd \le (a^p + c^p)^{1/p} (b^q + d^q)^{1/q},$$

where p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we get for

$$a = [S_{f,t}(x,y)]^{\alpha},\ b = [S_{f,t}(u,v)]^{\beta},\ c = [S_{f,1-t}(x,y)]^{\alpha},\ d = [S_{f,1-t}(u,v)]^{\beta}$$

and $p = \frac{1}{\alpha}$, $q = \frac{1}{\beta}$ that

$$\begin{aligned} &[S_{f,t}(x,y)]^{\alpha} [S_{f,t}(u,v)]^{\beta} + [S_{f,1-t}(x,y)]^{\alpha} [S_{f,1-t}(u,v)]^{\beta} \\ &\le \left[([S_{f,t}(x,y)]^{\alpha})^{1/\alpha} + ([S_{f,1-t}(x,y)]^{\alpha})^{1/\alpha} \right]^{\alpha} \\ &\times \left[\left([S_{f,t}(u,v)]^{\beta} \right)^{1/\beta} + \left([S_{f,1-t}(u,v)]^{\beta} \right)^{1/\beta} \right]^{\beta} \\ &= [S_{f,t}(x,y) + S_{f,1-t}(x,y)]^{\alpha} [S_{f,t}(u,v) + S_{f,1-t}(u,v)]^{\beta}. \end{aligned} \tag{2.7}$$

By making use of (2.6) and (2.7) we get

$$2T_{f,t}(\alpha(x,y) + \beta(u,v)) \le [2T_{f,t}(x,y)]^{\alpha} [2T_{f,t}(u,v)]^{\beta} = 2[T_{f,t}(x,y)]^{\alpha} [T_{f,t}(u,v)]^{\beta},$$

which proves the fact that $T_{f,t}$ is log-convex on C^2. □

For a Lebesgue integrable function $w : [0,1] \to [0,\infty)$ and a log-convex function $f : C \to (0,\infty)$ we consider the function

$$S_{f,w}(x,y) = \int_0^1 S_{f,t}(x,y)\, w(t)\, dt = \int_0^1 f(tx + (1-t)y)\, w(t)\, dt.$$

Theorem 7. *If the function $f : C \to (0,\infty)$ is a log-convex function on C and $w : [0,1] \to [0,\infty)$ a Lebesgue integrable function on $[0,1]$, then $S_{f,w}$ is log-convex on C^2.*

Proof. Let $\alpha,\ \beta>0$ with $\alpha+\beta=1$ and $(x,y)\,,\ (u,v)\in C^2$. Then, by Lemma 2 we have

$$\begin{aligned} S_{f,w}\left(\alpha\left(x,y\right)+\beta\left(u,v\right)\right) &= \int_0^1 S_{f,t}\left(\alpha\left(x,y\right)+\beta\left(u,v\right)\right)w\left(t\right)dt \\ &\leq \int_0^1 \left[S_{f,t}\left(x,y\right)\right]^{\alpha}\left[S_{f,t}\left(u,v\right)\right]^{\beta}w\left(t\right)dt. \end{aligned}$$

By Hölder's weighted integral inequality for $p=\frac{1}{\alpha}$, $q=\frac{1}{\beta}$ we have

$$\begin{aligned} &\int_0^1 \left[S_{f,t}\left(x,y\right)\right]^{\alpha}\left[S_{f,t}\left(u,v\right)\right]^{\beta}w\left(t\right)dt \\ &\leq \left(\int_0^1 \left(\left[S_{f,t}\left(x,y\right)\right]^{\alpha}\right)^{1/\alpha}w\left(t\right)dt\right)^{\alpha}\left(\int_0^1 \left(\left[S_{f,t}\left(u,v\right)\right]^{\beta}\right)^{1/\beta}w\left(t\right)dt\right)^{\beta} \\ &= \left(\int_0^1 S_{f,t}\left(x,y\right)w\left(t\right)dt\right)^{\alpha}\left(\int_0^1 S_{f,t}\left(u,v\right)w\left(t\right)dt\right)^{\beta} \\ &= \left[S_{f,w}\left(x,y\right)\right]^{\alpha}\left[S_{f,w}\left(u,v\right)\right]^{\beta}, \end{aligned}$$

which proves the log-convexity of $S_{f,w}$ on C^2. □

We denote by $[x,y]$ the closed segment defined by $\{(1-s)\,x+sy,\ s\in[0,1]\}$. We also define the functional

$$\Psi_{g,t}\left(x,y\right):=\left(1-t\right)g\left(x\right)+tg\left(y\right)-g\left(\left(1-t\right)x+ty\right)\geq 0$$

where $x,\ y\in C$, $x\neq y$ and $t\in[0,1]$.

In [5] we obtained among others the following result :

Lemma 3. *Let $g:C\subset X\to\mathbb{R}$ be a convex function on the convex set C. Then for each $x,\ y\in C$ and $z\in[x,y]$ we have*

$$(0\leq)\ \Psi_{g,t}\left(x,z\right)+\Psi_{g,t}\left(z,y\right)\leq\Psi_{g,t}\left(x,y\right) \tag{2.8}$$

for each $t\in[0,1]$, i.e., the functional $\Psi_{g,t}(\cdot,\cdot)$ is superadditive as a function of interval.

If $z,\ u\in[x,y]$, then

$$(0\leq)\ \Psi_{g,t}\left(z,u\right)\leq\Psi_{g,t}\left(x,y\right) \tag{2.9}$$

for each $t\in[0,1]$, i.e., the functional $\Psi_{g,t}(\cdot,\cdot)$ is nondecreasing as a function of interval.

For a log-convex function $f:C\to(0,\infty)$ and for $x,\ y\in C$, $x\neq y$ and $t\in[0,1]$ we consider the function $\Pi_{f,t}:C^2\to[1,\infty)$ defined by

$$\Pi_{f,t}\left(x,y\right):=\frac{\left[f\left(x\right)\right]^{1-t}\left[f\left(y\right)\right]^{t}}{f\left(\left(1-t\right)x+ty\right)}\geq 1. \tag{2.10}$$

We observe that

$$\Psi_{\ln f,t}\left(x,y\right):=\left(1-t\right)\ln f\left(x\right)+t\ln f\left(y\right)-\ln f\left(\left(1-t\right)x+ty\right)=\ln\Pi_{f,t}\left(x,y\right)$$

for $x,\ y\in C$, $x\neq y$ and $t\in[0,1]$.

We have:

Theorem 8. *Let $f : C \to (0, \infty)$ be a log-convex function. Then for each $x, y \in C$ and $z \in [x, y]$ we have*

$$(1 \leq) \Pi_{f,t}(x, z) \Pi_{f,t}(z, y) \leq \Pi_{f,t}(x, y) \tag{2.11}$$

for each $t \in [0, 1]$, i.e., the functional $\Pi_{f,t}(\cdot, \cdot)$ is supermultiplicative as a function of interval.

If $z, u \in [x, y]$, then

$$(1 \leq) \Pi_{f,t}(z, u) \leq \Pi_{f,t}(x, y) \tag{2.12}$$

for each $t \in [0, 1]$, i.e., the functional $\Pi_{f,t}(\cdot, \cdot)$ is nondecreasing as a function of interval.

For a log-convex function $f : C \to (0, \infty)$ and for $x, y \in C$, $x \neq y$ and $t \in [0, 1]$ we also consider the function $\Omega_{f,t} : C^2 \to [1, \infty)$ defined by

$$\Omega_{f,t}(x, y) := \Pi_{f,t}(x, y) \Pi_{f,1-t}(x, y) = \frac{f(x) f(y)}{f((1-t)x + ty) f(tx + (1-t)y)}.$$

Corollary 1. *Let $f : C \to (0, \infty)$ be a log-convex function. Then for each $x, y \in C$, $x \neq y$ and $z \in [x, y]$ we have*

$$(1 \leq) \Omega_{f,t}(x, z) \Omega_{f,t}(z, y) \leq \Omega_{f,t}(x, y) \tag{2.13}$$

for each $t \in [0, 1]$.

If $z, u \in [x, y]$, then

$$(1 \leq) \Omega_{f,t}(z, u) \leq \Omega_{f,t}(x, y) \tag{2.14}$$

for each $t \in [0, 1]$.

The proof follows by Theorem 8 written for t and $1 - t$ and multiplying the obtained inequalities.

3. Schur Convexity

For any $x = (x_1, ..., x_n) \in \mathbb{R}^n$, let $x_{[1]} \geq ... \geq x_{[n]}$ denote the components of x in decreasing order, and let $x_{\downarrow} = (x_{[1]}, ..., x_{[n]})$ denote the decreasing rearrangement of x. For $x, y \in \mathbb{R}^n$, $x \prec y$ if, by definition,

$$\begin{cases} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \ k = 1, ..., n-1; \\ \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}. \end{cases}$$

When $x \prec y$, x is said to be *majorized* by y (y *majorizes* x). This notation and terminology was introduced by Hardy, Littlewood and Pólya in 1934.

A real-valued function ϕ defined on a set $\mathcal{A} \subset \mathbb{R}^n$ is said to be *Schur-convex* on $\mathcal{A}$ if

$$x \prec y \text{ on } \mathcal{A} \Rightarrow \phi(x) \leq \phi(y). \tag{3.1}$$

If, in addition, $\phi(x) < \phi(y)$ whenever $x \prec y$ but x is not a permutation of y, then ϕ is said to be *strictly Schur-convex* on $\mathcal{A}$. If $\mathcal{A} = \mathbb{R}^n$, then ϕ is simply said to be Schur-convex or strictly Schur-convex.

For fundamental properties of Schur convexity see the monograph [16] and the references therein. For some recent results, see [1]-[3] and [17]-[19].

The following result is known in the literature as *Schur-Ostrowski theorem* [16, p. 84]:

Theorem 9. *Let $I \subset \mathbb{R}$ be an open interval and let $\phi : I^n \to \mathbb{R}$ be continuously differentiable. Necessary and sufficient conditions for ϕ to be Schur-convex on I^n are*

$$\phi \text{ is symmetric on } I^n, \tag{3.2}$$

and for all $i \neq j$, with $i, j \in \{1, ..., n\}$,

$$(z_i - z_j)\left[\frac{\partial \phi(z)}{\partial x_i} - \frac{\partial \phi(z)}{\partial x_j}\right] \geq 0 \text{ for all } z \in I^n, \tag{3.3}$$

where $\frac{\partial \phi}{\partial x_k}$ denotes the partial derivative of ϕ with respect to its k-th argument.

Let $\mathcal{A} \subset \mathbb{R}^n$ be a set with the following properties:

(i) $\mathcal{A}$ is *symmetric* in the sense that $x \in \mathcal{A} \Rightarrow x\Pi \in \mathcal{A}$ for all permutations Π of the coordinates.

(ii) $\mathcal{A}$ is convex and has a nonempty interior.

We have the following result, [16, p. 85].

Theorem 10. *If ϕ is continuously differentiable on the interior of $\mathcal{A}$ and continuous on $\mathcal{A}$, then necessary and sufficient conditions for ϕ to be Schur-convex on $\mathcal{A}$ are*

$$\phi \text{ is symmetric on } \mathcal{A} \tag{3.4}$$

and

$$(z_1 - z_2)\left[\frac{\partial \phi(z)}{\partial x_1} - \frac{\partial \phi(z)}{\partial x_2}\right] \geq 0 \text{ for all } z \in \mathcal{A}. \tag{3.5}$$

Another interesting characterization of Schur convex functions ϕ on $\mathcal{A}$ was obtained by C. Stępniak in [20]:

Theorem 11. *Let ϕ be any function defined on a symmetric convex set $\mathcal{A}$ in $\mathbb{R}^n$. Then the function ϕ is Schur convex on $\mathcal{A}$ if and only if*

$$\phi(x_1, ..., x_i, ..., x_j, ..., x_n) = \phi(x_1, ..., x_j, ..., x_i, ..., x_n) \tag{3.6}$$

for all $(x_1, ..., x_n) \in \mathcal{A}$ and $1 \leq i < j \leq n$ and

$$\phi(\lambda x_1 + (1-\lambda) x_2, \lambda x_2 + (1-\lambda) x_1, x_3, ..., x_n) \leq \phi(x_1, ..., x_n) \tag{3.7}$$

for all $(x_1, ..., x_n) \in \mathcal{A}$ and for all $\lambda \in (0, 1)$,

It is well known that any symmetric convex function defined on a symmetric convex set $\mathcal{A}$ is Schur convex, [16, p. 97]. If the function $\phi : \mathcal{A} \to \mathbb{R}$ is symmetric and quasi-convex, namely

$$\phi(\alpha u + (1-\alpha) v) \leq \max\{\phi(u), \phi(v)\}$$

for all $\alpha \in [0, 1]$ and $u, v \in \mathcal{A}$, a symmetric convex set, then ϕ is Schur convex on $\mathcal{A}$ [16, p. 98].

Let X be a linear space and $G \subset X^2 := X \times X$ a convex set. We say that G is *symmetric* if $(x, y) \in G$ implies that $(y, x) \in G$. If $C \subset X$ is a convex subset of X, then the Cartesian product $G := C^2 := C \times C$ is convex and symmetric in X^2.

Motivated by the characterization result of Stępniak above, we say that a function $\phi : G \to \mathbb{R}$ will be called *Schur convex* on the convex and symmetric set $G \subset X^2$ if

$$\phi(s(x, y) + (1-s)(y, x)) \leq \phi(x, y) \tag{3.8}$$

for all $(x,y) \in G$ and for all $s \in [0,1]$.

If $G = C^2$, then we recapture the general concept of Schur convexity introduced by Burai and Makó in 2016, [1].

We say that the function $\phi : G \to \mathbb{R}$ is symmetric on G if $\phi(x,y) = \phi(y,x)$ for all $(x,y) \in G$.

If $\phi : G \to \mathbb{R}$ is *Schur convex* on the convex and symmetric set $G \subset X^2$, then ϕ is symmetric on G. Indeed, if $(x,y) \in G$, then by (3.8) we get for $s = 0$ that $\phi(y,x) \le \phi(x,y)$. If we replace x with y then we also get $\phi(x,y) \le \phi(y,x)$ which shows that $\phi(x,y) = \phi(y,x)$ for all $(x,y) \in G$.

For a function $f : C \to (0,\infty)$ and $t \in [0,1]$ we define the associated symmetric functions $T_{f,t} : C^2 \to (0,\infty)$ and $M_{f,t} : C^2 \to (0,\infty)$ by

$$T_{f,t}(x,y) := \frac{f(x) f(y)}{f((1-t)x + ty) f(tx + (1-t)y)} \tag{3.9}$$

and

$$M_{f,t}(x,y) := \frac{f((1-t)x + ty) f(tx + (1-t)y)}{f^2\left(\frac{x+y}{2}\right)}. \tag{3.10}$$

Theorem 12. *Let $f : C \to (0,\infty)$ be a log-convex function and $t \in [0,1]$. The functions $T_{f,t}$ and $M_{f,t}$ are Schur convex on C^2.*

Proof. Let $(x,y) \in C^2$ and $s \in [0,1]$, $t \in [0,1]$. Then

$$\begin{aligned} & T_{f,t}(s(x,y) + (1-s)(y,x)) \\ & = T_{f,t}(sx + (1-s)y, sy + (1-s)x) \\ & = \frac{f((1-s)x + sy)}{f((1-t)((1-s)x + sy) + t(sx + (1-s)y))} \\ & \times \frac{f(sx + (1-s)y)}{f(t((1-s)x + sy) + (1-t)(sx + (1-s)y))}. \end{aligned} \tag{3.11}$$

If we take $u = (1-s)x + sy$, $v = sx + (1-s)y$ in (2.14), then we get

$$\begin{aligned} & \frac{f((1-s)x + sy)}{f((1-t)((1-s)x + sy) + t(sx + (1-s)y))} \\ & \times \frac{f(sx + (1-s)y)}{f(t((1-s)x + sy) + (1-t)(sx + (1-s)y))} \\ & \le \frac{f(x) f(y)}{f((1-t)x + ty) f(tx + (1-t)y)} = T_{f,t}(x,y). \end{aligned} \tag{3.12}$$

Therefore, by (3.11) and (3.12) we get

$$T_{f,t}(s(x,y) + (1-s)(y,x)) \le T_{f,t}(x,y),$$

for all $(x,y) \in C^2$ and $s \in [0,1]$, $t \in [0,1]$, which shows that $T_{f,t}$ is Schur convex.

Let $(x,y)\in C^2$ and $s\in[0,1]$, $t\in[0,1]$. Then

$$\begin{aligned}
&M_{f,t}\left(s\left(x,y\right)+\left(1-s\right)\left(y,x\right)\right)\\
&=M_{f,t}\left(sx+\left(1-s\right)y,sy+\left(1-s\right)x\right)\\
&=\frac{f\left(\left(1-t\right)\left(sx+\left(1-s\right)y\right)+t\left(sy+\left(1-s\right)x\right)\right)}{f\left(\frac{sx+(1-s)y+y+sy+(1-s)x}{2}\right)}\\
&\times\frac{f\left(t\left(sx+\left(1-s\right)y\right)+\left(1-t\right)\left(sy+\left(1-s\right)x\right)\right)}{f\left(\frac{sx+(1-s)y+y+sy+(1-s)x}{2}\right)}\\
&=\frac{f\left(s\left(\left(1-t\right)x+ty\right)+\left(1-s\right)\left(\left(1-t\right)y+tx\right)\right)}{f\left(\frac{x+y}{2}\right)}\\
&\times\frac{f\left(s\left(\left(1-t\right)y+tx\right)+\left(1-s\right)\left(\left(1-t\right)x+ty\right)\right)}{f\left(\frac{x+y}{2}\right)}.
\end{aligned}\tag{3.13}$$

By the log-convexity of f we have

$$\begin{aligned}
&f\left(s\left(\left(1-t\right)x+ty\right)+\left(1-s\right)\left(\left(1-t\right)y+tx\right)\right)\\
&\le\left[f\left(\left(1-t\right)x+ty\right)\right]^{s}\left[f\left(\left(1-t\right)y+tx\right)\right]^{1-s}
\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}
&f\left(s\left(\left(1-t\right)y+tx\right)+\left(1-s\right)\left(\left(1-t\right)x+ty\right)\right)\\
&\le\left[f\left(\left(1-t\right)y+tx\right)\right]^{s}\left[f\left(\left(1-t\right)x+ty\right)\right]^{1-s}
\end{aligned}\tag{3.15}$$

for all $(x,y)\in C^2$ and $s\in[0,1]$.

If we multiply (3.14) with (3.15) we get

$$\begin{aligned}
&\frac{f\left(s\left(\left(1-t\right)x+ty\right)+\left(1-s\right)\left(\left(1-t\right)y+tx\right)\right)}{f\left(\frac{x+y}{2}\right)}\\
&\times\frac{f\left(s\left(\left(1-t\right)y+tx\right)+\left(1-s\right)\left(\left(1-t\right)x+ty\right)\right)}{f\left(\frac{x+y}{2}\right)}\\
&\le\frac{f\left(\left(1-t\right)x+ty\right)f\left(\left(1-t\right)y+tx\right)}{f^2\left(\frac{x+y}{2}\right)}=M_{f,t}\left(x,y\right).
\end{aligned}\tag{3.16}$$

By making use of (3.13) and (3.16) we deduce that

$$M_{f,t}\left(s\left(x,y\right)+\left(1-s\right)\left(y,x\right)\right)\le M_{f,t}\left(x,y\right),$$

which shows that $M_{f,t}$ is Schur convex. $\square$

Remark 2. *We observe that the function*

$$J_{f,t}\left(x,y\right):=\frac{f\left(\left(1-t\right)x+ty\right)f\left(tx+\left(1-t\right)y\right)}{f\left(x\right)f\left(y\right)}$$

is Schur concave, namely

$$J_{f,t}\left(s\left(x,y\right)+\left(1-s\right)\left(y,x\right)\right)\ge J_{f,t}\left(x,y\right)$$

provided $f:C\to(0,\infty)$ is a log-convex function and $t\in[0,1]$.

If $r>0$, then the function $J_{f,t}^{r}$ is a Schur concave function and $M_{f,t}^{r}$ is a Schur convex function on C^2, provided $f:C\to(0,\infty)$ is a log-convex function and $t\in[0,1]$.

Theorem 13. *Let $f : C \to (0, \infty)$ be a log-convex function, $r > 0$ and $w : [0,1] \to [0, \infty)$ a Lebesgue integrable function. Then $J_{f,w,r}$ is Schur concave on C^2 and $M_{f,w,r}$ is Schur convex on C^2.*

Proof. Let $(x, y) \in C^2$ and $s \in [0, 1]$. Then

$$\begin{aligned} J_{f,w,r}\left(s\left(x,y\right)+\left(1-s\right)\left(y,x\right)\right) &= \int_0^1 J_{f,t}^r\left(s\left(x,y\right)+\left(1-s\right)\left(y,x\right)\right)w\left(t\right)dt \\ &\geq \int_0^1 J_{f,t}^r\left(x,y\right)w\left(t\right)dt = J_{f,w,r}\left(x,y\right) \end{aligned}$$

and

$$\begin{aligned} M_{f,w,r}\left(s\left(x,y\right)+\left(1-s\right)\left(y,x\right)\right) &= \int_0^1 M_{f,t}^r\left(s\left(x,y\right)+\left(1-s\right)\left(y,x\right)\right)w\left(t\right)dt \\ &\leq \int_0^1 M_{f,t}^r\left(x,y\right)w\left(t\right)dt = M_{f,w,r}\left(x,y\right), \end{aligned}$$

which proves the desired results. □

For a logarithmic convex function f defined on the interval I, by changing the variable $u = (1-t)x + ty$, $t \in [0, 1]$, $(x, y) \in I^2$, $y \neq x$, we have

$$J_{f,w,r}\left(x,y\right) = \begin{cases} \frac{\int_x^y f^r(u) f^r(x+y-u) w\left(\frac{u-x}{y-x}\right) du}{(y-x) f^r(x) f^r(y)}, & (x,y) \in I^2,\ y \neq x, \\ 1, & (x,y) \in I^2,\ y = x \end{cases} \tag{3.17}$$

and

$$M_{f,w,r}\left(x,y\right) = \begin{cases} \frac{\int_x^y f^r(u) f^r(x+y-u) w\left(\frac{u-x}{y-x}\right) du}{(y-x) f^{2r}\left(\frac{x+y}{2}\right)}, & (x,y) \in I^2,\ y \neq x, \\ 1, & (x,y) \in I^2,\ y = x \end{cases} \tag{3.18}$$

for a Lebesgue integrable function $w : [0, 1] \to [0, \infty)$ and $r > 0$.

In particular, for $w \equiv 1$ we put

$$J_{f,r}\left(x,y\right) = \begin{cases} \frac{\int_x^y f^r(u) f^r(x+y-u) du}{(y-x) f^r(x) f^r(y)}, & (x,y) \in I^2,\ y \neq x, \\ 1, & (x,y) \in I^2,\ y = x \end{cases} \tag{3.19}$$

and

$$M_{f,r}\left(x,y\right) = \begin{cases} \frac{\int_x^y f^r(u) f^r(x+y-u) du}{(y-x) f^{2r}\left(\frac{x+y}{2}\right)}, & (x,y) \in I^2,\ y \neq x, \\ 1, & (x,y) \in I^2,\ y = x. \end{cases} \tag{3.20}$$

Corollary 2. *Let $f : I \to (0, \infty)$ be a log-convex function on I, $r > 0$ and $w : [0, 1] \to [0, \infty)$ a Lebesgue integrable function. Then $J_{f,w,r}$ defined by (3.17) is Schur concave on I^2 and $M_{f,w,r}$ defined by (3.18) is Schur convex on I^2.*

In particular, $J_{f,r}$ defined by (3.19) is Schur concave on I^2 and $M_{f,r}$ defined by (3.20) is Schur convex on I^2.

Further, if $w:[0,1]\to[0,\infty)$ is symmetric on $[0,1]$, namely $w(1-t)=w(t)$ for all $t\in[0,1]$. In this situation

$$S_{g,w}(x,y)=\int_0^1 g(tx+(1-t)y)\,w(t)\,dt$$

is symmetric on C^2.

Indeed, we have

$$\begin{aligned}S_{g,w}(y,x)&=\int_0^1 g(ty+(1-t)x)\,w(t)\,dt=\int_0^1 g((1-s)y+sx)\,w(1-s)\,ds\\&=\int_0^1 g(sx+(1-s)y)\,w(s)\,ds=S_{g,w}(x,y)\end{aligned}$$

for all $(x,y)\in C^2$.

Theorem 14. *Let $f:C\to(0,\infty)$ be a log-convex function, $r>0$ and $w:[0,1]\to[0,\infty)$ a Lebesgue integrable symmetric function. Then $S_{f,w,r}$ is Schur convex on C^2, where*

$$S_{f,w,r}(x,y)=\int_0^1 f^r(tx+(1-t)y)\,w(t)\,dt.$$

Proof. Observe that

$$S_{f,w,r}(x,y)=\int_0^1 S_{f^r,t}(x,y)\,w(t)\,dt=S_{f^r,w}(x,y).$$

Since f^r is log-convex, f being log-convex on C, hence by Theorem 7 we get that $S_{f,w,r}$ is log-convex on C^2. Therefore, for $(x,y)\in C^2$, $s\in[0,1]$ we get

$$\begin{aligned}S_{f,w,r}(s(x,y)+(1-s)(y,x))&\le[S_{f,w,r}(x,y)]^s[S_{f,w,r}(y,x)]^{1-r}\\&=[S_{f,w,r}(x,y)]^s[S_{f,w,r}(x,y)]^{1-r}=S_{f,w,r}(x,y),\end{aligned}$$

which proves that $S_{f,w,r}$ is Schur convex on C^2. □

In the case when f is log-convex on the interval I, $r>0$ and $w:[0,1]\to[0,\infty)$ a Lebesgue integrable symmetric function, then

$$S_{f,w,r}(x,y)=\begin{cases}\frac{1}{y-x}\int_0^1 f^r(u)\,w\left(\frac{u-x}{y-x}\right)du, & (x,y)\in I^2,\ y\neq x,\\ \\ f^r(x)\int_0^1 w(t)\,dt, & (x,y)\in I^2,\ y=x\end{cases}$$

is *Schur convex on* I^2.

For $w(t)=\left|t-\frac{1}{2}\right|$ and $w(t)=t(1-t)$ we can consider the functions

$$S_{f,\left|\cdot-\frac{1}{2}\right|,r}(x,y)=\begin{cases}\frac{1}{(y-x)^2}\int_0^1 f^r(u)\left|u-\frac{x+y}{2}\right|dt, & (x,y)\in I^2,\ y\neq x,\\ \\ \frac{1}{4}f^r(x), & (x,y)\in I^2,\ y=x\end{cases}\quad\text{and}$$

$$S_{f,\cdot(1-\cdot),r}(x,y)=\begin{cases}\frac{1}{(y-x)^3}\int_0^1 f^r(u)(y-u)(u-x)\,dt, & (x,y)\in I^2,\ y\neq x,\\ \\ \frac{1}{6}f^r(x), & (x,y)\in I^2,\ y=x.\end{cases}$$

Therefore we conclude that $S_{f,\left|\cdot-\frac{1}{2}\right|,r}$ and $S_{f,\cdot(1-\cdot),r}$ are *Schur convex* on I provided f is log-convex on the interval I and $r>0$.

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