SOME HERMITE-HADAMARD TYPE INEQUALITIES VIA OPERATOR CONVEX FUNCTIONS OF TWO VARIABLES

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Abstract. For Lebesgue integrable functions p, $q : [0,1] \to [0,\infty)$ with $\int_0^1 p(t)\,dt = \int_0^1 q(t)\,dt = 1$ and $f : I \to \mathbb{R}$ an operator convex function on I, we show in this paper, among other Hermite-Hadamard type results, that

$$\begin{aligned}
& f\left(\frac{A+B}{2}\right) \\
& \leq \int_0^1 \left(\int_0^1 f\left((1-t-s+2ts)A + (s+t-2ts)B\right)\breve{p}(s)\,ds\right)\breve{q}(t)\,dt \\
& \leq \int_0^1 f\left((1-s)A + sB\right)\breve{p}(s)\,ds \\
& \leq \frac{1}{2}\left[f(A) + f(B)\right]
\end{aligned}$$

for any two Hilbert space operators A and B with spectra in I. Here $\breve{p}(t) := \frac{1}{2}\left[p(t) + p(1-t)\right]$, $t \in [0,1]$.

1. Introduction

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f\left((1-\lambda)A + \lambda B\right) \leq (\geq)\, (1-\lambda) f(A) + \lambda f(B) \tag{1.1}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [8] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0,\infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0,\infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0,\infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t) = -t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

In the recent paper [7] we obtained among others the following Hermite-Hadamard type inequalities for operator convex functions:

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Theorem 1. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I and for any $\lambda \in [0,1]$ we have the inequalities*

$$\begin{aligned} &f\left(\frac{A+B}{2}\right) \\ &\leq (1-\lambda) f\left[\frac{(1-\lambda)A+(1+\lambda)B}{2}\right] + \lambda f\left[\frac{(2-\lambda)A+\lambda B}{2}\right] \\ &\leq \int_0^1 f((1-s)A+sB)\,ds \\ &\leq \frac{1}{2}\left[f((1-\lambda)A+\lambda B)+(1-\lambda)f(B)+\lambda f(A)\right] \\ &\leq \frac{f(A)+f(B)}{2}. \end{aligned} \tag{1.2}$$

A similar result and with a different proof was obtained by B. Li in [12]. For $\lambda = \frac{1}{2}$ in (1.2) we recapture the result obtained in the earlier paper [6] by the author. For other similar inequalities for operator convex functions see [1] and [3]-[17].

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i E_i(d\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2] we define

$$f(A) = f(A_1, ..., A_n) = \int_{I_1 \times ... \times I_k} f(\lambda_1, ..., \lambda_1) E_1(d\lambda_1) \otimes ... \otimes E_k(d\lambda_k) \tag{1.3}$$

as a bounded selfadjoint operator on $H_1 \otimes ... \otimes H_k$.

The above function $f : I_1 \times ... \times I_k \to \mathbb{R}$ is said to be operator convex, if the operator inequality

$$f((1-\alpha)A+\alpha B) \leq (1-\alpha) f(A) + \alpha f(B) \tag{1.4}$$

for all $\alpha \in [0,1]$, for any Hilbert spaces $H_1, ..., H_k$ and any k-tuples of of selfadjoint operators $A = (A_1, ..., A_n)$, $B = (B_1, ..., B_n)$ on $H_1 \otimes ... \otimes H_k$ contained in the domain of f. The definition is meaningful since also the spectrum of $\alpha A_i + (1-\alpha)B_i$ is contained in the interval I_i for each $i = 1, ..., k$.

In the following we restrict ourself to the case $k = 1$, $I_1 = I_2 = I$ and $H_1 = H_1 = H$. The operator convexity of $f : I \times I \to \mathbb{R}$ in this case means, for instance,

$$f((1-\alpha)A_1+\alpha B_1, (1-\alpha)A_2+\alpha B_2) \leq (1-\alpha) f(A_1, A_2) + \alpha f(B_1, B_2) \tag{1.5}$$

or, equivalently

$$f((1-\alpha)(A_1, A_2)+\alpha(B_1, B_2)) \leq (1-\alpha) f(A_1, A_2) + \alpha f(B_1, B_2) \tag{1.6}$$

for all selfadjoint operators A_1, A_2, B_1, B_2 with spectra in I and for all $\alpha \in [0,1]$.

Motivated by the above results, in this paper we obtain via convex functions of two variables some Hermite-Hadamard type inequalities for operator convex functions on I.

For Lebesgue integrable functions $p, q : [0,1] \to [0,\infty)$ with $\int_0^1 p(t)\,dt = \int_0^1 q(t)\,dt = 1$ and $f : I \to \mathbb{R}$ an operator convex function on I, we show, among others, that

$$\begin{aligned}
&f\left(\frac{A+B}{2}\right)\\
&\le \int_0^1 \left(\int_0^1 f\left((1-t-s+2ts)A + (s+t-2ts)B\right)\breve{p}(s)\,ds\right)\breve{q}(t)\,dt\\
&\le \int_0^1 f\left((1-s)A + sB\right)\breve{p}(s)\,ds\\
&\le \frac{1}{2}\left[f(A) + f(B)\right]
\end{aligned}$$

for any two Hilbert space operators A and B with spectra in I. Here $\breve{p}(t) := \frac{1}{2}\left[p(t) + p(1-t)\right]$, $t \in [0,1]$.

2. Preliminary Results

For two selfadjoint operators A, B with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$ and f an operator convex function on I we define the auxiliary function $\phi_{f,(A,B)} : [0,1] \to \mathcal{SA}(H)$, the class of selfadjoint operators on H, by

$$\phi_{f,(A,B)}(t) = f\left((1-t)A + tB\right).$$

We observe that if $T, V \in \mathcal{SA}(H)$, then $\alpha T + \beta V \in \mathcal{SA}(H)$ for all real numbers α, β. In particular, $\mathcal{SA}(H)$ is a convex set in $\mathcal{B}(H)$, the Banach algebra of all bounded linear operators on H.

Lemma 1. *The function $f : I \to \mathbb{R}$ is operator convex on I if and only if for any selfadjoint operators A, B with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, the auxiliary function $\phi_{f,(A,B)} : [0,1] \to \mathcal{SA}(H)$ is convex, namely*

$$\phi_{f,(A,B)}(\alpha t + \beta s) \le \alpha \phi_{f,(A,B)}(t) + \beta \phi_{f,(A,B)}(s) \tag{2.1}$$

for all t, $s \in [0,1]$ and $\alpha + \beta = 1$ with α, $\beta \ge 0$, in the operator order.

Proof. Assume that $f : I \to \mathbb{R}$ is operator convex on I and t, $s \in [0,1]$ with $\alpha + \beta = 1$ and α, $\beta \ge 0$. Then

$$\begin{aligned}
\phi_{f,(A,B)}(\alpha t + \beta s) &= f\left((1-\alpha t - \beta s)A + (\alpha t + \beta s)B\right)\\
&= f\left((\alpha + \beta - \alpha t - \beta s)A + (\alpha t + \beta s)B\right)\\
&= f\left(\alpha\left[(1-t)A + tB\right] + \beta\left[(1-s)A + sB\right]\right)\\
&\le \alpha f\left((1-t)A + tB\right) + \beta f\left((1-s)A + sB\right)\\
&= \alpha \phi_{f,(A,B)}(t) + \beta \phi_{f,(A,B)}(s),
\end{aligned}$$

which proves (2.1).

Now consider selfadjoint operators A, B with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$. By using (2.1) for $t = 0$, $s = 1$, $\alpha = 1 - \beta$ and $\beta \ge 0$ we get

$$\begin{aligned}
f\left((1-\beta)A + \beta B\right) &= \phi_{f,(A,B)}(\beta) = \phi_{f,(A,B)}\left((1-\beta)\cdot 0 + \beta \cdot 1\right)\\
&\le (1-\beta)\phi_{f,(A,B)}(0) + \beta \phi_{f,(A,B)}(1)\\
&= (1-\beta) f(A) + \beta f(B)
\end{aligned}$$

for all $\beta \in [0,1]$, which proves that $f : I \to \mathbb{R}$ is operator convex on I. $\square$

For I an interval, we consider the set $\mathcal{SA}_I(H)$ of all selfadjoint operators with spectra in I. $\mathcal{SA}_I(H)$ is a convex set in $\mathcal{B}(H)$ since for A, B selfadjoints with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subset I$, $\alpha A + \beta B$ is selfadjoint with $\operatorname{Sp}(\alpha A + \beta B) \subset I$.

For $t \in (0,1)$ we define Φ_t, $\Psi_t : \mathcal{SA}_I(H) \times \mathcal{SA}_I(H) \to \mathcal{SA}(H)$ by

$$\Phi_t(A,B) = f((1-t)A + tB)$$

and

$$\begin{aligned}\Psi_t(A,B) &= \frac{1}{2}[\Phi_t(A,B) + \Phi_{1-t}(A,B)] \\ &= \frac{1}{2}[f((1-t)A+tB) + f(tA+(1-t)B)].\end{aligned}$$

These functions can be seen as the functions of operators in the sense of definition (1.3) generated by the scalar functions

$$\Phi_t(x,y) = f((1-t)x + ty)$$

and

$$\Psi_t(x,y) = \frac{1}{2}[f((1-t)x+ty) + f(tx+(1-t)y)]$$

defined on $I \times I$.

Lemma 2. *Assume that $f : I \to \mathbb{R}$ is operator convex on I and $t \in (0,1)$, then Φ_t and Ψ_t are operator convex as functions of two variables. Ψ_t is symmetric in the sense that $\Psi_t(A,B) = \Psi_t(B,A)$ for all $(A,B) \in \mathcal{SA}_I(H)$. Also $\Psi_t(A,B) = \Psi_{1-t}(A,B)$ for all $(A,B) \in \mathcal{SA}_I(H)$.*

Proof. Let A_1, A_2, B_1, B_2 be selfadjoint operators with spectra in I, $t \in (0,1)$ and $\alpha \in [0,1]$. Then

$$\begin{aligned}&\Phi_t((1-\alpha)A_1 + \alpha B_1, (1-\alpha)A_2 + \alpha B_2) \\ &= f((1-t)[(1-\alpha)A_1 + \alpha B_1] + t[(1-\alpha)A_2 + \alpha B_2]) \\ &= f((1-\alpha)[(1-t)A_1 + tA_2] + \alpha[(1-t)B_1 + tB_2]) \\ &\le (1-\alpha) f((1-t)A_1 + tA_2) + \alpha f((1-t)B_1 + tB_2) \\ &= (1-\alpha)\Phi_t(A_1,A_2) + \alpha\Phi_t(B_1,B_2),\end{aligned}$$

which shows that Φ_t is operator convex as a function of two variables.

The proof of the fact that Ψ_t is operator convex as a function of two variables follows from that fact that Φ_t has that property, as shown above.

The symmetry properties are obvious from the definition of Ψ_t. $\square$

For a Lebesgue integrable function $p : [0,1] \to [0,\infty)$ and an operator convex function $f : I \to \mathbb{R}$ we can define the functions

$$\Phi_{f,p}(A,B) := \int_0^1 \Phi_t(A,B)\, p(t)\, dt = \int_0^1 f((1-t)A + tB)\, p(t)\, dt$$

and

$$\begin{aligned}
\Psi_{f,p}(A,B) &:= \int_0^1 \Psi_t(A,B)\,p(t)\,dt = \frac{1}{2}\int_0^1 [\Phi_t(A,B)+\Phi_{1-t}(A,B)]\,p(t)\,dt \\
&= \frac{1}{2}\left[\int_0^1 \Phi_t(A,B)\,p(t)\,dt + \int_0^1 \Phi_{1-t}(A,B)\,p(t)\,dt\right] \\
&= \frac{1}{2}\left[\int_0^1 \Phi_t(A,B)\,p(t)\,dt + \int_0^1 \Phi_t(A,B)\,p(1-t)\,dt\right] \\
&= \int_0^1 \Phi_t(A,B)\,\breve{p}(t)\,dt = \int_0^1 f((1-t)A+tB)\,\breve{p}(t)\,dt \\
&= \Phi_{f,\breve{p}}(A,B)
\end{aligned}$$

where

$$\breve{p}(t) := \frac{1}{2}[p(t)+p(1-t)],\ t\in[0,1].$$

In particular, for $p\equiv 1$ we have

$$\Phi_f(A,B) := \int_0^1 f((1-t)A+tB)\,dt.$$

If p is symmetric, namely $p(1-t)=p(t)$ for $t\in[0,1]$, then $\Psi_{f,p}(A,B) = \Phi_{f,p}(A,B)$ for all $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$. If we consider the symmetric function $p_m:[0,1]\to[0,\infty)$, $p_m(t)=\left|t-\frac{1}{2}\right|$, then

$$\Phi_{f,p_m}(A,B) = \int_0^1 f((1-t)A+tB)\left|t-\frac{1}{2}\right|dt.$$

Also for $p_\beta:[0,1]\to[0,\infty)$, $p_\beta(t)=t(1-t)$, then

$$\Phi_{f,p_\beta}(A,B) = \int_0^1 f((1-t)A+tB)\,t(1-t)\,dt.$$

Theorem 2. *Assume that $f:I\to\mathbb{R}$ is operator convex on I and $p:[0,1]\to[0,\infty)$ a Lebesgue integrable function on $[0,1]$, then $\Phi_{f,p}$ and $\Psi_{f,p}$ are operator convex as functions of two variables. $\Psi_{f,p}$ is also symmetric on $\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$.*

Proof. Let A_1, A_2, B_1, B_2 be selfadjoint operators with spectra in I and $\alpha\in[0,1]$. Then by Lemma 2 we have

$$\begin{aligned}
&\Phi_{f,p}((1-\alpha)A_1+\alpha B_1,(1-\alpha)A_2+\alpha B_2) \\
&= \int_0^1 \Phi_t((1-\alpha)A_1+\alpha B_1,(1-\alpha)A_2+\alpha B_2)\,p(t)\,dt \\
&\le \int_0^1 [(1-\alpha)\Phi_t(A_1,A_2)+\alpha\Phi_t(B_1,B_2)]\,p(t)\,dt \\
&= (1-\alpha)\int_0^1 \Phi_t(A_1,A_2)\,p(t)\,dt + \alpha\int_0^1 \Phi_t(B_1,B_2)\,p(t)\,dt \\
&= (1-\alpha)\Phi_{f,p}(A_1,A_2)+\alpha\Phi_{f,p}(B_1,B_2),
\end{aligned}$$

which proves that $\Phi_{f,p}$ is operator convex as a function of two variables.

Since $\Psi_{f,p}=\Phi_{f,\breve{p}}$, hence $\Psi_{f,p}$ is also operator convex as a function of two variables.

If $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$, then $(B,A)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$ and

$$\Psi_{f,p}(B,A)=\int_0^1 f((1-t)B+tA)\breve{p}(t)\,dt=\int_0^1 f(sB+(1-s)A)\breve{p}(1-s)\,ds$$
$$=\int_0^1 f((1-s)A+sB)\breve{p}(s)\,ds=\Psi_{f,p}(A,B),$$

which proves the symmetry of $\Psi_{f,p}$. □

Theorem 3. *Assume that $f:I\to\mathbb{R}$ is operator convex on I, $t\in(0,1)$ and (A,B), $(C,D)\in\mathcal{SA}_I(H)$, then for any $\lambda\in[0,1]$ we have*

$$\begin{aligned}(2.2)\qquad & f\left((1-t)\frac{A+C}{2}+t\frac{B+D}{2}\right)\\ &\le(1-\lambda)f\left((1-t)\frac{(1-\lambda)A+(1+\lambda)C}{2}+t\frac{(1-\lambda)B+(1+\lambda)D}{2}\right)\\ &+\lambda f\left((1-t)\frac{(2-\lambda)A+\lambda C}{2}+t\frac{(2-\lambda)B+\lambda D}{2}\right)\\ &\le\int_0^1 f((1-t)((1-s)A+sC)+t((1-s)B+sD))\,ds\\ &\le\frac{1}{2}f((1-t)((1-\lambda)A+\lambda C)+t((1-\lambda)B+\lambda D))\\ &+\frac{1}{2}[(1-\lambda)f((1-t)C+tD)+\lambda f((1-t)A+tB)]\\ &\le\frac{1}{2}[f((1-t)A+tB)+f((1-t)C+tD)].\end{aligned}$$

Proof. If we write the inequality (1.2) for the operator convex function Φ_t and (A,B), $(C,D)\in\mathcal{SA}_I(H)$ then we get

$$\begin{aligned}(2.3)\qquad & \Phi_t\left(\frac{(A,B)+(C,D)}{2}\right)\\ &\le(1-\lambda)\Phi_t\left[\frac{(1-\lambda)(A,B)+(1+\lambda)(C,D)}{2}\right]\\ &+\lambda\Phi_t\left[\frac{(2-\lambda)(A,B)+\lambda(C,D)}{2}\right]\\ &\le\int_0^1\Phi_t((1-s)(A,B)+s(C,D))\,ds\\ &\le\frac{1}{2}[\Phi_t((1-\lambda)(A,B)+\lambda(C,D))+(1-\lambda)\Phi_t(C,D)+\lambda\Phi_t(A,B)]\\ &\le\frac{\Phi_t(A,B)+\Phi_t(C,D)}{2}.\end{aligned}$$

Observe that

$$\Phi_t\left(\frac{(A,B)+(C,D)}{2}\right)=\Phi_t\left(\frac{A+C}{2},\frac{B+D}{2}\right)$$
$$=f\left((1-t)\frac{A+C}{2}+t\frac{B+D}{2}\right),$$

$$\begin{aligned}
&\Phi_t\left[\frac{(1-\lambda)(A,B)+(1+\lambda)(C,D)}{2}\right] \\
&=\Phi_t\left(\frac{(1-\lambda)A+(1+\lambda)C}{2},\frac{(1-\lambda)B+(1+\lambda)D}{2}\right) \\
&=f\left((1-t)\frac{(1-\lambda)A+(1+\lambda)C}{2}+t\frac{(1-\lambda)B+(1+\lambda)D}{2}\right),
\end{aligned}$$

$$\begin{aligned}
&\Phi_t\left[\frac{(2-\lambda)(A,B)+\lambda(C,D)}{2}\right] \\
&=\Phi_t\left(\frac{(2-\lambda)A+\lambda C}{2},\frac{(2-\lambda)B+\lambda D}{2}\right) \\
&=f\left((1-t)\frac{(2-\lambda)A+\lambda C}{2}+t\frac{(2-\lambda)B+\lambda D}{2}\right),
\end{aligned}$$

$$\begin{aligned}
&\Phi_t((1-s)(A,B)+s(C,D)) \\
&=\Phi_t((1-s)A+sC,(1-s)B+sD) \\
&=f((1-t)((1-s)A+sC)+t((1-s)B+sD))
\end{aligned}$$

and

$$\begin{aligned}
&\Phi_t((1-\lambda)(A,B)+\lambda(C,D)) \\
&=\Phi_t((1-\lambda)A+\lambda C,(1-\lambda)B+\lambda D) \\
&=f((1-t)((1-\lambda)A+\lambda C)+t((1-\lambda)B+\lambda D)).
\end{aligned}$$

By making use of (2.3) we then deduce the desired inequality (2.2). □

We have the following particular case of interest:

Corollary 1. *Assume that $f:I\to\mathbb{R}$ is operator convex on I, $t\in(0,1)$ and $(A,B)\in\mathcal{SA}_I(H)$, then for any $\lambda\in[0,1]$ we have*

$$\begin{aligned}
&f\left(\frac{A+B}{2}\right) \\
&\le(1-\lambda)f\left(\frac{1-\lambda+2t\lambda}{2}A+\frac{1+\lambda-2t\lambda}{2}B\right) \\
&+\lambda f\left(\frac{2-\lambda-2t+2t\lambda}{2}A+\frac{\lambda+2t-2t\lambda}{2}B\right) \\
&\le\int_0^1 f((1-t-s+2ts)A+(s+t-2ts)B)\,ds \\
&\le\frac{1}{2}f((1-t-\lambda+2t\lambda)A+(\lambda+t-2t\lambda)B) \\
&+\frac{1}{2}[(1-\lambda)f((1-t)B+tA)+\lambda f((1-t)A+tB)] \\
&\le\frac{1}{2}[f((1-t)A+tB)+f((1-t)B+tA)] \\
&\le\frac{f(A)+f(B)}{2}.
\end{aligned}\tag{2.4}$$

Proof. If we replace C with B and D with A, we get

$$\begin{aligned}
(2.5)\qquad & f\left((1-t)\frac{A+B}{2}+t\frac{B+A}{2}\right) \\
& \leq (1-\lambda) f\left((1-t)\frac{(1-\lambda)A+(1+\lambda)B}{2}+t\frac{(1-\lambda)B+(1+\lambda)A}{2}\right) \\
& + \lambda f\left((1-t)\frac{(2-\lambda)A+\lambda B}{2}+t\frac{(2-\lambda)B+\lambda A}{2}\right) \\
& \leq \int_0^1 f\left((1-t)((1-s)A+sB)+t((1-s)B+sA)\right)ds \\
& \leq \frac{1}{2} f\left((1-t)((1-\lambda)A+\lambda B)+t((1-\lambda)B+\lambda A)\right) \\
& + \frac{1}{2}\left[(1-\lambda) f((1-t)B+tA)+\lambda f((1-t)A+tB)\right] \\
& \leq \frac{1}{2}\left[f((1-t)A+tB)+f((1-t)B+tA)\right].
\end{aligned}$$

Since

$$\begin{aligned}
& f\left((1-t)\frac{(1-\lambda)A+(1+\lambda)B}{2}+t\frac{(1-\lambda)B+(1+\lambda)A}{2}\right) \\
& = f\left(\frac{(1-t)(1-\lambda)+t(1+\lambda)}{2}A+\frac{(1-t)(1+\lambda)+t(1-\lambda)}{2}B\right) \\
& = f\left(\frac{1-\lambda+2t\lambda}{2}A+\frac{1+\lambda-2t\lambda}{2}B\right),
\end{aligned}$$

$$\begin{aligned}
& f\left((1-t)\frac{(2-\lambda)A+\lambda B}{2}+t\frac{(2-\lambda)B+\lambda A}{2}\right) \\
& = f\left(\frac{(1-t)(2-\lambda)+t\lambda}{2}A+\frac{(1-t)\lambda+t(2-\lambda)}{2}B\right) \\
& = f\left(\frac{2-\lambda-2t+2t\lambda}{2}A+\frac{\lambda+2t-2t\lambda}{2}B\right),
\end{aligned}$$

$$\begin{aligned}
& f((1-t)((1-s)A+sB)+t((1-s)B+sA)) \\
& = f([(1-t)(1-s)+ts]A+[(1-t)s+t(1-s)]B) \\
& = f((1-t-s+2ts)A+(s+t-2ts)B)
\end{aligned}$$

and

$$\begin{aligned}
& f((1-t)((1-\lambda)A+\lambda B)+t((1-\lambda)B+\lambda A)) \\
& = f([(1-t)(1-\lambda)+t\lambda]A+[(1-t)\lambda+t(1-\lambda)]B) \\
& = f((1-t-\lambda+2t\lambda)A+(\lambda+t-2t\lambda)B),
\end{aligned}$$

hence by (2.5) we get the desired result (2.4). $\square$

Remark 1. *If we take in (2.4) $\lambda = \frac{1}{2}$, then we get for all $t \in [0,1]$ that*

$$\begin{aligned}
(2.6) \qquad & f\left(\frac{A+B}{2}\right) \\
& \leq \frac{1}{2} f\left(\frac{1+2t}{4} A + \frac{3-2t}{4} B\right) + \frac{1}{2} f\left(\frac{3-2t}{4} A + \frac{1+2t}{4} B\right) \\
& \leq \int_0^1 f\left((1-t-s+2ts) A + (s+t-2ts) B\right) ds \\
& \leq \frac{1}{2} f\left(\frac{A+B}{2}\right) + \frac{1}{4}\left[f\left((1-t) B + tA\right) + f\left((1-t) A + tB\right)\right] \\
& \leq \frac{1}{2}\left[f\left((1-t) A + tB\right) + f\left((1-t) B + tA\right)\right] \\
& \leq \frac{f(A) + f(B)}{2}.
\end{aligned}$$

By taking the integral over $t \in [0,1]$ in (2.6) we get

$$\begin{aligned}
& f\left(\frac{A+B}{2}\right) \\
& \leq \frac{1}{2} \int_0^1 \left[f\left(\frac{1+2t}{4} A + \frac{3-2t}{4} B\right) + f\left(\frac{3-2t}{4} A + \frac{1+2t}{4} B\right)\right] dt \\
& \leq \int_0^1 \left(\int_0^1 f\left((1-t-s+2ts) A + (s+t-2ts) B\right) ds\right) dt \\
& \leq \frac{1}{2} f\left(\frac{A+B}{2}\right) + \frac{1}{4} \int_0^1 \left[f\left((1-t) B + tA\right) + f\left((1-t) A + tB\right)\right] dt \\
& \leq \frac{1}{2} \int_0^1 \left[f\left((1-t) A + tB\right) + f\left((1-t) B + tA\right)\right] dt
\end{aligned}$$

which, upon a change of variable, gives

$$\begin{aligned}
(2.7) \qquad f\left(\frac{A+B}{2}\right) & \leq \int_0^1 f\left(\frac{1+2t}{4} A + \frac{3-2t}{4} B\right) dt \\
& \leq \int_0^1 \left(\int_0^1 f\left((1-t-s+2ts) A + (s+t-2ts) B\right) ds\right) dt \\
& \leq \frac{1}{2}\left[f\left(\frac{A+B}{2}\right) + \int_0^1 f\left((1-t) A + tB\right) dt\right] \\
& \leq \int_0^1 f\left((1-t) A + tB\right) dt.
\end{aligned}$$

We have:

Theorem 4. *Assume that $f : I \to \mathbb{R}$ is operator convex on I and $p : [0,1] \to [0,\infty)$ a Lebesgue integrable function on $[0,1]$, then for $(A,B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$*

$$(2.8) \qquad f\left(\frac{A+B}{2}\right) \int_0^1 p(t)\, dt \leq \Psi_{f,p}(A,B) \leq \frac{1}{2}\left[f(A) + f(B)\right] \int_0^1 p(t)\, dt.$$

Proof. From the operator convexity of f we have for $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$ that

$$\begin{aligned}f\left(\frac{A+B}{2}\right)&\leq\frac{1}{2}\left[f\left((1-t)A+tB\right)+f\left(tA+(1-t)B\right)\right]=\Psi_t(A,B)\\&\leq\frac{1}{2}\left[f(A)+f(B)\right].\end{aligned}$$

By multiplying this inequality with $\breve{p}(t)\geq 0,\ t\in[0,1]$ and integrate on $[0,1]$, we get

$$f\left(\frac{A+B}{2}\right)\int_0^1\breve{p}(t)\,dt\leq\int_0^1\Psi_t(A,B)\,\breve{p}(t)\,dt\leq\frac{1}{2}\left[f(A)+f(B)\right]\int_0^1\breve{p}(t)\,dt$$

and since $\int_0^1\breve{p}(t)\,dt=\int_0^1 p(t)\,dt$, hence the inequality (2.8) is proved. □

3. Double Integral Inequalities

We have the following result as well:

Theorem 5. *Assume that $f:I\to\mathbb{R}$ is operator convex on I, $\lambda\in[0,1]$ and $p:[0,1]\to[0,\infty)$ a Lebesgue integrable function on $[0,1]$ with $\int_0^1 p(t)\,dt=1$, then for $(A,B),(C,D)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$ we have*

$$\begin{aligned}&f\left(\frac{A+B+C+D}{4}\right)\\&\leq\int_0^1 f\left((1-t)\frac{A+C}{2}+t\frac{B+D}{2}\right)\breve{p}(t)\,dt\\&\leq(1-\lambda)\\&\times\int_0^1 f\left((1-t)\frac{(1-\lambda)A+(1+\lambda)C}{2}+t\frac{(1-\lambda)B+(1+\lambda)D}{2}\right)\breve{p}(t)\,dt\\&+\lambda\int_0^1 f\left((1-t)\frac{(2-\lambda)A+\lambda C}{2}+t\frac{(2-\lambda)B+\lambda D}{2}\right)\breve{p}(t)\,dt\\&\leq\int_0^1\left(\int_0^1 f\left((1-t)\left((1-s)A+sC\right)+t\left((1-s)B+sD\right)\right)\breve{p}(t)\,dt\right)ds\\&\leq\frac{1}{2}\int_0^1 f\left((1-t)\left((1-\lambda)A+\lambda C\right)+t\left((1-\lambda)B+\lambda D\right)\right)\breve{p}(t)\,dt\\&+\frac{1}{2}(1-\lambda)\int_0^1 f\left((1-t)C+tD\right)\breve{p}(t)\,dt\\&+\frac{1}{2}\lambda\int_0^1 f\left((1-t)A+tB\right)\breve{p}(t)\,dt\\&\leq\frac{1}{2}\frac{\left[f\left((1-\lambda)A+\lambda C\right)+f\left((1-\lambda)B+\lambda D\right)\right]}{2}\\&+\frac{1}{2}\left[(1-\lambda)\frac{f(C)+f(D)}{2}+\lambda\frac{f(A)+f(B)}{2}\right]\\&\leq\frac{1}{4}\left[f(A)+f(B)+f(C)+f(D)\right].\end{aligned}\tag{3.1}$$

Proof. Let $(A,B)\,,\,(C,D)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)\,.$ If we write the inequality (1.2) for the two variables convex function $\Psi_{f,p}$ we get

$$\Psi_{f,p}\left(\frac{(A,B)+(C,D)}{2}\right)\leq(1-\lambda)\,\Psi_{f,p}\left[\frac{(1-\lambda)\,(A,B)+(1+\lambda)\,(C,D)}{2}\right]+\lambda\Psi_{f,p}\left[\frac{(2-\lambda)\,(A,B)+\lambda\,(C,D)}{2}\right]\tag{3.2}$$

$$\leq\int_0^1\Psi_{f,p}\left((1-s)\,(A,B)+s\,(C,D)\right)ds$$
$$\leq\frac{1}{2}\left[\Psi_{f,p}\left((1-\lambda)\,(A,B)+\lambda\,(C,D)\right)+(1-\lambda)\,\Psi_{f,p}\,(C,D)+\lambda\Phi_t\,(A,B)\right].$$

Observe that

$$\Psi_{f,p}\left(\frac{(A,B)+(C,D)}{2}\right)=\Psi_{f,p}\left(\frac{A+C}{2},\frac{B+D}{2}\right)=\int_0^1 f\left((1-t)\,\frac{A+C}{2}+t\frac{B+D}{2}\right)\breve{p}\,(t)\,dt,$$

$$\Psi_{f,p}\left[\frac{(1-\lambda)\,(A,B)+(1+\lambda)\,(C,D)}{2}\right]$$
$$=\Psi_{f,p}\left(\frac{(1-\lambda)\,A+(1+\lambda)\,C}{2},\frac{(1-\lambda)\,B+(1+\lambda)\,D}{2}\right)$$
$$=\int_0^1 f\left((1-t)\,\frac{(1-\lambda)\,A+(1+\lambda)\,C}{2}+t\frac{(1-\lambda)\,B+(1+\lambda)\,D}{2}\right)\breve{p}\,(t)\,dt,$$

$$\Psi_{f,p}\left[\frac{(2-\lambda)\,(A,B)+\lambda\,(C,D)}{2}\right]$$
$$=\Psi_{f,p}\left(\frac{(2-\lambda)\,A+\lambda C}{2},\frac{(2-\lambda)\,B+\lambda D}{2}\right)$$
$$=\int_0^1 f\left((1-t)\,\frac{(2-\lambda)\,A+\lambda C}{2}+t\frac{(2-\lambda)\,B+\lambda D}{2}\right)\breve{p}\,(t)\,dt,$$

$$\Psi_{f,p}\left((1-s)\,(A,B)+s\,(C,D)\right)$$
$$=\Psi_{f,p}\left((1-s)\,A+sC,(1-s)\,B+sD\right)$$
$$=\int_0^1 f\left((1-t)\left((1-s)\,A+sC\right)+t\left((1-s)\,B+sD\right)\right)\breve{p}\,(t)\,dt$$

and

$$\Psi_{f,p}\left((1-\lambda)\,(A,B)+\lambda\,(C,D)\right)$$
$$=\Psi_{f,p}\left((1-\lambda)\,A+\lambda C,(1-\lambda)\,B+\lambda D\right)$$
$$=\int_0^1 f\left((1-t)\left((1-\lambda)\,A+\lambda C\right)+t\left((1-\lambda)\,B+\lambda D\right)\right)\breve{p}\,(t)\,dt.$$

By making use of (3.2) we get the second, third and forth inequality in (3.1).

The rest follows by Theorem 5 and the operator convex property of function f. □

Corollary 2. *With the assumptions of Theorem 5 we have for all* $\lambda \in [0,1]$ *that*

$$\begin{aligned}
(3.3)\qquad & f\left(\frac{A+B}{2}\right) \\
& \leq (1-\lambda) \\
& \times \int_0^1 f\left(\left(\frac{1-\lambda}{2}+\lambda t\right)A+\left(\frac{1+\lambda}{2}-\lambda t\right)B\right)\breve{p}(t)\,dt \\
& +\lambda \int_0^1 f\left(\left[\frac{2-\lambda}{2}-(1-\lambda)t\right]A+\left[\frac{\lambda}{2}+(1-\lambda)t\right]B\right)\breve{p}(t)\,dt \\
& \leq \int_0^1 \left(\int_0^1 f\left((1-t-s+2ts)A+(s+t-2ts)B\right)\breve{p}(t)\,dt\right)ds \\
& \leq \frac{1}{2}\int_0^1 f\left((1-t-\lambda+2\lambda t)A+(t+\lambda-2\lambda t)B\right)\breve{p}(t)\,dt \\
& +\frac{1}{2}\int_0^1 f\left((1-t)A+tB\right)\breve{p}(t)\,dt \\
& \leq \frac{1}{2}\frac{\left[f\left((1-\lambda)A+\lambda B\right)+f\left((1-\lambda)B+\lambda A\right)\right]}{2}+\frac{1}{2}\frac{f(A)+f(B)}{2} \\
& \leq \frac{1}{2}\left[f(A)+f(B)\right].
\end{aligned}$$

In particular, for $\lambda = \frac{1}{2}$ *we get*

$$\begin{aligned}
(3.4)\qquad & f\left(\frac{A+B}{2}\right) \\
& \leq \frac{1}{2}\int_0^1 f\left(\frac{1+2t}{4}A+\frac{3-2t}{4}B\right)\breve{p}(t)\,dt \\
& +\frac{1}{2}\int_0^1 f\left(\frac{3-2t}{4}A+\frac{1+2t}{4}B\right)\breve{p}(t)\,dt \\
& \leq \int_0^1 \left(\int_0^1 f\left((1-t-s+2ts)A+(s+t-2ts)B\right)\breve{p}(t)\,dt\right)ds \\
& \leq \frac{1}{2}\left[f\left(\frac{A+B}{2}\right)+\int_0^1 f\left((1-t)A+tB\right)\breve{p}(t)\,dt\right] \\
& \leq \frac{1}{2}\left[f(A)+f(B)\right].
\end{aligned}$$

We also have:

Theorem 6. *Assume that* $f : I \to \mathbb{R}$ *is operator convex on* I *and* p, $q : [0,1] \to [0,\infty)$ *Lebesgue integrable functions on* $[0,1]$ *with* $\int_0^1 p(t)\,dt = \int_0^1 q(t)\,dt = 1$, *then* *for* $(A,B), (C,D) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$ *we have*

$$\begin{aligned} (3.5)\quad & f\left(\frac{A+B+C+D}{4}\right) \\ & \leq \int_0^1 f\left((1-s)\frac{A+C}{2}+s\frac{B+D}{2}\right)\breve{p}(s)\,ds \\ & \leq \int_0^1 \left(\int_0^1 f((1-s)((1-t)A+tC)+s((1-t)B+tD))\,\breve{p}(s)\,ds\right)\breve{q}(t)\,dt \\ & \leq \int_0^1 \frac{f((1-s)A+sB)+f((1-s)C+sD)}{2}\breve{p}(s)\,ds \\ & \leq \frac{1}{4}\left[f(A)+f(B)+f(C)+f(D)\right]. \end{aligned}$$

Proof. Let $(A,B), (C,D) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$. Using an argument similar to the one from Theorem 4 we have for the operator convex function $\Psi_{f,p}$ that

$$\begin{aligned} (3.6)\qquad \Psi_{f,p}\left(\frac{(A,B)+(C,D)}{2}\right) & \leq \int_0^1 \Psi_{f,p}((1-t)(A,B)+t(C,D))\,\breve{q}(t)\,dt \\ & \leq \frac{1}{2}\left[\Psi_{f,p}(A,B)+\Psi_{f,p}(C,D)\right]. \end{aligned}$$

Observe that

$$\begin{aligned} \Psi_{f,p}\left(\frac{(A,B)+(C,D)}{2}\right) & = \Psi_{f,p}\left(\frac{A+C}{2},\frac{B+D}{2}\right) \\ & = \int_0^1 f\left((1-t)\frac{A+C}{2}+t\frac{B+D}{2}\right)\breve{p}(t)\,dt, \end{aligned}$$

and

$$\begin{aligned} & \Psi_{f,p}((1-t)(A,B)+t(C,D)) \\ & = \Psi_{f,p}((1-t)A+tC,(1-t)B+tD) \\ & = \int_0^1 f((1-s)((1-t)A+tC)+s((1-t)B+tD))\,\breve{p}(s)\,ds, \end{aligned}$$

which, via (3.6), proves the second and third inequalities in (3.5).

The rest is obvious and we omit the details. □

Corollary 3. *With the assumptions of Theorem 6 we have for* $(A,B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$ *that*

$$\begin{aligned} (3.7)\qquad & f\left(\frac{A+B}{2}\right) \\ & \leq \int_0^1 \left(\int_0^1 f((1-t-s+2ts)A+(s+t-2ts)B)\,\breve{p}(s)\,ds\right)\breve{q}(t)\,dt \\ & \leq \int_0^1 f((1-s)A+sB)\,\breve{p}(s)\,ds \\ & \leq \frac{1}{2}\left[f(A)+f(B)\right]. \end{aligned}$$

4. Some Examples

The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\le r\le 2$ or $-1\le r\le 0$ and is operator concave on $(0,\infty)$ if $0\le r\le 1$. The logarithmic function $f(t)=\ln t$ is operator concave on $(0,\infty)$. The entropy function $f(t)=t\ln t$ is operator convex on $(0,\infty)$.

From the inequalities (2.4) and (2.7) we get for $1\le r\le 2$ or $-1\le r<0$ that

$$\begin{aligned}\left(\frac{A+B}{2}\right)^r &\le (1-\lambda)\left(\frac{1-\lambda+2t\lambda}{2}A+\frac{1+\lambda-2t\lambda}{2}B\right)^r \\ &\quad+\lambda\left(\frac{2-\lambda-2t+2t\lambda}{2}A+\frac{\lambda+2t-2t\lambda}{2}B\right)^r \\ &\le \int_0^1 ((1-t-s+2ts)A+(s+t-2ts)B)^r\,ds \\ &\le \frac{1}{2}((1-t-\lambda+2t\lambda)A+(\lambda+t-2t\lambda)B)^r \\ &\quad+\frac{1}{2}[(1-\lambda)((1-t)B+tA)^r+\lambda((1-t)A+tB)^r] \\ &\le \frac{1}{2}[((1-t)A+tB)^r+((1-t)B+tA)^r]\le\frac{A^r+B^r}{2},\end{aligned} \tag{4.1}$$

where $t,\ \lambda\in[0,1]$ and

$$\begin{aligned}\left(\frac{A+B}{2}\right)^r &\le \int_0^1 f\left(\frac{1+2t}{4}A+\frac{3-2t}{4}B\right)^r dt \\ &\le \int_0^1\left(\int_0^1 ((1-t-s+2ts)A+(s+t-2ts)B)^r\,ds\right)dt \\ &\le \frac{1}{2}\left[\left(\frac{A+B}{2}\right)^r+\int_0^1((1-t)A+tB)^r\,dt\right] \\ &\le \int_0^1((1-t)A+tB)^r\,dt\end{aligned} \tag{4.2}$$

for any selfadjoint operators $A,\ B>0$. If $0<r\le 1$ then the sign of inequality reverses in (4.1) and (4.2).

Let $p:[0,1]\to[0,\infty)$ be a Lebesgue integrable function on $[0,1]$ with $\int_0^1 p(t)\,dt=1$, then by (3.4)

$$\begin{aligned}\left(\frac{A+B}{2}\right)^r &\le \frac{1}{2}\int_0^1\left(\frac{1+2t}{4}A+\frac{3-2t}{4}B\right)^r \breve{p}(t)\,dt \\ &\quad+\frac{1}{2}\int_0^1\left(\frac{3-2t}{4}A+\frac{1+2t}{4}B\right)^r \breve{p}(t)\,dt \\ &\le \int_0^1\left(\int_0^1((1-t-s+2ts)A+(s+t-2ts)B)^r\,\breve{p}(t)\,dt\right)ds \\ &\le \frac{1}{2}\left[\left(\frac{A+B}{2}\right)^r+\int_0^1((1-t)A+tB)^r\,\breve{p}(t)\,dt\right]\le\frac{A^r+B^r}{2},\end{aligned} \tag{4.3}$$

for $1\le r\le 2$ or $-1\le r<0$ and for any selfadjoint operators $A,\ B>0$. If $0<r\le 1$ then the sign of inequality reverses in (4.3).

Let $p, q : [0,1] \to [0,\infty)$ be Lebesgue integrable functions on $[0,1]$ with $\int_0^1 p(t)\,dt = \int_0^1 q(t)\,dt = 1$, then by (3.7)

$$\begin{aligned}
&\left(\frac{A+B}{2}\right)^r \\
&\leq \int_0^1 \left(\int_0^1 ((1-t-s+2ts)A + (s+t-2ts)B)^r\, \breve{p}(s)\,ds\right) \breve{q}(t)\,dt \\
&\leq \int_0^1 ((1-s)A + sB)^r\, \breve{p}(s)\,ds \leq \frac{A^r + B^r}{2},
\end{aligned} \tag{4.4}$$

for $1 \leq r \leq 2$ or $-1 \leq r < 0$ and for any selfadjoint operators A, $B > 0$. If $0 < r \leq 1$ then the sign of inequality reverses in (4.3).

From the inequalities (2.4) and (2.7) for the operator concave function $f(t) = \ln t$, $t > 0$ we get

$$\begin{aligned}
\ln\left(\frac{A+B}{2}\right) &\geq (1-\lambda)\ln\left(\frac{1-\lambda+2t\lambda}{2}A + \frac{1+\lambda-2t\lambda}{2}B\right) \\
&\quad + \lambda \ln\left(\frac{2-\lambda-2t+2t\lambda}{2}A + \frac{\lambda+2t-2t\lambda}{2}B\right) \\
&\geq \int_0^1 \ln((1-t-s+2ts)A + (s+t-2ts)B)\,ds \\
&\geq \frac{1}{2}\ln((1-t-\lambda+2t\lambda)A + (\lambda+t-2t\lambda)B) \\
&\quad + \frac{1}{2}[(1-\lambda)\ln((1-t)B + tA) + \lambda\ln((1-t)A + tB)] \\
&\geq \frac{1}{2}[\ln((1-t)A + tB) + \ln((1-t)B + tA)] \\
&\geq \frac{\ln A + \ln B}{2},
\end{aligned} \tag{4.5}$$

where t, $\lambda \in [0,1]$ and

$$\begin{aligned}
\ln\left(\frac{A+B}{2}\right) &\geq \int_0^1 \ln\left(\frac{1+2t}{4}A + \frac{3-2t}{4}B\right) dt \\
&\geq \int_0^1 \left(\int_0^1 \ln((1-t-s+2ts)A + (s+t-2ts)B)\,ds\right) dt \\
&\geq \frac{1}{2}\left[\ln\left(\frac{A+B}{2}\right) + \int_0^1 \ln((1-t)A + tB)\,dt\right] \\
&\geq \int_0^1 \ln((1-t)A + tB)\,dt
\end{aligned} \tag{4.6}$$

for any selfadjoint operators A, $B > 0$.

Let $p:[0,1]\to[0,\infty)$ be a Lebesgue integrable function on $[0,1]$ with $\int_0^1 p(t)\,dt=1$, then by (3.4) for $f(t)=\ln t$, $t>0$

$$
\begin{aligned}
\ln\left(\frac{A+B}{2}\right) &\geq \frac{1}{2}\int_0^1 \ln\left(\frac{1+2t}{4}A+\frac{3-2t}{4}B\right)\breve{p}(t)\,dt \\
&\quad+\frac{1}{2}\int_0^1 \ln\left(\frac{3-2t}{4}A+\frac{1+2t}{4}B\right)\breve{p}(t)\,dt \\
&\geq \int_0^1\left(\int_0^1 \ln\left((1-t-s+2ts)A+(s+t-2ts)B\right)\breve{p}(t)\,dt\right)ds \\
&\geq \frac{1}{2}\left[\ln\left(\frac{A+B}{2}\right)+\int_0^1 \ln\left((1-t)A+tB\right)\breve{p}(t)\,dt\right] \\
&\geq \frac{\ln A+\ln B}{2},
\end{aligned} \tag{4.7}
$$

for any selfadjoint operators A, $B>0$.

Let $p,q:[0,1]\to[0,\infty)$ be Lebesgue integrable functions on $[0,1]$ with $\int_0^1 p(t)\,dt=\int_0^1 q(t)\,dt=1$, then by (3.7) we obtain

$$
\begin{aligned}
&\ln\left(\frac{A+B}{2}\right) \\
&\geq \int_0^1\left(\int_0^1 \ln\left((1-t-s+2ts)A+(s+t-2ts)B\right)\breve{p}(s)\,ds\right)\breve{q}(t)\,dt \\
&\geq \int_0^1 \ln\left((1-s)A+sB\right)\breve{p}(s)\,ds \geq \frac{\ln A+\ln B}{2},
\end{aligned} \tag{4.8}
$$

for any selfadjoint operators A, $B>0$.

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