OPERATOR SCHUR CONVEXITY AND SOME INTEGRAL INEQUALITIES

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Abstract. A continuous function $f : I \times I \to \mathbb{R}$ is called *operator Schur convex*, if f is symmetric, namely $f(x,y) = f(y,x)$ for all x, $y \in I$ and

$$f(tA + (1-t)B, tB + (1-t)A) \leq f(A,B)$$

in the operator order, for all $(A,B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$ and $t \in [0,1]$, where $\mathcal{SA}_I(H)$ is the convex set of all selfadjoint operators on Hilbert space H with spectra in I.

In this paper we investigate the main properties of such functions, establish some integral inequalities of Hermite-Hadamard, Čebyšev and Grüss' type and give some general classes of examples of operator Schur convex functions.

1. Introduction

For any $x = (x_1, ..., x_n) \in \mathbb{R}^n$, let $x_{[1]} \geq ... \geq x_{[n]}$ denote the components of x in decreasing order, and let $x_\downarrow = \left(x_{[1]}, ..., x_{[n]}\right)$ denote the decreasing rearrangement of x. For x, $y \in \mathbb{R}^n$, $x \prec y$ if, by definition,

$$\begin{cases} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \ k = 1, ..., n-1; \\ \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}. \end{cases}$$

When $x \prec y$, x is said to be *majorized* by y (y *majorizes* x). This notation and terminology was introduced by Hardy, Littlewood and Pólya in 1934.

Functions that preserve the ordering of majorization are said to be Schur-convex, [19, p.80]. A real-valued function ϕ defined on a set $\mathcal{A} \subset \mathbb{R}^n$ is said to be *Schur-convex* on $\mathcal{A}$ if

$$x \prec y \text{ on } \mathcal{A} \Rightarrow \phi(x) \leq \phi(y). \tag{1.1}$$

If, in addition, $\phi(x) < \phi(y)$ whenever $x \prec y$ but x is not a permutation of y, then ϕ is said to be *strictly Schur-convex* on $\mathcal{A}$. If $\mathcal{A} = \mathbb{R}^n$, then ϕ is simply said to be Schur-convex or strictly Schur-convex.

For fundamental properties of Schur convexity see the monograph [19] and the references therein. For some recent results, see [5]-[11], [13], [20] and [22]-[24].

The following result is known in the literature as *Schur-Ostrowski theorem* [19, p. 84]:

Theorem 1. *Let $I \subset \mathbb{R}$ be an open interval and let $\phi : I^n \to \mathbb{R}$ be continuously differentiable. Necessary and sufficient conditions for ϕ to be Schur-convex on I^n*

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are

$$\phi \text{ is symmetric on } I^n, \tag{1.2}$$

and for all $i \neq j$, *with* $i, j \in \{1, ..., n\}$,

$$(z_i - z_j)\left[\frac{\partial\phi(z)}{\partial x_i} - \frac{\partial\phi(z)}{\partial x_j}\right] \geq 0 \text{ for all } z \in I^n, \tag{1.3}$$

where $\frac{\partial\phi}{\partial x_k}$ *denotes the partial derivative of* ϕ *with respect to its k-th argument.*

Let $\mathcal{A} \subset \mathbb{R}^n$ be a set with the following properties:

(i) $\mathcal{A}$ is *symmetric* in the sense that $x \in \mathcal{A} \Rightarrow x\Pi \in \mathcal{A}$ for all permutations Π of the coordinates.

(ii) $\mathcal{A}$ is convex and has a nonempty interior.

We have the following result, [19, p. 85].

Theorem 2. *If* ϕ *is continuously differentiable on the interior of* $\mathcal{A}$ *and continuous on* $\mathcal{A}$, *then necessary and sufficient conditions for* ϕ *to be Schur-convex on* $\mathcal{A}$ *are*

$$\phi \text{ is symmetric on } \mathcal{A} \tag{1.4}$$

and

$$(z_1 - z_2)\left[\frac{\partial\phi(z)}{\partial x_1} - \frac{\partial\phi(z)}{\partial x_2}\right] \geq 0 \text{ for all } z \in \mathcal{A}. \tag{1.5}$$

Another interesting characterization of Schur convex functions ϕ on $\mathcal{A}$ was obtained by C. Stępniak in [24]:

Theorem 3. *Let* ϕ *be any function defined on a symmetric convex set* $\mathcal{A}$ *in* $\mathbb{R}^n$. *Then the function* ϕ *is Schur convex on* $\mathcal{A}$ *if and only if*

$$\phi(x_1, ..., x_i, ..., x_j, ..., x_n) = \phi(x_1, ..., x_j, ..., x_i, ..., x_n) \tag{1.6}$$

for all $(x_1, ..., x_n) \in \mathcal{A}$ *and* $1 \leq i < j \leq n$ *and*

$$\phi(\lambda x_1 + (1-\lambda) x_2, \lambda x_2 + (1-\lambda) x_1, x_3, ..., x_n) \leq \phi(x_1, ..., x_n) \tag{1.7}$$

for all $(x_1, ..., x_n) \in \mathcal{A}$ *and for all* $\lambda \in (0, 1)$,

It is well known that any symmetric convex function defined on a symmetric convex set $\mathcal{A}$ is Schur convex, [19, p. 97]. If the function $\phi : \mathcal{A} \to \mathbb{R}$ is symmetric and quasi-convex, namely

$$\phi(\alpha u + (1-\alpha) v) \leq \max\{\phi(u), \phi(v)\}$$

for all $\alpha \in [0, 1]$ and $u, v \in \mathcal{A}$, a symmetric convex set, then ϕ is Schur convex on $\mathcal{A}$ [19, p. 98].

In order to extend the above concept to continuous functions of selfadjoint operators on complex Hilbert space we need some preparations as follow.

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda) A + \lambda B) \leq (\geq) (1-\lambda) f(A) + \lambda f(B) \tag{1.8}$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [14] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0, \infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0, \infty)$. The entropy function $f(t) = -t \ln t$ is operator concave on $(0, \infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone. For recent inequalities for operator convex functions see [1], [3], [6], [7], [8], [10]-[18] and [25]-[29].

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i E_i(d\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2] we define

$$f(A) = f(A_1, ..., A_n) = \int_{I_1 \times ... \times I_k} f(\lambda_1, ..., \lambda_1) E_1(d\lambda_1) \otimes ... \otimes E_k(d\lambda_k) \tag{1.9}$$

as a bounded selfadjoint operator on $H_1 \otimes ... \otimes H_k$.

The above function $f : I_1 \times ... \times I_k \to \mathbb{R}$ is said to be operator convex, if the operator inequality

$$f((1-\alpha) A + \alpha B) \leq (1-\alpha) f(A) + \alpha f(B) \tag{1.10}$$

for all $\alpha \in [0, 1]$, for any Hilbert spaces $H_1, ..., H_k$ and any k-tuples of of selfadjoint operators $A = (A_1, ..., A_n)$, $B = (B_1, ..., B_n)$ on $H_1 \otimes ... \otimes H_k$ contained in the domain of f. The definition is meaningful since also the spectrum of $\alpha A_i + (1-\alpha) B_i$ is contained in the interval I_i for each $i = 1, ..., k$.

In the following we restrict ourself to the case $k = 1$, $I_1 = I_2 = I$ and $H_1 = H_1 = H$. The operator convexity of $f : I \times I \to \mathbb{R}$ in this case means, for instance,

$$f((1-\alpha) A_1 + \alpha B_1, (1-\alpha) A_2 + \alpha B_2) \leq (1-\alpha) f(A_1, A_2) + \alpha f(B_1, B_2) \tag{1.11}$$

or, equivalently,

$$f((1-\alpha)(A_1, A_2) + \alpha (B_1, B_2)) \leq (1-\alpha) f(A_1, A_2) + \alpha f(B_1, B_2) \tag{1.12}$$

for all selfadjoint operators A_1, A_2, B_1, B_2 with spectra in I and for all $\alpha \in [0, 1]$.

In this paper we introduce the concept of *operator Schur convex* functions, investigate their main properties, establish some integral inequalities of Hermite-Hadamard, Čebyšev and Grüss' type and give some general classes of examples of such functions.

2. Operator Schur Convex Functions

For I an interval, we consider the set $\mathcal{SA}_I(H)$ of all selfadjoint operators with spectra in I. $\mathcal{SA}_I(H)$ is a convex set in $\mathcal{B}(H)$ since for A, B selfadjoints with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, $\alpha A+\beta B$ is selfadjoint with $\operatorname{Sp}(\alpha A+\beta B) \subset I$, where $\alpha, \beta \geq 0$ and $\alpha+\beta=1$. Motivated by the Stępniak's result for functions of real variables, we can introduce the following concept:

Definition 1. *We say that the function $f: I \times I \rightarrow \mathbb{R}$ is called operator Schur convex, if f is symmetric, namely $f(x, y)=f(y, x)$ for all $x, y \in I$ and*

$$f(tA+(1-t)B, tB+(1-t)A) \leq f(A, B)$$

or, equivalently,

$$f(t(A, B)+(1-t)(B, A)) \leq f(A, B)$$

in the operator order, for all $(A, B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$ and $t \in [0,1]$. The function f is called operator Schur concave if $-f$ is operator Schur convex.

For $(A, B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$, let us define the following auxiliary function $\varphi_{(A, B)}:[0,1] \rightarrow \mathcal{SA}(H \otimes H)$, the set of all selfadjoint operators on $H \otimes H$, by

$$\begin{aligned} \varphi_{f,(A, B)}(t) & =f(t(A, B)+(1-t)(B, A)) \\ & =f(tA+(1-t)B, tB+(1-t)A) . \end{aligned} \tag{2.1}$$

A function $f: J \rightarrow \mathcal{SA}(K)$ defined of an interval of real numbers J with self adjoint operator values on a Hilbert space K is called *operator monotone increasing* on J if

$$f(t) \leq f(s) \text{ in the operator order}$$

for all $s, t \in J$ with $t<s$.

The following characterization of operator Schur convexity holds:

Theorem 4. *Let $f: I \times I \rightarrow \mathbb{R}$ be a continuous symmetric function on $I \times I$. Then f is operator Schur convex on $I \times I$ if and only if for all arbitrarily fixed $(A, B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$ the function $\varphi_{f,(A, B)}$ is operator monotone decreasing on $[0,1/2)$, operator monotone increasing on $(1/2,1]$, and $\varphi_{f,(A, B)}$ has a global minimum at $1/2$ in the operator order.*

Proof. Assume that f is operator Schur convex on $I \times I$. Then for all $(C, D) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$ and $t \in [0,1]$ we have

$$f(t(C, D)+(1-t)(D, C)) \leq f(C, D) . \tag{2.2}$$

Let $(A, B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$ and for $0 \leq r<s<\frac{1}{2}$ and put $C=rA+(1-r)B$, $D=rB+(1-r)A$ and $t=\frac{s-r}{1-2r}$. Then $(C, D)=r(A, B)+(1-r)(B, A) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$, which is a convex set. By (2.2) we have

$$\begin{aligned} \varphi_{f,(A, B)}(r) & =f(r(A, B)+(1-r)(B, A))=f(C, D) \\ & \geq f\left(\frac{s-r}{1-2r}(C, D)+\left(1-\frac{s-r}{1-2r}\right)(D, C)\right)=: \beta . \end{aligned} \tag{2.3}$$

Observe that

$$\begin{aligned}
&\frac{s-r}{1-2r}(C,D)+\left(1-\frac{s-r}{1-2r}\right)(D,C)\\
&=\frac{s-r}{1-2r}\left[r(A,B)+(1-r)(B,A)\right]\\
&+\left(\frac{1-r-s}{1-2r}\right)\left[r(B,A)+(1-r)(A,B)\right]\\
&=\left[\left(\frac{s-r}{1-2r}\right)r+\left(\frac{1-r-s}{1-2r}\right)(1-r)\right](A,B)\\
&+\left[\frac{s-r}{1-2r}(1-r)+\left(\frac{1-r-s}{1-2r}\right)r\right](B,A)\\
&=\left(\frac{1-s-2r+2rs}{1-2r}\right)(A,B)+\left(\frac{s-2rs}{1-2r}\right)(B,A)\\
&=(1-s)(A,B)+s(B,A).
\end{aligned}$$

Then

$$\beta=f((1-s)(A,B)+s(B,A))=\varphi_{f,(A,B)}(s)$$

and by (2.3) we get that $\varphi_{f,(A,B)}(r)\geq\varphi_{f,(A,B)}(s)$ for $0\leq r<s<\frac{1}{2}$, which shows that the function $\varphi_{f,(A,B)}$ is operator monotone decreasing on $[0,1/2)$.

Observe that, by the symmetry of f on $\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$, we have

$$\begin{aligned}
\varphi_{f,(A,B)}(1-t)&=f((1-t)(A,B)+t(B,A))\\
&=f((1-t)A+tB,(1-t)B+tA)\\
&=f((1-t)B+tA,(1-t)A+tB)\\
&=f(t(A,B)+(1-t)(B,A))=\varphi_{f,(A,B)}(t)
\end{aligned}$$

for all $t\in[0,1]$.

This shows that the function $\varphi_{f,(A,B)}$ is also operator monotone increasing on $(1/2,1]$.

From (2.2) we get for $t=\frac{1}{2}$ that

$$f\left(\frac{C+D}{2},\frac{C+D}{2}\right)\leq f(C,D) \tag{2.4}$$

for all $(C,D)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$. If $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$ and we take $C=tA+(1-t)B$, $D=tB+(1-t)A$, $t\in[0,1]$ then $(C,D)=t(A,B)+(1-t)(B,A)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$, $\frac{C+D}{2}=\frac{A+B}{2}$ and by (2.4) we get $\varphi_{f,(A,B)}(1/2)\leq\varphi_{f,(A,B)}(t)$ for all $t\in[0,1]$, showing that $\varphi_{f,(A,B)}$ has a global minimum at $1/2$ in the operator order.

Now, for fixed $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$, assume that the function $\varphi_{f,(A,B)}$ is operator monotone decreasing on $[0,1/2)$, operator monotone increasing on $(1/2,1]$, and has a global minimum at $1/2$ in the operator order.

Then for $t\in[0,1/2)$ we have

$$f(t(A,B)+(1-t)(B,A))=\varphi_{f,(A,B)}(t)\leq\varphi_{f,(A,B)}(0)=f(B,A)=f(A,B)$$

and for $t\in(1/2,1]$ we have

$$f(t(A,B)+(1-t)(B,A))=\varphi_{f,(A,B)}(t)\leq\varphi_{f,(A,B)}(1)=f(A,B).$$

Therefore, for all $t\in[0,1]$ we have $\varphi_{f,(A,B)}(t)\le f(A,B)$, which shows that f is operator Schur convex on $\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$. $\square$

We have the following integral inequality in the operator order:

Theorem 5. *Assume that the function $f:I\times I\to\mathbb{R}$ is operator Schur convex on $I\times I$. Then for any Lebesgue integrable function $p:[0,1]\to[0,\infty)$ with $\int_0^1 p(t)\,dt=1$ we have*

$$f\left(\frac{A+B}{2},\frac{A+B}{2}\right)dt\le\int_0^1 f(tA+(1-t)B,tB+(1-t)A)\,p(t)\,dt \le f(A,B) \tag{2.5}$$

for all $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$.

In particular, we have

$$f\left(\frac{A+B}{2},\frac{A+B}{2}\right)\le\int_0^1 f(tA+(1-t)B,tB+(1-t)A)\,dt\le f(A,B) \tag{2.6}$$

for all $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$.

Proof. Using Theorem 4 we have

$$f\left(\frac{A+B}{2},\frac{A+B}{2}\right)\le f(t(A,B)+(1-t)(B,A))\le f(A,B)$$

for all $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$ and $t\in[0,1]$.

If we multiply this inequality by $p(t)\ge0$ and integrate on $[0,1]$ we deduce the desired result (2.5). $\square$

For scalar inequalities of Hermite-Hadamard type see the monograph online [12] and the recent survey paper [9].

If some monotonicity information is available for the function p we also have:

Theorem 6. *Assume that the function $f:I\times I\to\mathbb{R}$ is operator Schur convex on $I\times I$. If $p:[0,1]\to\mathbb{R}$ is symmetric towards $1/2$, namely $p(1-t)=p(t)$ for all $t\in[0,1]$ and monotonic decreasing (increasing) on $[0,1/2]$, then*

$$\int_0^1 f(tA+(1-t)B,tB+(1-t)A)\,p(t)\,dt \ge(\le)\int_0^1 p(t)\,dt\int_0^1 f(tA+(1-t)B,tB+(1-t)A)\,dt. \tag{2.7}$$

Proof. Let $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$. Since the functions $\varphi_{f,(A,B)}$ and p are symmetric on $[0,1]$, then

$$\int_0^1 f(t(A,B)+(1-t)(B,A))\,p(t)\,dt=2\int_0^{1/2} f(t(A,B)+(1-t)(B,A))\,p(t)\,dt.$$

Let $x\in H$. Then the function $\varphi_{f,(A,B),x}(t):[0,1]\to\mathbb{R}$ defined by

$$\varphi_{f,(A,B),x}(t)=\left\langle\varphi_{f,(A,B)}(t)\,x,x\right\rangle$$

where $\langle\cdot,\cdot\rangle$ is the inner product on H, is monotone decreasing as a real valued function on $[0,1/2]$.

Assume that p is monotone decreasing on $[0,1/2]$, then by Čebyšev's inequality for synchronous functions $h,\ g:[a,b]\to\mathbb{R}$, namely

$$\frac{1}{b-a}\int_a^b h(t)\,g(t)\,dt \geq \frac{1}{b-a}\int_a^b h(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt,$$

we have

$$\begin{aligned} &2\int_0^{1/2}\langle f(t(A,B)+(1-t)(B,A))\,x,x\rangle\, p(t)\,dt \\ &\geq 2\int_0^{1/2}\langle f(t(A,B)+(1-t)(B,A))\,x,x\rangle\, dt\cdot 2\int_0^{1/2}p(t)\,dt \end{aligned} \tag{2.8}$$

and since, by symmetry,

$$\begin{aligned} &2\int_0^{1/2}\langle f(t(A,B)+(1-t)(B,A))\,x,x\rangle\, dt \\ &=\int_0^{1}\langle f(t(A,B)+(1-t)(B,A))\,x,x\rangle\, dt \end{aligned}$$

and

$$2\int_0^{1/2}p(t)\,dt=\int_0^1 p(t)\,dt,$$

hence by (2.8) we get

$$\begin{aligned} &\left\langle\left(\int_0^1 f(t(A,B)+(1-t)(B,A))\,p(t)\,dt\right)x,x\right\rangle \\ &\geq\left\langle\left(\int_0^1 p(t)\,dt\int_0^1 f(t(A,B)+(1-t)(B,A))\,dt\right)x,x\right\rangle, \end{aligned}$$

which is equivalent to the desired result (2.7). □

We can prove the following refinement of (2.5):

Corollary 1. *Assume that the function $f:I\times I\to\mathbb{R}$ is operator Schur convex on $I\times I$ and $p:[0,1]\to\mathbb{R}$ is symmetric towards $1/2$ with $\int_0^1 p(t)=1$.*

(i) *If p is monotone decreasing on $[0,1/2]$, then*

$$\begin{aligned} f\left(\frac{A+B}{2},\frac{A+B}{2}\right) &\leq \int_0^1 f(tA+(1-t)B,tB+(1-t)A)\,dt \\ &\leq \int_0^1 f(tA+(1-t)B,tB+(1-t)A)\,p(t)\,dt \\ &\leq f(A,B) \end{aligned} \tag{2.9}$$

for all $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$.

(ii) *If p is monotone increasing on $[0,1/2]$, then*

$$\begin{aligned} f\left(\frac{A+B}{2},\frac{A+B}{2}\right) &\leq \int_0^1 f(tA+(1-t)B,tB+(1-t)A)\,p(t)\,dt \\ &\leq \int_0^1 f(tA+(1-t)B,tB+(1-t)A)\,dt \\ &\leq f(A,B) \end{aligned} \tag{2.10}$$

for all $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$.

Proof. (i). From (2.7) we get

$$\frac{1}{\int_0^1 p(t)\,dt}\int_0^1 f(tA+(1-t)B, tB+(1-t)A)\,p(t)\,dt$$
$$\geq \int_0^1 f(tA+(1-t)B, tB+(1-t)A)\,dt$$

and by (2.5) and (2.6) we get the desired result (2.9).

(ii). The proof goes in a similar way. □

Remark 1. *If we consider the weight* $p(t)=4\left|t-\frac{1}{2}\right|$, *then* $\int_0^1 p(t)\,dt=1$ *and by (2.9) we get*

$$f\left(\frac{A+B}{2},\frac{A+B}{2}\right)\leq\int_0^1 f(tA+(1-t)B, tB+(1-t)A)\,dt \tag{2.11}$$
$$\leq 4\int_0^1 f(tA+(1-t)B, tB+(1-t)A)\left|t-\frac{1}{2}\right|dt$$
$$\leq f(A,B)$$

for any function $f:I\times I\to\mathbb{R}$ *that is operator Schur convex and for all* $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$.

If we consider the weight $p(t)=6t(1-t)$, *then* $\int_0^1 p(t)\,dt=1$ *and by (2.10) we get*

$$f\left(\frac{A+B}{2},\frac{A+B}{2}\right)\leq\int_0^1 f(tA+(1-t)B, tB+(1-t)A)\,dt \tag{2.12}$$
$$\leq 6\int_0^1 f(tA+(1-t)B, tB+(1-t)A)\,t(1-t)\,dt$$
$$\leq f(A,B)$$

for any function $f:I\times I\to\mathbb{R}$ *that is operator Schur convex and for all* $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$.

We also have:

Theorem 7. *Assume that the function* $f:I\times I\to\mathbb{R}$ *is operator Schur convex on* $I\times I$ *and* $p:[0,1]\to\mathbb{R}$ *is symmetric towards* $1/2$.

If $p:[0,1]\to\mathbb{R}$ *is monotonic decreasing on* $[0,1/2]$, *then*

$$0\leq\int_0^1 f(tA+(1-t)B, tB+(1-t)A)\,p(t)\,dt \tag{2.13}$$
$$-\int_0^1 p(t)\,dt\int_0^1 f(tA+(1-t)B, tB+(1-t)A)\,dt$$
$$\leq\frac{1}{4}\left[p(0)-p\left(\frac{1}{2}\right)\right]\left[f(A,B)-f\left(\frac{A+B}{2},\frac{A+B}{2}\right)\right]$$

for all $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$.

If p is monotonic increasing on $[0,1/2]$, then

$$0 \leq \int_0^1 p(t)\,dt \int_0^1 f(tA+(1-t)B, tB+(1-t)A)\,dt - \int_0^1 f(tA+(1-t)B, tB+(1-t)A)\,p(t)\,dt \leq \frac{1}{4}\left[p\left(\frac{1}{2}\right)-p(0)\right]\left[f(A,B)-f\left(\frac{A+B}{2},\frac{A+B}{2}\right)\right] \tag{2.14}$$

for all $(A,B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$.

Proof. Recall the famous *Grüss' inequality* that provides an upper bound for the distance between the integral mean of the product and the product of integral means, more precisely

$$\left|\frac{1}{b-a}\int_a^b h(t)\,k(t)\,dt - \frac{1}{b-a}\int_a^b h(t)\,dt \frac{1}{b-a}\int_a^b k(t)\,dt\right| \leq \frac{1}{4}(M-m)(N-n) \tag{2.15}$$

provided the functions h, k are measurable on $[a,b]$ and $-\infty < m \leq h(t) \leq M < \infty$, $-\infty < n \leq k(t) \leq N < \infty$, for almost every $t \in [a,b]$. The constant $\frac{1}{4}$ is best possible in (2.15).

Let $x \in H$. Then the function $\varphi_{f,(A,B),x}(t) : [0,1] \to \mathbb{R}$ defined by

$$\varphi_{f,(A,B),x}(t) = \left\langle \varphi_{f,(A,B)}(t)\,x, x\right\rangle$$

is monotone decreasing as a real valued function on $[0,1/2]$ and

$$\left\langle f\left(\frac{A+B}{2},\frac{A+B}{2}\right)x,x\right\rangle \leq \varphi_{f,(A,B),x}(t) \leq \langle f(A,B)\,x,x\rangle$$

for all $t \in [0,1/2]$.

Assume that p is monotonic decreasing on $[0,1/2]$. Then

$$p\left(\frac{1}{2}\right) \leq p(t) \leq p(0),\ \ t \in [0,1/2].$$

Therefore, by (2.15) we have

$$\begin{aligned}
0 &\leq 2\int_0^{1/2} \langle f(t(A,B)+(1-t)(B,A))\,x,x\rangle\, p(t)\,dt \\
&- 2\int_0^{1/2} \langle f(t(A,B)+(1-t)(B,A))\,x,x\rangle\,dt \cdot 2\int_0^{1/2} p(t)\,dt \\
&= \left|2\int_0^{1/2} \langle f(t(A,B)+(1-t)(B,A))\,x,x\rangle\, p(t)\,dt\right. \\
&\left.-2\int_0^{1/2} \langle f(t(A,B)+(1-t)(B,A))\,x,x\rangle\,dt \cdot 2\int_0^{1/2} p(t)\,dt\right| \\
&\leq \frac{1}{4}\left[p(0)-p\left(\frac{1}{2}\right)\right]\left[\langle f(A,B)\,x,x\rangle - \left\langle f\left(\frac{A+B}{2},\frac{A+B}{2}\right)x,x\right\rangle\right],
\end{aligned}$$

namely

$$\begin{aligned}0 &\le \left\langle \left(\int_0^1 f\left(t\left(A,B\right)+\left(1-t\right)\left(B,A\right)\right) p\left(t\right) dt \right) x, x \right\rangle \\ &- \left\langle \left(\int_0^1 p\left(t\right) dt \int_0^1 f\left(t\left(A,B\right)+\left(1-t\right)\left(B,A\right)\right) dt \right) x, x \right\rangle \\ &\le \frac{1}{4} \left\langle \left[p\left(0\right) - p\left(\frac{1}{2}\right) \right] \left[f\left(A,B\right) - f\left(\frac{A+B}{2}, \frac{A+B}{2} \right) \right] x, x \right\rangle\end{aligned}$$

for all $x \in H$, which is equivalent to the operator order inequality (2.13). □

Remark 2. *Assume that the function $f : I \times I \to \mathbb{R}$ is operator Schur convex on $I \times I$. Then we have the inequalities*

$$\begin{aligned}0 &\le \int_0^1 f\left(tA+\left(1-t\right)B, tB+\left(1-t\right)A\right) \left| t-\frac{1}{2} \right| dt \\ &- \frac{1}{4} \int_0^1 f\left(tA+\left(1-t\right)B, tB+\left(1-t\right)A\right) dt \\ &\le \frac{1}{8} \left[f\left(A,B\right) - f\left(\frac{A+B}{2}, \frac{A+B}{2} \right) \right]\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}0 &\le \frac{1}{6} \int_0^1 f\left(tA+\left(1-t\right)B, tB+\left(1-t\right)A\right) dt \\ &- \int_0^1 f\left(tA+\left(1-t\right)B, tB+\left(1-t\right)A\right) t\left(1-t\right) dt \\ &\le \frac{1}{16} \left[f\left(A,B\right) - f\left(\frac{A+B}{2}, \frac{A+B}{2} \right) \right]\end{aligned} \tag{2.17}$$

for all $(A,B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$.

In 1970, A. M. Ostrowski proved amongst others the following result

$$\begin{aligned}&\left| \frac{1}{b-a} \int_a^b h\left(t\right) k\left(t\right) dt - \frac{1}{b-a} \int_a^b h\left(t\right) dt \frac{1}{b-a} \int_a^b k\left(t\right) dt \right| \\ &\le \frac{1}{8} \left(b-a\right) \left(M-m\right) \left\| k' \right\|_\infty ,\end{aligned} \tag{2.18}$$

provided h is Lebesgue integrable on $[a,b]$ and satisfying $-\infty < m \le h(t) \le M < \infty$ while $k : [a,b] \to \mathbb{R}$ is absolutely continuous and $k' \in L_\infty [a,b]$. The constant $\frac{1}{8}$ in (2.18) is also sharp.

We can prove the following similar result as well:

Theorem 8. *Assume that the function $f : I \times I \to \mathbb{R}$ is operator Schur convex on $I \times I$ and $p : [0,1] \to \mathbb{R}$ is symmetric towards $1/2$ and absolutely continuous with $p' \in L_\infty [0,1]$.*

If $p:[0,1]\to\mathbb{R}$ *is monotonic decreasing on* $[0,1/2]$, *then*

$$
\begin{aligned}
(2.19)\qquad 0 &\le \int_0^1 f\left(tA+(1-t)B,tB+(1-t)A\right)p(t)\,dt \\
&- \int_0^1 p(t)\,dt\int_0^1 f\left(tA+(1-t)B,tB+(1-t)A\right)dt \\
&\le \frac{1}{8}\left\|p'\right\|_\infty,\left[f(A,B)-f\left(\frac{A+B}{2},\frac{A+B}{2}\right)\right]
\end{aligned}
$$

for all $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$.

If p *is monotonic increasing on* $[0,1/2]$, *then*

$$
\begin{aligned}
(2.20)\qquad 0 &\le \int_0^1 p(t)\,dt\int_0^1 f\left(tA+(1-t)B,tB+(1-t)A\right)dt \\
&- \int_0^1 f\left(tA+(1-t)B,tB+(1-t)A\right)p(t)\,dt \\
&\le \frac{1}{8}\left\|p'\right\|_\infty\left[f(A,B)-f\left(\frac{A+B}{2},\frac{A+B}{2}\right)\right]
\end{aligned}
$$

for all $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$.

3. Some Examples

Let $f:I\to\mathbb{R}$ be a continuous function on the interval I. For $t\in(0,1)$ we define the auxiliary function $f_t:I\times I\to\mathbb{R}$ by

$$
f_t(x,y):=\frac{1}{2}\left[f\left((1-t)x+ty\right)+f\left((1-t)y+tx\right)\right].
$$

We observe that f_t is continuous on $I\times I$ and symmetric, namely $f_t(x,y)=f_t(y,x)$ for all $(x,y)\in I\times I$.

Proposition 1. *Let* $f:I\to\mathbb{R}$ *be a continuous function on the interval* I. *If* f *is operator convex on* I *then* f_t *is operator Schur convex on* $I\times I$.

Proof. Let $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$, $s\in[0,1]$ and $t\in(0,1)$. By the operator convexity of f we have

$$
\begin{aligned}
&f_t\left(sA+(1-s)B,sB+(1-s)A\right) \\
&=\frac{1}{2}f\left((1-t)\left[sA+(1-s)B\right]+t\left[sB+(1-s)A\right]\right) \\
&+\frac{1}{2}f\left((1-t)\left[sB+(1-s)A\right]+t\left[sA+(1-s)B\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2} f\left(s\left[(1-t)A + tB\right] + (1-s)\left[(1-t)B + tA\right]\right) \\
&+ \frac{1}{2} f\left(s\left[(1-t)B + tA\right] + (1-s)\left[(1-t)A + tB\right]\right) \\
&\leq \frac{1}{2} s f\left((1-t)A + tB\right) + \frac{1}{2}(1-s) f\left((1-t)B + tA\right) \\
&+ \frac{1}{2} s f\left((1-t)B + tA\right) + \frac{1}{2}(1-s) f\left((1-t)A + tB\right) \\
&= \frac{1}{2}\left[f\left((1-t)A + tB\right) + f\left((1-t)B + tA\right)\right] \\
&= f_t(A, B),
\end{aligned}
$$

which shows that f_t is operator Schur convex on $I \times I$. □

For a Lebesgue integrable function $p : [0,1] \to [0,\infty)$ and $f : I \to \mathbb{R}$ a continuous function on the interval I we consider the function $F_p : I \times I \to \mathbb{R}$ defined by

$$
\begin{aligned}
F_p(x,y) &:= \int_0^1 f_t(x,y)\, p(t)\, dt \\
&= \frac{1}{2}\int_0^1 \left[f\left((1-t)x + ty\right) + f\left((1-t)y + tx\right)\right] p(t)\, dt \\
&= \int_0^1 f\left((1-t)x + ty\right) \breve{p}(t)\, dt,
\end{aligned}
$$

where $\breve{p}(t) := \frac{1}{2}\left[p(t) + p(1-t)\right],\ t \in [0,1]$.

In particular, for $p \equiv 1$ we put

$$
F(x,y) := \int_0^1 f\left((1-t)x + ty\right) dt
$$

for $(x,y) \in I \times I$.

We have:

Proposition 2. *Let $f : I \to \mathbb{R}$ be a continuous function on the interval I and $p : [0,1] \to [0,\infty)$ a Lebesgue integrable function on $[0,1]$. If f is operator convex on I then F_p is operator Schur convex on $I \times I$. In particular, F is operator Schur convex.*

Proof. Let $(A,B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$, $s \in [0,1]$. By the operator Schur convexity of f_t we have

$$
\begin{aligned}
&F_p\left(sA + (1-s)B, sB + (1-s)A\right) \\
&= \int_0^1 f_t\left(sA + (1-s)B, sB + (1-s)A\right) p(t)\, dt \\
&\leq \int_0^1 f_t(A,B)\, p(t)\, dt = F_p(A,B),
\end{aligned}
$$

which proves that F_p is operator Schur convex. □

By making the change of variable $u=(1-t)x+ty$, $t\in[0,1]$ for $x\neq y$ we have $du=(y-x)\,dt$, $t=\frac{u-x}{y-x}$, $1-t=\frac{y-u}{y-x}$ and

$$F_p(x,y) = \begin{cases} \frac{1}{2(y-x)}\int_x^y f(u)\left[p\left(\frac{u-x}{y-x}\right)+p\left(\frac{y-u}{y-x}\right)\right]du, & (x,y)\in I\times I,\ x\neq y, \\ \\ f(x)\int_0^1 p(t)\,dt, & (x,y)\in I\times I,\ x=y. \end{cases} \tag{3.1}$$

In particular

$$F(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(u)\,du, & (x,y)\in I\times I,\ x\neq y, \\ \\ f(x), & (x,y)\in I\times I,\ x=y. \end{cases} \tag{3.2}$$

If we consider $p_m(t):=\left|t-\frac{1}{2}\right|,\ t\in[0,1]$, then

$$F_{p_m}(x,y) = \begin{cases} \frac{1}{(y-x)^2}\int_x^y f(u)\left|u-\frac{x+y}{2}\right|du, & (x,y)\in I\times I,\ x\neq y, \\ \\ \frac{1}{4}f(x), & (x,y)\in I\times I,\ x=y. \end{cases} \tag{3.3}$$

If we consider $p_g(t):=t(1-t),\ t\in[0,1]$, then

$$F_{p_g}(x,y) = \begin{cases} \frac{1}{(y-x)^3}\int_x^y f(u)(u-x)(y-u)\,du, & (x,y)\in I\times I,\ x\neq y, \\ \\ \frac{1}{6}f(x), & (x,y)\in I\times I,\ x=y. \end{cases} \tag{3.4}$$

Therefore, if f is operator convex on I, then the functions defined by (3.1)-(3.4) are *operator Schur convex* on $I\times I$.

Since the function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\le r\le 2$ or $-1\le r\le 0$ and is operator concave on $(0,\infty)$ if $0\le r\le 1$, hence for $p:[0,1]\to[0,\infty)$ a Lebesgue integrable function on $[0,1]$,

$$F_{p,r}(x,y) := \begin{cases} \frac{1}{2(y-x)}\int_x^y u^r\left[p\left(\frac{u-x}{y-x}\right)+p\left(\frac{y-u}{y-x}\right)\right]du, \\ (x,y)\in(0,\infty)\times(0,\infty),\ x\neq y, \\ \\ x^r\int_0^1 p(t)\,dt, \quad (x,y)\in(0,\infty)\times(0,\infty),\ x=y, \end{cases} \tag{3.5}$$

is *operator Schur convex* on $(0,\infty)\times(0,\infty)$ if either $1\le r\le 2$ or $-1\le r\le 0$ and is *operator Schur concave* on $(0,\infty)\times(0,\infty)$ if $0\le r\le 1$.

In particular,

$$F_r(x,y) := \begin{cases} \frac{y^{r+1}-y^{r+1}}{(r+1)(y-x)}, & (x,y)\in(0,\infty)\times(0,\infty),\ x\neq y, \\ \\ x^r, & (x,y)\in(0,\infty)\times(0,\infty),\ x=y. \end{cases} \tag{3.6}$$

is *operator Schur convex* on $(0,\infty)\times(0,\infty)$ if either $1\le r\le 2$ or $-1<r\le 0$ and is *operator Schur concave* on $(0,\infty)\times(0,\infty)$ if $0\le r\le 1$.

For $r=-1$, if we put

$$F_{-1}(x,y):=\begin{cases}\frac{\ln y-\ln x}{y-x}, & (x,y)\in(0,\infty)\times(0,\infty),\ x\neq y,\\ \\ x^{-1}, & (x,y)\in(0,\infty)\times(0,\infty),\ x=y,\end{cases} \tag{3.7}$$

then we conclude that F_{-1} is *operator Schur convex* on $(0,\infty)\times(0,\infty)$.

Since $f(t)=\ln t$, $t\in(0,\infty)$ is operator concave, then for $p:[0,1]\to[0,\infty)$, a Lebesgue integrable function on $[0,1]$,

$$\begin{aligned} &F_{p,\ln}(x,y) \\ &=\begin{cases}\frac{1}{2(y-x)}\int_x^y\left[p\left(\frac{u-x}{y-x}\right)+p\left(\frac{y-u}{y-x}\right)\right]\ln u du,\\ (x,y)\in(0,\infty)\times(0,\infty),\ x\neq y,\\ \\ f(x)\int_0^1p(t)\,dt, \quad (x,y)\in(0,\infty)\times(0,\infty),\ x=y,\end{cases}\end{aligned} \tag{3.8}$$

is *operator Schur concave* on $(0,\infty)\times(0,\infty)$.

In particular, if we put

$$F_{\ln}(x,y):=\begin{cases}\frac{y\ln y-x\ln x}{y-x}-1, & (x,y)\in(0,\infty)\times(0,\infty),\ x\neq y,\\ \\ \ln x, & (x,y)\in(0,\infty)\times(0,\infty),\ x=y,\end{cases} \tag{3.9}$$

then we conclude that $F_{\ln}$ is *operator Schur concave* on $(0,\infty)\times(0,\infty)$.

References

[1] R. P. Agarwal and S. S. Dragomir, A survey of Jensen type inequalities for functions of selfadjoint operators in Hilbert spaces. *Comput. Math. Appl.* 5**9** (2010), no. 12, 3785–3812.

[2] H. Araki and F. Hansen, Jensen's operator inequality for functions of several variables, *Proc. Amer. Math. Soc.* **128** (2000), Number 7, 2075-2084.

[3] V. Bacak, T. Vildan and R. Türkmen, Refinements of Hermite-Hadamard type inequalities for operator convex functions. *J. Inequal. Appl.* **2013**, 2013:262, 10 pp.

[4] Y. Chu, G. Wang, X. Zhang, Schur convexity and Hadamard's inequality, *Math. Inequal. Appl.* **13** (4) (2010) 725–731.

[5] V. Čuljak, A remark on Schur-convexity of the mean of a convex function. *J. Math. Inequal.* **9** (2015), No. 4, 1133–1142.

[6] V. Darvish, S. S. Dragomir, H. M. Nazari and A. Taghavi, Some inequalities associated with the Hermite-Hadamard inequalities for operator h-convex functions. *Acta Comment. Univ. Tartu. Math.* **21** (2017), no. 2, 287–297.

[7] S. S. Dragomir, Hermite-Hadamard's type inequalities for operator convex functions. *Appl. Math. Comput.* **218** (2011), no. 3, 766–772.

[8] S. S. Dragomir, *Operator Inequalities of the Jensen, Čebyšev and Grüss Type.* Springer Briefs in Mathematics. Springer, New York, 2012. xii+121 pp. ISBN: 978-1-4614-1520-6.

[9] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results. *Aust. J. Math. Anal. Appl.* **14** (2017), No. 1, Art. 1, 283 pp. [Online https://ajmaa.org/cgi-bin/paper.pl?string=v14n1/V14I1P1.tex].

[10] S. S. Dragomir, Some Hermite-Hadamard type inequalities for operator convex functions and positive maps. *Spec. Matrices* **7** (2019), 38–51. Preprint *RGMIA Res. Rep. Coll.* **19** (2016), Art. 80. [Online http://rgmia.org/papers/v19/v19a80.pdf].

[11] S. S. Dragomir and K. Nikodem, Functions generating (m,M,Ψ)-Schur-convex sums. *Aequationes Math.* **93** (2019), No. 1, 79–90.

[12] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, 2000. [Online https://rgmia.org/monographs/hermite_hadamard.html].

[13] N. Elezović and J. Pečarić, A note on Schur convex fuctions, *Rocky Mountain J. Math.* **30** (2000), No. 3, 853-956.

[14] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.
[15] A. G. Ghazanfari, The Hermite-Hadamard type inequalities for operator s-convex functions. *J. Adv. Res. Pure Math.* **6** (2014), no. 3, 52–61.
[16] A. G. Ghazanfari, Hermite-Hadamard type inequalities for functions whose derivatives are operator convex. *Complex Anal. Oper. Theory* **10** (2016), no. 8, 1695–1703.
[17] J. Han and J. Shi, Refinements of Hermite-Hadamard inequality for operator convex function. *J. Nonlinear Sci. Appl.* **10** (2017), no. 11, 6035–6041.
[18] B. Li, Refinements of Hermite-Hadamard's type inequalities for operator convex functions. *Int. J. Contemp. Math. Sci.* **8** (2013), no. 9-12, 463–467.
[19] A. W. Marshall, I. Olkin and B. C. Arnold, *Inequalities: Theory of Majorization and Its Applications,* Second Edition, Springer New York Dordrecht Heidelberg London, 2011.
[20] K. Nikodem, T. Rajba and S. Wąsowicz, Functions generating strongly Schur-convex sums. *Inequalities and applications* 2010, 175–182, Internat. Ser. Numer. Math., 161, Birkhäuser/Springer, Basel, 2012.
[21] A. M. Ostrowski, On an integral inequality, *Aequat. Math.,* **4** (1970), 358-373.
[22] J. Qi and W. Wang, Schur convex functions and the Bonnesen style isoperimetric inequalities for planar convex polygons. *J. Math. Inequal.* **12** (2018), no. 1, 23–29.
[23] H.-N. Shi and J. Zhang, Compositions involving Schur harmonically convex functions. *J. Comput. Anal. Appl.* **22** (2017), no. 5, 907–922.
[24] C. Stępniak, An effective characterization of Schur-convex functions with applications, *Journal of Convex Analysis*, **14** (2007), No. 1, 103–108.pp.
[25] A. Taghavi, V. Darvish, H. M. Nazari and S. S. Dragomir, Hermite-Hadamard type inequalities for operator geometrically convex functions. *Monatsh. Math.* **181** (2016), no. 1, 187–203.
[26] M. Vivas Cortez, H. Hernández and E. Jorge, Refinements for Hermite-Hadamard type inequalities for operator h-convex function. *Appl. Math. Inf. Sci.* **11** (2017), no. 5, 1299–1307.
[27] M. Vivas Cortez, H. Hernández and E. Jorge, On some new generalized Hermite-Hadamard-Fejér inequalities for product of two operator h-convex functions. A*ppl. Math. Inf. Sci.* **11** (2017), no. 4, 983–992.
[28] S.-H. Wang, Hermite-Hadamard type inequalities for operator convex functions on the co-ordinates. *J. Nonlinear Sci. Appl.* **10** (2017), no. 3, 1116–1125
[29] S.-H. Wang, New integral inequalities of Hermite-Hadamard type for operator m-convex and (α, m)-convex functions. *J. Comput. Anal. Appl.* **22** (2017), no. 4, 744–753.

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