SOME NEW PROPERTIES OF AH-CONVEX FUNCTIONS DEFINED ON CONVEX SUBSETS IN LINEAR SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For a Lebesgue integrable function $p : [0, 1] \to [0, \infty)$ we consider the symmetric functions

$$\Delta_{f,p}(x, y) = \int_0^1 \frac{\check{p}(t)\, dt}{f((1-t)x + ty)} - \frac{f(x) + f(y)}{2f(x)f(y)} \int_0^1 p(t)\, dt$$

and

$$\Theta_{f,p}(x, y) := \frac{1}{f\left(\frac{x+y}{2}\right)} \int_0^1 p(t)\, dt - \int_0^1 \frac{\check{p}(t)\, dt}{f((1-t)x + ty)},$$

where $f : C \to (0, \infty)$ is a AH-convex function defined on the convex subset C of a linear space X and $\check{p}(t) := \frac{1}{2}[p(t) + p(1-t)]$, $t \in [0, 1]$.

In this paper we show among others that $\Delta_{f,p}$ and $\Theta_{f,p}$ are Schur convex on $C \times C$. Some examples for AH-convex functions of a real variable are also given.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)dx \le \frac{f(a) + f(b)}{2}, \quad a,\ b \in \mathbb{R},\ a < b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [14]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [1]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [14]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

For related results, see [12] and [9].

Let X be a vector space over the real or complex number field $\mathbb{K}$ and $x, y \in X$, $x \neq y$. Define the segment

$$[x, y] := \{(1-t)x + ty,\ t \in [0, 1]\}.$$

We consider the function $f : [x, y] \to \mathbb{R}$ and the associated function

$$g(x, y) : [0, 1] \to \mathbb{R},\ g(x, y)(t) := f[(1-t)x + ty],\ t \in [0, 1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0, 1]$.

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For any convex function defined on a segment $[x, y] \subset X$, we have the *Hermite-Hadamard integral inequality* (see [5, p. 2], [6, p. 2])

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]dt \leq \frac{f(x)+f(y)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x, y) : [0, 1] \to \mathbb{R}$.

Let X be a linear space and C a convex subset in X. A function $f : C \to \mathbb{R}\setminus\{0\}$ is called *AH-convex (concave)* on the convex set C if the following inequality holds

$$f\left((1-\lambda)x + \lambda y\right) \leq (\geq) \frac{1}{(1-\lambda)\frac{1}{f(x)} + \lambda\frac{1}{f(y)}} = \frac{f(x)f(y)}{(1-\lambda)f(y) + \lambda f(x)} \tag{AH}$$

for any $x, y \in C$ and $\lambda \in [0, 1]$.

An important case which provides many examples is that one in which the function is assumed to be positive for any $x \in C$. In that situation the inequality (AH) is equivalent to

$$(1-\lambda)\frac{1}{f(x)} + \lambda\frac{1}{f(y)} \leq (\geq) \frac{1}{f\left((1-\lambda)x + \lambda y\right)}$$

for any $x, y \in C$ and $\lambda \in [0, 1]$.

Therefore we can state the following fact:

Criterion 1. *Let X be a linear space and C a convex subset in X. The function $f : C \to (0, \infty)$ is AH-convex (concave) on C if and only if $\frac{1}{f}$ is concave (convex) on C in the usual sense.*

If we apply the Hermite-Hadamard inequality (1.2) for the function $\frac{1}{f}$ then we state the following result:

Proposition 1. *Let X be a linear space and C a convex subset in X. If the function $f : C \to (0, \infty)$ is AH-convex (concave) on C, then*

$$\frac{f(x)+f(y)}{2f(x)f(y)} \leq (\geq) \int_0^1 \frac{d\lambda}{f\left((1-\lambda)x + \lambda y\right)} \leq (\geq) \frac{1}{f\left(\frac{x+y}{2}\right)} \tag{1.3}$$

for any $x, y \in C$.

Motivated by the above results, in this paper we establish some new Hermite-Hadamard type inequalities for AH-convex (concave) functions, first in the general setting of linear spaces and then in the particular case of functions of a real variable. Some examples for special means are provided as well.

Recently we obtained following results for AH-convex defined on convex subsets in linear spaces [8]:

Theorem 1. *Let X be a linear space and C a convex subset in X. If the function $f : C \to (0, \infty)$ is AH-convex (concave) on C, then for any $x, y \in C$ we have*

$$\int_0^1 f\left((1-\lambda)x + \lambda y\right) d\lambda \leq (\geq) \frac{G^2\left(f(x), f(y)\right)}{L\left(f(x), f(y)\right)}, \tag{1.4}$$

where the Logarithmic mean of positive numbers a, b is defined as

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \\ \\ a & \text{if } a = b, \end{cases}$$

and the geometric mean is $G=\sqrt{ab}$.

Remark 1. *Using the following well known inequalities*

$$H(a,b)\leq G(a,b)\leq L(a,b)$$

we have

$$\int_0^1 f((1-\lambda)x+\lambda y)\,d\lambda \leq \frac{G^2(f(x),f(y))}{L(f(x),f(y))}\leq G(f(x),f(y)) \tag{1.5}$$

for any $x,\, y\in C$, *provided that* $f:C\to(0,\infty)$ *is AH-convex.*

If $f:C\to(0,\infty)$ *is AH-concave, then*

$$\begin{aligned}\int_0^1 f((1-\lambda)x+\lambda y)\,d\lambda &\geq \frac{G^2(f(x),f(y))}{L(f(x),f(y))}\\ &\geq \frac{G(f(x),f(y))}{L(f(x),f(y))}H(f(x),f(y))\end{aligned} \tag{1.6}$$

for any $x,\, y\in C$.

Theorem 2. *Let* X *be a linear space and* C *a convex subset in* X. *If the function* $f:C\to(0,\infty)$ *is AH-convex (concave) on* C, *then for any* $x,\, y\in C$ *we have*

$$f\left(\frac{x+y}{2}\right)\leq(\geq)\frac{\int_0^1 f((1-\lambda)x+\lambda y)f(\lambda x+(1-\lambda)y)\,d\lambda}{\int_0^1 f((1-\lambda)x+\lambda y)\,d\lambda}. \tag{1.7}$$

Remark 2. *By the Cauchy-Bunyakovsky-Schwarz integral inequality we have*

$$\begin{aligned}&\int_0^1 f((1-\lambda)x+\lambda y)f(\lambda x+(1-\lambda)y)\,d\lambda\\ &\leq\left[\int_0^1 f^2((1-\lambda)x+\lambda y)\,d\lambda\int_0^1 f^2(\lambda x+(1-\lambda)y)\,d\lambda\right]^{1/2}\\ &=\int_0^1 f^2((1-\lambda)x+\lambda y)\,d\lambda\end{aligned} \tag{1.8}$$

for any $x,\, y\in C$.

If the function $f:C\to(0,\infty)$ *is AH-convex on* C, *then we have*

$$\begin{aligned}f\left(\frac{x+y}{2}\right)&\leq\frac{\int_0^1 f((1-\lambda)x+\lambda y)f(\lambda x+(1-\lambda)y)\,d\lambda}{\int_0^1 f((1-\lambda)x+\lambda y)\,d\lambda}\\ &\leq\frac{\int_0^1 f^2((1-\lambda)x+\lambda y)\,d\lambda}{\int_0^1 f((1-\lambda)x+\lambda y)\,d\lambda}.\end{aligned} \tag{1.9}$$

If the function $\psi_{x,y}(t)=f((1-t)x+ty)$, *for some given* $x,\, y\in C$ *with* $x\neq y$, *is monotonic nondecreasing on* $[0,1]$, *then* $\chi_{x,y}(t)=f(tx+(1-t)y)$ *is monotonic nonincreasing on* $[0,1]$ *and by Čebyšev's inequality for monotonic opposite functions we have*

$$\int_0^1 f((1-\lambda)x+\lambda y)f(\lambda x+(1-\lambda)y)\,d\lambda\leq\left(\int_0^1 f((1-\lambda)x+\lambda y)\,d\lambda\right)^2.$$

So, for some given $x,\ y\in C$ with $x\neq y$, $\psi_{x,y}(t)=f((1-t)x+ty)$ is monotonic nondecreasing (nonincreasing) on $[0,1]$ and if the function $f:C\to(0,\infty)$ is AH-convex on C, then we have

$$f\left(\frac{x+y}{2}\right)\leq\frac{\int_0^1 f((1-\lambda)x+\lambda y)f(\lambda x+(1-\lambda)y)\,d\lambda}{\int_0^1 f((1-\lambda)x+\lambda y)\,d\lambda}\leq\int_0^1 f((1-\lambda)x+\lambda y)\,d\lambda. \tag{1.10}$$

If $(X,\|\cdot\|)$ is a normed space, then the function $g:X\to[0,\infty)$, $g(x)=\|x\|^p$, $p\geq 1$ is convex and then the function $f:C\subset X\to(0,\infty)$, $f(x)=\frac{1}{\|x\|^p}$ is *AH-concave* on any convex subset of X which does not contain $\{0\}$.

Utilising (1.4) we have

$$\int_0^1\frac{d\lambda}{\|(1-\lambda)x+\lambda y\|^p}\geq\frac{1}{L(\|x\|^p,\|y\|^p)}, \tag{1.11}$$

for any linearly independent $x,\ y\in X$ and $p\geq 1$.

Making use of (1.7) we also have

$$\int_0^1\frac{d\lambda}{\|(1-\lambda)x+\lambda y\|^p}\geq\left\|\frac{x+y}{2}\right\|^p\int_0^1\frac{d\lambda}{\|(1-\lambda)x+\lambda y\|^p\,\|\lambda x+(1-\lambda)y\|^p} \tag{1.12}$$

for any linearly independent $x,\ y\in X$ and $p\geq 1$.

2. More on *AH*-Convex Functions

We consider the function $f:C\to\mathbb{R}$ defined on the convex subset C of the linear space X and for each $(x,y)\in C^2:=C\times C$ we introduce the auxiliary function $\varphi_{(x,y)}:[0,1]\to\mathbb{R}$ defined by

$$\varphi_{(x,y)}(t):=f((1-t)x+ty). \tag{2.1}$$

It is well known that the function f is convex on C if and only if for each $(x,y)\in C^2$ the auxiliary function $\varphi_{(x,y)}$ is convex on $[0,1]$.

Lemma 1. *Consider the function $f:C\to(0,\infty)$. The function f is AH-convex on C if and only if for all $(x,y)\in C^2$ the auxiliary function $\varphi_{(x,y)}$ is AH-convex on $[0,1]$.*

Proof. Assume that f is AH-convex on C and $(x,y)\in C^2$. Let $\alpha,\ \beta>0$ with $\alpha+\beta=1$ and $t_1,\ t_2\in[0,1]$ then

$$\begin{aligned}\varphi_{(x,y)}(\alpha t_1+\beta t_2)&=f((\alpha t_1+\beta t_2)x+(1-\alpha t_1-\beta t_2)y)\\&=f((\alpha t_1+\beta t_2)x+(\alpha+\beta-\alpha t_1-\beta t_2)y)\\&=f(\alpha[t_1x+(1-t_1)y]+\beta[t_2x+(1-t_2)y])\\&\leq\frac{1}{\frac{\alpha}{f(t_1x+(1-t_1)y)}+\frac{\beta}{f(t_2x+(1-t_2)y)}}\\&=\frac{1}{\frac{\alpha}{\varphi_{(x,y)}(t_1)}+\frac{\beta}{\varphi_{(x,y)}(t_2)}},\end{aligned}$$

which shows that $\varphi_{(x,y)}$ is AH-convex on $[0,1]$.

Let $(x,y)\in C^2$ and $t\in[0,1]$, then by the log-convexity of $\varphi_{(x,y)}$ we have

$$\begin{aligned} f(tx+(1-t)y) &= \varphi_{(x,y)}(t)=\varphi_{(x,y)}(t\cdot 1+(1-t)\cdot 0) \\ &\leq \frac{1}{\frac{t}{\varphi_{(x,y)}(1)}+\frac{1-t}{\varphi_{(x,y)}(0)}}=\frac{1}{\frac{t}{f(x)}+\frac{1-t}{f(y)}}, \end{aligned}$$

which proves the AH-convexity of f on C. □

Now, for $t\in[0,1]$ we define the function $S_t : C^2 \to (0,\infty)$ by

$$S_{f,t}(x,y)=f(tx+(1-t)y). \tag{2.2}$$

Lemma 2. *If $f : C\to(0,\infty)$ is a AH-convex function on C and $t\in(0,1)$, then $S_{f,t}$ is AH-convex on C^2.*

Proof. Let $\alpha,\ \beta>0$ with $\alpha+\beta=1$ and $(x,y),(u,v)\in C^2$. Then

$$\begin{aligned} S_{f,t}(\alpha(x,y)+\beta(u,v)) &= S_{f,t}(\alpha x+\beta u,\alpha y+\beta v) \\ &= f(t(\alpha x+\beta u)+(1-t)(\alpha y+\beta v)) \\ &= f(\alpha[tx+(1-t)y]+\beta[tu+(1-t)v]) \\ &\leq \frac{1}{\frac{\alpha}{f(tx+(1-t)y)}+\frac{\beta}{f(tu+(1-t)v)}} \\ &= \frac{1}{\frac{\alpha}{S_{f,t}(x,y)}+\frac{\beta}{S_{f,t}(u,v)}}, \end{aligned}$$

which shows that $S_{f,t}$ is AH-convex on C^2. □

Lemma 3. *The function $\phi : (0,\infty)^2\to(0,\infty)$, defined by*

$$\phi(x,y)=\frac{xy}{x+y}=\frac{1}{\frac{1}{x}+\frac{1}{y}} \tag{2.3}$$

is concave on $(0,\infty)^2$.

Proof. The first partial derivatives are

$$\frac{\partial\phi(x,y)}{\partial x}=\frac{y(x+y)-xy}{(x+y)^2}=\frac{y^2}{(x+y)^2}$$

and

$$\frac{\partial\phi(x,y)}{\partial y}=\frac{x(x+y)-xy}{(x+y)^2}=\frac{x^2}{(x+y)^2}$$

for $x,\ y>0$.

The second partial derivatives are

$$\frac{\partial^2\phi(x,y)}{\partial x^2}=y^2\frac{\partial}{\partial x}\left[(x+y)^{-2}\right]=-2y^2(x+y)^{-3}=-2\frac{y^2}{(x+y)^3},$$

$$\begin{aligned} \frac{\partial^2\phi(x,y)}{\partial y\partial x} &= \frac{\partial}{\partial y}\left[\frac{y^2}{(x+y)^2}\right]=\frac{2y(x+y)^2-2y^2(x+y)}{(x+y)^4} \\ &= 2\frac{yx+y^2-y^2}{(x+y)^3}=2\frac{xy}{(x+y)^3} \end{aligned}$$

and

$$\frac{\partial^2 \phi(x,y)}{\partial y^2} = x^2 \frac{\partial}{\partial y}\left((x+y)^{-2}\right) = -2x^2 (x+y)^{-3} = -2\frac{x^2}{(x+y)^3}$$

and the Hessian is

$$\begin{pmatrix} -2\frac{y^2}{(x+y)^3} & 2\frac{xy}{(x+y)^3} \\ 2\frac{xy}{(x+y)^3} & -2\frac{x^2}{(x+y)^3} \end{pmatrix}$$

for $x,\ y > 0$.

We have

$$-2\frac{y^2}{(x+y)^3} < 0 \text{ and } \begin{vmatrix} -2\frac{y^2}{(x+y)^3} & 2\frac{xy}{(x+y)^3} \\ 2\frac{xy}{(x+y)^3} & -2\frac{x^2}{(x+y)^3} \end{vmatrix} = 0$$

for $x,\ y > 0$, which shows that the Hessian is negative semidefinite and therefore the function ϕ is globally concave on $(0,\infty)^2$. □

Corollary 1. *Let $\lambda \in (0,1)$ and consider the function (the λ-Harmonic mean)*

$$\phi_\lambda(x,y) = \frac{1}{\frac{1-\lambda}{x} + \frac{\lambda}{y}} = \frac{xy}{\lambda x + (1-\lambda) y} \tag{2.4}$$

for $x,\ y > 0$. The function ϕ_λ is concave on $(0,\infty)^2$.

In particular, the Harmonic mean

$$\phi_{1/2}(x,y) = \frac{2xy}{x+y}$$

is concave on $(0,\infty)^2$.

Proof. Observe that

$$\phi_\lambda(x,y) = \frac{1}{\lambda(1-\lambda)} \frac{\lambda x (1-\lambda) y}{\lambda x + (1-\lambda) y} = \frac{1}{\lambda(1-\lambda)} \phi(\lambda x, (1-\lambda) y).$$

Let $(x,y),\ (u,v) \in (0,\infty)^2$ and $\alpha,\ \beta > 0$ with $\alpha + \beta = 1$. Then

$$\begin{aligned} \phi_\lambda[\alpha(x,y) + \beta(u,v)] &= \phi_\lambda(\alpha x + \beta u, \alpha y + \beta v) \\ &= \frac{1}{\lambda(1-\lambda)} \phi(\lambda(\alpha x + \beta u), (1-\lambda)(\alpha y + \beta v)) \\ &= \frac{1}{\lambda(1-\lambda)} \phi(\alpha\lambda x + \beta\lambda u, \alpha(1-\lambda) y + \beta(1-\lambda) v) \\ &= \frac{1}{\lambda(1-\lambda)} \phi[\alpha(\lambda x, (1-\lambda) y) + \beta(\lambda u, (1-\lambda) v)] \\ &\quad \text{by the concavity of } \phi \\ &\geq \frac{1}{\lambda(1-\lambda)} [\alpha\phi(\lambda x, (1-\lambda) y) + \beta\phi(\lambda u, (1-\lambda) v)] \\ &= \alpha \frac{1}{\lambda(1-\lambda)} \phi(\lambda x, (1-\lambda) y) + \beta \frac{1}{\lambda(1-\lambda)} \phi(\lambda u, (1-\lambda) v) \\ &= \alpha\phi_\lambda(x,y) + \beta\phi_\lambda(u,v), \end{aligned}$$

which shows that ϕ_λ is globally concave on $(0,\infty)^2$. □

Lemma 4. *Let $g:(0,\infty)^2\to(0,\infty)$ be a concave function on $(0,\infty)^2$ and $x,\ y,$ $w:[a,b]\subset\mathbb{R}\to(0,\infty)$ be Lebesgue integrable on $[a,b]$. Then we have*

$$\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,g(x(t),y(t))\,dt \le g\left(\frac{\int_a^b x(s)\,w(s)\,ds}{\int_a^b w(s)\,ds},\frac{\int_a^b y(s)\,w(s)\,ds}{\int_a^b w(s)\,ds}\right). \tag{2.5}$$

Proof. Since g is concave on $(0,\infty)^2$ then for all $(x,y),\ (u,v)\in(0,\infty)^2$ we have the gradient inequality

$$g(x,y)-g(u,v)\le\frac{\partial g(u,v)}{\partial x}(x-u)+\frac{\partial g(u,v)}{\partial y}(y-v).$$

If we take in this inequality

$$u=\frac{\int_a^b x(s)\,w(s)\,ds}{\int_a^b w(s)\,ds},\ \ v=\frac{\int_a^b y(s)\,w(s)\,ds}{\int_a^b w(s)\,ds}$$

we get

$$\begin{aligned}
&g(x(t),y(t))-g\left(\frac{\int_a^b x(s)\,w(s)\,ds}{\int_a^b w(s)\,ds},\frac{\int_a^b y(s)\,w(s)\,ds}{\int_a^b w(s)\,ds}\right)\\
&\le\frac{\partial g}{\partial x}\left(\frac{\int_a^b x(s)\,w(s)\,ds}{\int_a^b w(s)\,ds},\frac{\int_a^b y(s)\,w(s)\,ds}{\int_a^b w(s)\,ds}\right)\left(x(t)-\frac{\int_a^b x(s)\,w(s)\,ds}{\int_a^b w(s)\,ds}\right)\\
&+\frac{\partial g}{\partial y}\left(\frac{\int_a^b x(s)\,w(s)\,ds}{\int_a^b w(s)\,ds},\frac{\int_a^b y(s)\,w(s)\,ds}{\int_a^b w(s)\,ds}\right)\left(y(t)-\frac{\int_a^b y(s)\,w(s)\,ds}{\int_a^b w(s)\,ds}\right)
\end{aligned}$$

for all $t\in[a,b]$.

If we multiply this inequality by $w(t)>0$ and integrate over $t\in[a,b]$ we get

$$\int_a^b w(t)\,g(x(t),y(t))\,dt-g\left(\frac{\int_a^b x(s)\,w(s)\,ds}{\int_a^b w(s)\,ds},\frac{\int_a^b y(s)\,w(s)\,ds}{\int_a^b w(s)\,ds}\right)\int_a^b w(t)\,dt\le 0$$

that is equivalent to (2.5). □

We have the following integral inequality for harmonic mean:

Corollary 2. *Let $x,\ y,\ w:[a,b]\subset\mathbb{R}\to(0,\infty)$ be Lebesgue integrable on $[a,b]$. Then we have*

$$\frac{1}{\int_a^b w(t)\,dt}\int_a^b\frac{w(t)}{\frac{1-\lambda}{x(t)}+\frac{\lambda}{y(t)}}dt\le\frac{1}{\frac{(1-\lambda)\int_a^b w(s)ds}{\int_a^b x(s)w(s)ds}+\frac{\lambda\int_a^b w(s)ds}{\int_a^b y(s)w(s)ds}} \tag{2.6}$$

or,

$$\int_a^b\frac{w(t)}{\frac{1-\lambda}{x(t)}+\frac{\lambda}{y(t)}}dt\le\frac{1}{\frac{1-\lambda}{\int_a^b x(s)w(s)ds}+\frac{\lambda}{\int_a^b y(s)w(s)ds}} \tag{2.7}$$

or, equivalently,

$$\int_a^b\frac{y(t)\,x(t)\,w(t)}{(1-\lambda)\,y(t)+\lambda x(t)}dt\le\frac{\int_a^b y(s)\,w(s)\,ds\int_a^b x(s)\,w(s)\,ds}{\int_a^b[(1-\lambda)\,y(s)+\lambda x(s)]\,w(s)\,ds}. \tag{2.8}$$

We define now the following function $S_{f,p}:C^2\to\mathbb{R}$,

$$S_{f,p}(x,y)=\int_0^1 S_{f,t}(x,y)\,p(t)\,dt=\int_0^1 f(tx+(1-t)y)\,p(t)\,dt$$

for a Lebesgue integrable function $p:[0,1]\to(0,\infty)$, and provided that the integral exists.

Theorem 3. *If $f:C\to(0,\infty)$ is a AH-convex function on C and $p:[0,1]\to(0,\infty)$ is Lebesgue integrable on $[0,1]$, then $S_{f,p}$ is a AH-convex function on C^2.*

Proof. Let $\alpha,\ \beta>0$ with $\alpha+\beta=1$ and $(x,y),\ (u,v)\in C^2$. Then by Lemma 2 we have

$$S_{f,p}(\alpha(x,y)+\beta(u,v))=\int_0^1 S_{f,t}(\alpha(x,y)+\beta(u,v))\,p(t)\,dt \leq\int_0^1\frac{p(t)\,dt}{\frac{\alpha}{S_{f,t}(x,y)}+\frac{\beta}{S_{f,t}(u,v)}}. \tag{2.9}$$

By Corollary 2 we also have

$$\int_0^1\frac{p(t)\,dt}{\frac{\alpha}{S_{f,t}(x,y)}+\frac{\beta}{S_{f,t}(u,v)}}\leq\frac{1}{\frac{\alpha}{\int_a^b S_{f,s}(x,y)p(s)ds}+\frac{\beta}{\int_a^b S_{f,s}(u,v)p(s)ds}} =\frac{1}{\frac{\alpha}{S_{f,p}(x,y)}+\frac{\beta}{S_{f,p}(u,v)}}. \tag{2.10}$$

By (2.9) and (2.10) we get

$$S_{f,p}(\alpha(x,y)+\beta(u,v))\leq\frac{1}{\frac{\alpha}{S_{f,p}(x,y)}+\frac{\beta}{S_{f,p}(u,v)}},$$

which shows that $S_{f,p}$ is a AH-convex function on C^2. □

For for $t\in[0,1]$ we define the function $T_{f,t}:C^2\to(0,\infty)$ by

$$T_{f,t}(x,y)=\frac{S_{f,t}(x,y)+S_{f,1-t}(x,y)}{2} =\frac{f(tx+(1-t)y)+f((1-t)x+ty)}{2}. \tag{2.11}$$

We observe that $T_{f,t}$ is symmetric on C^2, namely $T_{f,t}(x,y)=T_{f,t}(y,x)$ for all $(x,y)\in C^2$.

Lemma 5. *If $f:C\to(0,\infty)$ is a AH-convex function on C and $t\in(0,1)$, then $T_{f,t}$ is AH-convex on C^2.*

Proof. Let $\alpha,\ \beta>0$ with $\alpha+\beta=1$ and $(x,y),\ (u,v)\in C^2$. Then by the AH-convexity of $S_{f,t}$ and $S_{f,1-t}$, with $t\in(0,1)$, we get

$$\begin{aligned}
(2.12)\qquad & T_{f,t}(\alpha(x,y)+\beta(u,v)) \\
&= \frac{1}{2}\left[S_{f,t}(\alpha(x,y)+\beta(u,v))+S_{f,1-t}(\alpha(x,y)+\beta(u,v))\right] \\
&\leq \frac{1}{2}\left[\frac{1}{\frac{\alpha}{S_{f,t}(x,y)}+\frac{\beta}{S_{f,t}(u,v)}}+\frac{1}{\frac{\alpha}{S_{f,1-t}(x,y)}+\frac{\beta}{S_{f,1-t}(u,v)}}\right] \\
&= \frac{1}{2}\left[\phi_\beta(S_{f,t}(x,y),S_{f,t}(u,v))+\phi_\beta(S_{f,1-t}(x,y),S_{f,1-t}(u,v))\right].
\end{aligned}$$

By the global concavity of ϕ_β (see Corollary 1), we have

$$\begin{aligned}
(2.13)\qquad & \frac{1}{2}\left[\phi_\beta(S_{f,t}(x,y),S_{f,t}(u,v))+\phi_\beta(S_{f,1-t}(x,y),S_{f,1-t}(u,v))\right] \\
&\leq \phi_\beta\left(\frac{S_{f,t}(x,y)+S_{f,1-t}(x,y)}{2},\frac{S_{f,t}(u,v)+S_{f,1-t}(u,v)}{2}\right) \\
&= \frac{1}{\frac{\alpha}{\frac{S_{f,t}(x,y)+S_{f,1-t}(x,y)}{2}}+\frac{\beta}{\frac{S_{f,t}(u,v)+S_{f,1-t}(u,v)}{2}}} \\
&= \frac{1}{\frac{\alpha}{T_{f,t}(x,y)}+\frac{\beta}{T_{f,t}(u,v)}}.
\end{aligned}$$

By utilising the inequalities (2.12) and (2.13) we get

$$T_{f,t}(\alpha(x,y)+\beta(u,v))\leq\frac{1}{\frac{\alpha}{T_{f,t}(x,y)}+\frac{\beta}{T_{f,t}(u,v)}}$$

for $\alpha,\ \beta>0$ with $\alpha+\beta=1$ and $(x,y),\ (u,v)\in C^2$, which shows that $T_{f,t}$ is AH-convex on C^2. □

We define now the following function $T_{f,p}:C^2\to\mathbb{R}$,

$$\begin{aligned}
(2.14)\qquad T_{f,p}(x,y) &= \int_0^1 T_{f,t}(x,y)\,p(t)\,dt=\int_0^1\frac{S_{f,t}(x,y)+S_{f,1-t}(x,y)}{2}p(t)\,dt \\
&= \int_0^1\frac{f(tx+(1-t)y)+f((1-t)x+ty)}{2}p(t)\,dt \\
&= \int_0^1 f(tx+(1-t)y)\,\breve{p}(t)\,dt=S_{f,\breve{p}}(x,y)
\end{aligned}$$

for a Lebesgue integrable function $p:[0,1]\to(0,\infty)$, where $\breve{p}(t)=\frac{1}{2}[p(t)+p(1-t)]$ and provided that the integral exists.

We have:

Theorem 4. *If $f:C\to(0,\infty)$ is a AH-convex function on C and $p:[0,1]\to(0,\infty)$ is Lebesgue integrable on $[0,1]$, then $T_{f,p}$ is symmetric and AH-convex function on C^2.*

We have

$$T_{f,p}(y,x) = \int_0^1 f(ty+(1-t)x)\breve{p}(t)\,dt = \int_0^1 f((1-s)y+sx)\breve{p}(1-s)\,ds$$
$$= \int_0^1 f((1-s)y+sx)\breve{p}(s)\,ds = T_{f,p}(x,y),$$

for all $(x,y)\in C^2$.

The AH-convexity of $T_{f,p}$ follows by the identity (2.14) and by Theorem 3.

3. Schur Convexity

For any $x=(x_1,...,x_n)\in\mathbb{R}^n$, let $x_{[1]}\geq ...\geq x_{[n]}$ denote the components of x in decreasing order, and let $x_{\downarrow}=\left(x_{[1]},...,x_{[n]}\right)$ denote the decreasing rearrangement of x. For x, $y\in\mathbb{R}^n$, $x\prec y$ if, by definition,

$$\begin{cases} \sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]},\ k=1,...,n-1; \\ \sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]}. \end{cases}$$

When $x\prec y$, x is said to be *majorized* by y (y *majorizes* x). This notation and terminology was introduced by Hardy, Littlewood and Pólya in 1934.

A real-valued function ϕ defined on a set $\mathcal{A}\subset\mathbb{R}^n$ is said to be *Schur-convex* on $\mathcal{A}$ if

$$x\prec y \text{ on } \mathcal{A}\Rightarrow \phi(x)\leq\phi(y). \tag{3.1}$$

If, in addition, $\phi(x)<\phi(y)$ whenever $x\prec y$ but x is not a permutation of y, then ϕ is said to be *strictly Schur-convex* on $\mathcal{A}$. If $\mathcal{A}=\mathbb{R}^n$, then ϕ is simply said to be Schur-convex or strictly Schur-convex.

For fundamental properties of Schur convexity see the monograph [13] and the references therein. For some recent results, see [2]-[4] and [15]-[17].

The following result is known in the literature as *Schur-Ostrowski theorem* [13, p. 84]:

Theorem 5. *Let $I\subset\mathbb{R}$ be an open interval and let $\phi: I^n\to\mathbb{R}$ be continuously differentiable. Necessary and sufficient conditions for ϕ to be Schur-convex on I^n are*

$$\phi \text{ is symmetric on } I^n, \tag{3.2}$$

and for all $i\neq j$, with i, $j\in\{1,...,n\}$,

$$(z_i-z_j)\left[\frac{\partial\phi(z)}{\partial x_i}-\frac{\partial\phi(z)}{\partial x_j}\right]\geq 0 \text{ for all } z\in I^n, \tag{3.3}$$

where $\frac{\partial\phi}{\partial x_k}$ denotes the partial derivative of ϕ with respect to its k-th argument.

Let $\mathcal{A}\subset\mathbb{R}^n$ be a set with the following properties:

(i) $\mathcal{A}$ is *symmetric* in the sense that $x\in\mathcal{A}\Rightarrow x\Pi\in\mathcal{A}$ for all permutations Π of the coordinates.

(ii) $\mathcal{A}$ is convex and has a nonempty interior.

We have the following result, [13, p. 85].

Theorem 6. *If ϕ is continuously differentiable on the interior of $\mathcal{A}$ and continuous on $\mathcal{A}$, then necessary and sufficient conditions for ϕ to be Schur-convex on $\mathcal{A}$ are*

$$\phi \text{ is symmetric on } \mathcal{A} \tag{3.4}$$

and

$$(z_1 - z_2)\left[\frac{\partial\phi(z)}{\partial x_1} - \frac{\partial\phi(z)}{\partial x_2}\right] \geq 0 \text{ for all } z \in \mathcal{A}. \tag{3.5}$$

Another interesting characterization of Schur convex functions ϕ on $\mathcal{A}$ was obtained by C. Stępniak in [18]:

Theorem 7. *Let ϕ be any function defined on a symmetric convex set $\mathcal{A}$ in $\mathbb{R}^n$. Then the function ϕ is Schur convex on $\mathcal{A}$ if and only if*

$$\phi(x_1, ..., x_i, ..., x_j, ..., x_n) = \phi(x_1, ..., x_j, ..., x_i, ..., x_n) \tag{3.6}$$

for all $(x_1, ..., x_n) \in \mathcal{A}$ and $1 \leq i < j \leq n$ and

$$\phi(\lambda x_1 + (1-\lambda)x_2, \lambda x_2 + (1-\lambda)x_1, x_3, ..., x_n) \leq \phi(x_1, ..., x_n) \tag{3.7}$$

for all $(x_1, ..., x_n) \in \mathcal{A}$ and for all $\lambda \in (0,1)$,

It is well known that any symmetric convex function defined on a symmetric convex set $\mathcal{A}$ is Schur convex, [13, p. 97]. If the function $\phi : \mathcal{A} \to \mathbb{R}$ is symmetric and quasi-convex, namely

$$\phi(\alpha u + (1-\alpha)v) \leq \max\{\phi(u), \phi(v)\}$$

for all $\alpha \in [0,1]$ and u, $v \in \mathcal{A}$, a symmetric convex set, then ϕ is Schur convex on $\mathcal{A}$ [13, p. 98].

Let X be a linear space and $G \subset X^2 := X \times X$ a convex set. We say that G is *symmetric* if $(x,y) \in G$ implies that $(y,x) \in G$. If $C \subset X$ is a convex subset of X, then the Cartesian product $G := C^2 := C \times C$ is convex and symmetric in X^2.

Motivated by the characterization result of Stępniak above, we say that a function $\phi : G \to \mathbb{R}$ will be called *Schur convex* on the convex and symmetric set $G \subset X^2$ if

$$\phi(s(x,y) + (1-s)(y,x)) \leq \phi(x,y) \tag{3.8}$$

for all $(x,y) \in G$ and for all $s \in [0,1]$.

If $G = C^2$, then we recapture the general concept of Schur convexity introduced by Burai and Makó in 2016, [2].

We say that the function $\phi : G \to \mathbb{R}$ is symmetric on G if $\phi(x,y) = \phi(y,x)$ for all $(x,y) \in G$.

If $\phi : G \to \mathbb{R}$ is *Schur convex* on the convex and symmetric set $G \subset X^2$, then ϕ is symmetric on G. Indeed, if $(x,y) \in G$, then by (3.8) we get for $s = 0$ that $\phi(y,x) \leq \phi(x,y)$. If we replace x with y then we also get $\phi(x,y) \leq \phi(y,x)$ which shows that $\phi(x,y) = \phi(y,x)$ for all $(x,y) \in G$.

We denote by $[x,y]$ the closed segment defined by $\{(1-s)x + sy,\ s \in [0,1]\}$. We also define the functional

$$\Psi_{g,t}(x,y) := (1-t)g(x) + tg(y) - g((1-t)x + ty) \geq 0$$

where x, $y \in C$, $x \neq y$ and $t \in [0,1]$.

In [7] we obtained among others the following result :

Lemma 6. *Let $g : C \subset X \to \mathbb{R}$ be a convex function on the convex set C. Then for each x, $y \in C$ and $z \in [x, y]$ we have*

$$(0 \leq) \Psi_{g,t}(x, z) + \Psi_{g,t}(z, y) \leq \Psi_{g,t}(x, y) \tag{3.9}$$

for each $t \in [0, 1]$, i.e., the functional $\Psi_{g,t}(\cdot, \cdot)$ is superadditive as a function of interval.

If z, $u \in [x, y]$, then

$$(0 \leq) \Psi_{g,t}(z, u) \leq \Psi_{g,t}(x, y) \tag{3.10}$$

for each $t \in [0, 1]$, i.e., the functional $\Psi_{g,t}(\cdot, \cdot)$ is nondecreasing as a function of interval.

For a AH-convex function $f : C \to (0, \infty)$ and for x, $y \in C$, $x \neq y$ and $t \in [0, 1]$ we consider the function $\Lambda_{f,t} : C^2 \to [1, \infty)$ defined by

$$\Lambda_{f,t}(x, y) := \frac{1}{f((1-t)x + ty)} - \frac{1-t}{f(x)} - \frac{t}{f(y)} \geq 0.$$

We observe that

$$\Psi_{-\frac{1}{f},t}(x, y) = \Lambda_{f,t}(x, y)$$

for x, $y \in C$, $x \neq y$ and $t \in [0, 1]$.

We have:

Theorem 8. *Let $f : C \subset X \to \mathbb{R}$ be a convex function on the convex set C. Then for each x, $y \in C$, $x \neq y$ and $z \in [x, y]$ we have*

$$(0 \leq) \Lambda_{f,t}(x, z) + \Lambda_{f,t}(z, y) \leq \Lambda_{f,t}(x, y) \tag{3.11}$$

for each $t \in [0, 1]$, i.e., the functional $\Lambda_{f,t}(\cdot, \cdot)$ is superadditive as a function of interval.

If z, $u \in [x, y]$, then

$$(0 \leq) \Lambda_{f,t}(z, u) \leq \Lambda_{f,t}(x, y) \tag{3.12}$$

for each $t \in [0, 1]$, i.e., the functional $\Lambda_{f,t}(\cdot, \cdot)$ is nondecreasing as a function of interval.

The proof follows by Lemma 6 by observing that if f is AH-convex on C, then $-\frac{1}{f}$ is convex on C.

For a AH-convex function $f : C \to (0, \infty)$ and for x, $y \in C$, $x \neq y$ and $t \in [0, 1]$ we consider the function $\Delta_{f,t} : C^2 \to [1, \infty)$ defined by

$$\begin{aligned} \Delta_{f,t}(x, y) &:= \Lambda_{f,t}(x, y) + \Lambda_{f,1-t}(x, y) \\ &= \frac{1}{f((1-t)x + ty)} + \frac{1}{f(tx + (1-t)y)} - \frac{1}{f(x)} - \frac{1}{f(y)}. \end{aligned} \tag{3.13}$$

Corollary 3. *Let $f : C \subset X \to \mathbb{R}$ be a convex function on the convex set C. Then for each x, $y \in C$, $x \neq y$ and $z \in [x, y]$ we have*

$$(0 \leq) \Delta_{f,t}(x, z) + \Delta_{f,t}(z, y) \leq \Delta_{f,t}(x, y) \tag{3.14}$$

for each $t \in [0, 1]$, i.e., the functional $\Delta_{f,t}(\cdot, \cdot)$ is superadditive as a function of interval.

Theorem 9. *If $z,\, u \in [x, y]$, then*

$$(0 \le)\, \Delta_{f,t}(z, u) \le \Delta_{f,t}(x, y) \tag{3.15}$$

for each $t \in [0, 1]$, i.e., the functional $\Delta_{f,t}(\cdot, \cdot)$ is nondecreasing as a function of interval.

For a AH-convex function $f : C \to (0, \infty)$ and for $x,\, y \in C$, $x \ne y$ and $t \in [0, 1]$ we consider the function $\Theta_{f,t} : C^2 \to [1, \infty)$ defined by

$$\Theta_{f,t}(x, y) := \frac{2}{f\left(\frac{x+y}{2}\right)} - \frac{1}{f((1-t)x + ty)} - \frac{1}{f(tx + (1-t)y)}. \tag{3.16}$$

Theorem 10. *Let $f : C \to (0, \infty)$ be a AH-convex function and $t \in [0, 1]$. The functions $\Delta_{f,t}$ and $\Theta_{f,t}$ are Schur convex on C^2.*

Proof. Let $(x, y) \in C^2$ and $s \in [0, 1]$, $t \in [0, 1]$. Then

$$\begin{aligned}
&\Delta_{f,t}(s(x, y) + (1-s)(y, x)) \\
&= \Delta_{f,t}(sx + (1-s)y, sy + (1-s)x) \\
&= \frac{1}{f((1-t)(sx + (1-s)y) + t(sy + (1-s)x))} \\
&+ \frac{1}{f(t(sx + (1-s)y) + (1-t)(sy + (1-s)x))} \\
&- \frac{1}{f(sx + (1-s)y)} - \frac{1}{f(sy + (1-s)x)}.
\end{aligned} \tag{3.17}$$

If we take $u = (1-s)x + sy$, $v = sx + (1-s)y$ in (3.15), then we get

$$\begin{aligned}
&\frac{1}{f((1-t)(sx + (1-s)y) + t(sy + (1-s)x))} \\
&+ \frac{1}{f(t(sx + (1-s)y) + (1-t)(sy + (1-s)x))} \\
&- \frac{1}{f(sx + (1-s)y)} - \frac{1}{f(sy + (1-s)x)} \\
&\le \frac{1}{f((1-t)x + ty)} + \frac{1}{f(tx + (1-t)y)} - \frac{1}{f(x)} - \frac{1}{f(y)} \\
&= \Delta_{f,t}(x, y).
\end{aligned} \tag{3.18}$$

Therefore, by (3.17) and (3.18) we get

$$\Delta_{f,t}(s(x, y) + (1-s)(y, x)) \le \Delta_{f,t}(x, y),$$

$(x, y) \in C^2$ and $s \in [0, 1]$, $t \in [0, 1]$, which shows that $\Delta_{f,t}$ is Schur convex.

Let $(x,y)\in C^2$ and $s\in[0,1]$, $t\in[0,1]$. Then

$$\begin{aligned}
&\Theta_{f,t}\left(s\left(x,y\right)+\left(1-s\right)\left(y,x\right)\right)\\
&=\Theta_{f,t}\left(sx+\left(1-s\right)y,sy+\left(1-s\right)x\right)\\
&=\frac{2}{f\left(\frac{x+y}{2}\right)}\\
&-\frac{1}{f\left(\left(1-t\right)\left(sx+\left(1-s\right)y\right)+t\left(sy+\left(1-s\right)x\right)\right)}\\
&-\frac{1}{f\left(t\left(sx+\left(1-s\right)y\right)+\left(1-t\right)\left(sy+\left(1-s\right)x\right)\right)}\\
&=\frac{2}{f\left(\frac{x+y}{2}\right)}\\
&-\frac{1}{f\left(s\left(\left(1-t\right)x+ty\right)+\left(1-s\right)\left(\left(1-t\right)y+tx\right)\right)}\\
&-\frac{1}{f\left(s\left(\left(1-t\right)y+tx\right)+\left(1-s\right)\left(\left(1-t\right)x+ty\right)\right)}.
\end{aligned}\tag{3.19}$$

By the AH-convexity of f we have

$$\begin{aligned}
&\frac{1}{f\left(s\left(\left(1-t\right)x+ty\right)+\left(1-s\right)\left(\left(1-t\right)y+tx\right)\right)}\\
&\geq\frac{s}{f\left(\left(1-t\right)x+ty\right)}+\frac{1-s}{f\left(\left(1-t\right)y+tx\right)}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{f\left(s\left(\left(1-t\right)y+tx\right)+\left(1-s\right)\left(\left(1-t\right)x+ty\right)\right)}\\
&\geq\frac{s}{f\left(\left(1-t\right)y+\right)tx}+\frac{1-s}{f\left(\left(1-t\right)x+ty\right)}.
\end{aligned}$$

If we add these two inequalities we get

$$\begin{aligned}
&\frac{1}{f\left(s\left(\left(1-t\right)x+ty\right)+\left(1-s\right)\left(\left(1-t\right)y+tx\right)\right)}\\
&+\frac{1}{f\left(s\left(\left(1-t\right)y+tx\right)+\left(1-s\right)\left(\left(1-t\right)x+ty\right)\right)}\\
&\geq\frac{1}{f\left(\left(1-t\right)y+\right)tx}+\frac{1}{f\left(\left(1-t\right)y+tx\right)}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\frac{2}{f\left(\frac{x+y}{2}\right)}\\
&-\frac{1}{f\left(s\left(\left(1-t\right)x+ty\right)+\left(1-s\right)\left(\left(1-t\right)y+tx\right)\right)}\\
&-\frac{1}{f\left(s\left(\left(1-t\right)y+tx\right)+\left(1-s\right)\left(\left(1-t\right)x+ty\right)\right)}\\
&\leq\frac{2}{f\left(\frac{x+y}{2}\right)}-\frac{1}{f\left(\left(1-t\right)y+\right)tx}+\frac{1}{f\left(\left(1-t\right)y+tx\right)}\\
&=\Theta_{f,t}\left(x,y\right)
\end{aligned}\tag{3.20}$$

for all $(x,y)\in C^2$ and $s\in[0,1]$, $t\in[0,1]$.

Using (3.19) and (3.20) we deduce that

$$\Theta_{f,t}\left(s\left(x,y\right)+\left(1-s\right)\left(y,x\right)\right)\leq\Theta_{f,t}\left(x,y\right)$$

for all $(x,y)\in C^2$ and $s\in[0,1]$, which shows that $\Theta_{f,t}$ is Schur convex. □

Reconsider the function $T_{f,p}:C^2\to\mathbb{R}$, defined by (2.14)

$$T_{f,p}\left(x,y\right)=\int_0^1 f\left(tx+\left(1-t\right)y\right)\breve{p}\left(t\right)dt=S_{f,\breve{p}}\left(x,y\right)$$

for a Lebesgue integrable function $p:[0,1]\to(0,\infty)$, where $\breve{p}\left(t\right)=\frac{1}{2}\left[p\left(t\right)+p\left(1-t\right)\right]$ and provided that the integral exists.

Theorem 11. *Let $f:C\to(0,\infty)$ be a AH-convex function and $p:[0,1]\to(0,\infty)$ a Lebesgue integrable function, then $T_{f,p}$ is Schur convex on C^2.*

Proof. Let $(x,y)\in C^2$ and $s\in[0,1]$. Then by Theorem 4 we have that

$$\begin{aligned}T_{f,p}\left(s\left(x,y\right)+\left(1-s\right)\left(y,x\right)\right)&\leq\frac{1}{\frac{s}{T_{f,p}(x,y)}+\frac{1-s}{T_{f,p}(y,x)}}\\&=\frac{1}{\frac{s}{T_{f,p}(x,y)}+\frac{1-s}{T_{f,p}(x,y)}}=T_{f,p}\left(x,y\right),\end{aligned}$$

which shows that $T_{f,p}$ is Schur convex on C^2. □

We can also consider the function $\Delta_{f,p}:C^2\to\mathbb{R}$, defined by

$$\begin{aligned}\Delta_{f,p}\left(x,y\right)&:=\frac{1}{2}\int_0^1\Delta_{f,t}\left(x,y\right)p\left(t\right)dt\\&=\frac{1}{2}\int_0^1\left(\frac{1}{f\left(\left(1-t\right)x+ty\right)}+\frac{1}{f\left(tx+\left(1-t\right)y\right)}\right)p\left(t\right)dt\\&-\frac{f\left(x\right)+f\left(y\right)}{2f\left(x\right)f\left(y\right)}\int_0^1 p\left(t\right)dt\\&=\int_0^1\frac{\breve{p}\left(t\right)dt}{f\left(\left(1-t\right)x+ty\right)}-\frac{f\left(x\right)+f\left(y\right)}{2f\left(x\right)f\left(y\right)}\int_0^1 p\left(t\right)dt\end{aligned}\tag{3.21}$$

and the function $\Theta_{f,p}:C^2\to\mathbb{R}$, defined by

$$\begin{aligned}\Theta_{f,p}\left(x,y\right)&:=\frac{1}{2}\int_0^1\Theta_{f,t}\left(x,y\right)p\left(t\right)dt\\&=\frac{1}{f\left(\frac{x+y}{2}\right)}\int_0^1 p\left(t\right)dt\\&-\frac{1}{2}\int_0^1 p\left(t\right)\left(\frac{1}{f\left(\left(1-t\right)x+ty\right)}+\frac{1}{f\left(tx+\left(1-t\right)y\right)}\right)dt\\&=\frac{1}{f\left(\frac{x+y}{2}\right)}\int_0^1 p\left(t\right)dt-\int_0^1\frac{\breve{p}\left(t\right)dt}{f\left(\left(1-t\right)x+ty\right)}.\end{aligned}\tag{3.22}$$

Theorem 12. *Let $f:C\to(0,\infty)$ be a AH-convex function and $p:[0,1]\to(0,\infty)$ a Lebesgue integrable function, then $\Delta_{f,p}$ and $\Theta_{f,p}$ are Schur convex on C^2.*

Proof. Let $(x,y)\in C^2$ and $s\in[0,1]$. Then by Theorem 10 we have

$$\Delta_{f,p}\left(s\left(x,y\right)+\left(1-s\right)\left(y,x\right)\right)=\frac{1}{2}\int_0^1\Delta_{f,t}\left(s\left(x,y\right)+\left(1-s\right)\left(y,x\right)\right)p\left(t\right)dt$$
$$\leq\frac{1}{2}\int_0^1\Delta_{f,t}\left(x,y\right)w\left(t\right)dt=\Delta_{f,p}\left(x,y\right),$$

which proves the Schur convexity of $\Delta_{f,p}$.

The proof for the function $\Theta_{f,p}$ is similar. □

For a AH-convex function f defined on the interval I, by changing the variable $u=(1-t)x+ty$, $t\in[0,1]$, $(x,y)\in I^2$, $y\neq x$, we have

$$T_{f,p}\left(x,y\right)=\frac{1}{2}\frac{1}{y-x}\int_x^y f\left(u\right)\left[p\left(\frac{u-x}{y-x}\right)+p\left(\frac{y-u}{y-x}\right)\right]du,\tag{3.23}$$
$$T_{f,p}\left(x,x\right):=f\left(x\right)\int_0^1 p\left(t\right)dt;$$

$$\Delta_{f,p}\left(x,y\right)=\frac{1}{2}\frac{1}{y-x}\int_x^y\frac{1}{f\left(u\right)}\left[p\left(\frac{u-x}{y-x}\right)+p\left(\frac{y-u}{y-x}\right)\right]du-\frac{f\left(x\right)+f\left(y\right)}{2f\left(x\right)f\left(y\right)}\int_0^1 p\left(t\right)dt,\tag{3.24}$$
$$\Delta_{f,p}\left(x,x\right):=0;$$

and

$$\Theta_{f,p}\left(x,y\right)=\frac{1}{f\left(\frac{x+y}{2}\right)}\int_0^1 p\left(t\right)dt-\frac{1}{2}\frac{1}{y-x}\int_x^y\frac{1}{f\left(u\right)}\left[p\left(\frac{u-x}{y-x}\right)+p\left(\frac{y-u}{y-x}\right)\right]du,\tag{3.25}$$
$$\Theta_{f,p}\left(x,x\right):=0;$$

where $p:[0,1]\to(0,\infty)$ is a Lebesgue integrable function.

For $p\equiv1$ in (3.23)-(3.25) we get

$$T_f\left(x,y\right)=\begin{cases}\frac{1}{y-x}\int_x^y f\left(u\right)du, & (x,y)\in I^2,\ y\neq x\\ f\left(x\right), & (x,y)\in I^2,\ y=x,\end{cases},\tag{3.26}$$

$$\Delta_f\left(x,y\right)=\begin{cases}\frac{1}{y-x}\int_x^y\frac{du}{f(u)}-\frac{f(x)+f(y)}{2f(x)f(y)}, & (x,y)\in I^2,y\neq x\\ 0, & (x,y)\in I^2,\ y=x,\end{cases}\tag{3.27}$$

and

$$\Theta_f\left(x,y\right)=\begin{cases}\frac{1}{f\left(\frac{x+y}{2}\right)}-\frac{1}{y-x}\int_x^y\frac{du}{f(u)}, & (x,y)\in I^2,y\neq x\\ 0, & (x,y)\in I^2,\ y=x,\end{cases}.\tag{3.28}$$

For $p_m(t) = \left|t - \frac{1}{2}\right|$, $t \in [0,1]$, we have

$$T_{f,p_m}(x,y) = \begin{cases} \frac{1}{(y-x)^2}\int_0^1 f(u)\left|u - \frac{x+y}{2}\right| du, & (x,y) \in I^2, y \neq x \\ \frac{1}{4}f(x), & (x,y) \in I^2,\ y = x, \end{cases} , \tag{3.29}$$

$$\Delta_{f,p_m}(x,y) = \begin{cases} \frac{1}{(y-x)^2}\int_x^y \left|u - \frac{x+y}{2}\right| \frac{du}{f(u)} - \frac{f(x)+f(y)}{8f(x)f(y)}, & (x,y) \in I^2, y \neq x \\ 0, & (x,y) \in I^2,\ y = x, \end{cases} \tag{3.30}$$

and

$$\Theta_{f,p_m}(x,y) = \begin{cases} \frac{1}{4f\left(\frac{x+y}{2}\right)} - \frac{1}{(y-x)^2}\int_x^y \left|u - \frac{x+y}{2}\right| \frac{du}{f(u)}, & (x,y) \in I^2, y \neq x \\ 0, & (x,y) \in I^2,\ y = x. \end{cases} \tag{3.31}$$

Finally, we can state the following result that provides many example of Schur convex functions on I^2 originating from AH-convex functions on the interval I.

Proposition 2. *Let $f : I \to (0,\infty)$ be a AH-convex function on the interval I and $p : [0,1] \to (0,\infty)$ a Lebesgue integrable function. Then $T_{f,p}$, $\Delta_{f,p}$ and $\Theta_{f,p}$ defined by (3.23)-(3.25) are Schur convex on I^2. In particular, the functions T_f, Δ_f and Θ_f defined by (3.26)-(3.28) are Schur convex on I^2 and the functions T_{f,p_m}, Δ_{f,p_m} and Θ_{f,p_m} defined by (3.29)-(3.31) are also Schur convex on I^2.*

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa