OPERATOR SCHUR CONVEXITY OF SOME FUNCTIONS ASSOCIATED TO HERMITE-HADAMARD INEQUALITY

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Abstract. A continuous function $f: I \times I \to \mathbb{R}$ is called *operator Schur convex*, if f is symmetric, namely $f(x,y) = f(y,x)$ for all x, $y \in I$ and

$$f(tA + (1-t)B, tB + (1-t)A) \le f(A,B)$$

in the operator order, for all $(A,B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$ and $t \in [0,1]$, where $\mathcal{SA}_I(H)$ is the convex set of all selfadjoint operators on Hilbert space H with spectra in I.

In this paper we investigate the operator Schur convexity of some functions associated to the Hermite-Hadamard inequality for operator convex functions. Some particular examples of interest are also given.

1. Introduction

For any $x = (x_1, ..., x_n) \in \mathbb{R}^n$, let $x_{[1]} \ge ... \ge x_{[n]}$ denote the components of x in decreasing order, and let $x_\downarrow = (x_{[1]}, ..., x_{[n]})$ denote the decreasing rearrangement of x. For x, $y \in \mathbb{R}^n$, $x \prec y$ if, by definition,

$$\begin{cases} \sum_{i=1}^k x_{[i]} \le \sum_{i=1}^k y_{[i]}, \ k = 1, ..., n-1; \\ \\ \sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]}. \end{cases}$$

When $x \prec y$, x is said to be *majorized* by y (y *majorizes* x). This notation and terminology was introduced by Hardy, Littlewood and Pólya in 1934.

Functions that preserve the ordering of majorization are said to be Schur-convex, [21, p.80]. A real-valued function ϕ defined on a set $\mathcal{A} \subset \mathbb{R}^n$ is said to be *Schur-convex* on $\mathcal{A}$ if

$$x \prec y \text{ on } \mathcal{A} \Rightarrow \phi(x) \le \phi(y). \tag{1.1}$$

If, in addition, $\phi(x) < \phi(y)$ whenever $x \prec y$ but x is not a permutation of y, then ϕ is said to be *strictly Schur-convex* on $\mathcal{A}$. If $\mathcal{A} = \mathbb{R}^n$, then ϕ is simply said to be Schur-convex or strictly Schur-convex.

For fundamental properties of Schur convexity see the monograph [21] and the references therein. For some recent results, see [5]-[13], [15], [22] and [24]-[26].

The following result is known in the literature as *Schur-Ostrowski theorem* [21, p. 84]:

Theorem 1. *Let $I \subset \mathbb{R}$ be an open interval and let $\phi: I^n \to \mathbb{R}$ be continuously differentiable. Necessary and sufficient conditions for ϕ to be Schur-convex on I^n*

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are

$$\phi \text{ is symmetric on } I^n, \tag{1.2}$$

and for all $i \neq j$, *with* $i, j \in \{1, ..., n\}$,

$$(z_i - z_j)\left[\frac{\partial\phi(z)}{\partial x_i} - \frac{\partial\phi(z)}{\partial x_j}\right] \geq 0 \text{ for all } z \in I^n, \tag{1.3}$$

where $\frac{\partial\phi}{\partial x_k}$ *denotes the partial derivative of* ϕ *with respect to its k-th argument.*

Let $\mathcal{A} \subset \mathbb{R}^n$ be a set with the following properties:

(i) $\mathcal{A}$ is *symmetric* in the sense that $x \in \mathcal{A} \Rightarrow x\Pi \in \mathcal{A}$ for all permutations Π of the coordinates.

(ii) $\mathcal{A}$ is convex and has a nonempty interior.

We have the following result, [21, p. 85].

Theorem 2. *If* ϕ *is continuously differentiable on the interior of* $\mathcal{A}$ *and continuous on* $\mathcal{A}$, *then necessary and sufficient conditions for* ϕ *to be Schur-convex on* $\mathcal{A}$ *are*

$$\phi \text{ is symmetric on } \mathcal{A} \tag{1.4}$$

and

$$(z_1 - z_2)\left[\frac{\partial\phi(z)}{\partial x_1} - \frac{\partial\phi(z)}{\partial x_2}\right] \geq 0 \text{ for all } z \in \mathcal{A}. \tag{1.5}$$

Another interesting characterization of Schur convex functions ϕ on $\mathcal{A}$ was obtained by C. Stępniak in [26]:

Theorem 3. *Let* ϕ *be any function defined on a symmetric convex set* $\mathcal{A}$ *in* $\mathbb{R}^n$. *Then the function* ϕ *is Schur convex on* $\mathcal{A}$ *if and only if*

$$\phi(x_1, ..., x_i, ..., x_j, ..., x_n) = \phi(x_1, ..., x_j, ..., x_i, ..., x_n) \tag{1.6}$$

for all $(x_1, ..., x_n) \in \mathcal{A}$ *and* $1 \leq i < j \leq n$ *and*

$$\phi(\lambda x_1 + (1-\lambda)x_2, \lambda x_2 + (1-\lambda)x_1, x_3, ..., x_n) \leq \phi(x_1, ..., x_n) \tag{1.7}$$

for all $(x_1, ..., x_n) \in \mathcal{A}$ *and for all* $\lambda \in (0,1)$,

It is well known that any symmetric convex function defined on a symmetric convex set $\mathcal{A}$ is Schur convex, [21, p. 97]. If the function $\phi : \mathcal{A} \to \mathbb{R}$ is symmetric and quasi-convex, namely

$$\phi(\alpha u + (1-\alpha)v) \leq \max\{\phi(u), \phi(v)\}$$

for all $\alpha \in [0,1]$ and $u, v \in \mathcal{A}$, a symmetric convex set, then ϕ is Schur convex on $\mathcal{A}$ [21, p. 98].

In order to extend the above concept to continuous functions of selfadjoint operators on complex Hilbert space we need some preparations as follow.

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda)f(A) + \lambda f(B) \tag{1.8}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [16] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0,\infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0,\infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0,\infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t) = -t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

In [7] we obtained among others the following Hermite-Hadamard type inequalities for operator convex functions $f : I \to \mathbb{R}$

$$f\left(\frac{A+B}{2}\right) \leq \int_0^1 f((1-s)A + sB)\,ds \leq \frac{f(A)+f(B)}{2}, \tag{1.9}$$

where A, B are selfadjoint operators with spectra included in I.

If $p : [0,1] \to [0,\infty)$ is symmetric in the sense that $p(1-t) = p(t)$ for all $t \in [0,1]$, p is Lebesgue integrable with $\int_0^1 p(s)\,ds > 0$ and $f : I \to \mathbb{R}$ is operator convex function, then we also have the weighted operator inequality (see for instance [12])

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leq \frac{1}{\int_0^1 p(s)\,ds}\int_0^1 f((1-s)A+sB)\,p(s)\,ds \\ &\leq \frac{f(A)+f(B)}{2}, \end{aligned} \tag{1.10}$$

where A, B are selfadjoint operators with spectra included in I.

For recent inequalities for operator convex functions see [1], [3], [6], [7], [8], [10]-[20] and [27]-[31].

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i E_i(d\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2] we define

$$f(A) = f(A_1, ..., A_n) = \int_{I_1\times...\times I_k} f(\lambda_1, ..., \lambda_1)\,E_1(d\lambda_1) \otimes ... \otimes E_k(d\lambda_k) \tag{1.11}$$

as a bounded selfadjoint operator on $H_1 \otimes ... \otimes H_k$.

The above function $f : I_1 \times ... \times I_k \to \mathbb{R}$ is said to be operator convex, if the operator inequality

$$f((1-\alpha)A + \alpha B) \leq (1-\alpha)f(A) + \alpha f(B) \tag{1.12}$$

holds for all $\alpha \in [0,1]$, for any Hilbert spaces $H_1, ..., H_k$ and any k-tuples of of selfadjoint operators $A = (A_1, ..., A_n)$, $B = (B_1, ..., B_n)$ on $H_1 \otimes ... \otimes H_k$ contained in the domain of f. The definition is meaningful since also the spectrum of αA_i $+(1-\alpha)B_i$ is contained in the interval I_i for each $i = 1, ..., k$.

In the following we restrict ourself to the case $k=1$, $I_1=I_2=I$ and $H_1=H_1=H$. The operator convexity of $f:I\times I\to\mathbb{R}$ in this case means, for instance,

$$f\left(\left(1-\alpha\right)A_1+\alpha B_1,\left(1-\alpha\right)A_2+\alpha B_2\right)\leq\left(1-\alpha\right)f\left(A_1,A_2\right)+\alpha f\left(B_1,B_2\right) \tag{1.13}$$

or, equivalently,

$$f\left(\left(1-\alpha\right)\left(A_1,A_2\right)+\alpha\left(B_1,B_2\right)\right)\leq\left(1-\alpha\right)f\left(A_1,A_2\right)+\alpha f\left(B_1,B_2\right) \tag{1.14}$$

for all selfadjoint operators A_1, A_2, B_1, B_2 with spectra in I and for all $\alpha\in[0,1]$.

In this paper we investigate the operator Schur convexity of some functions associated to the Hermite-Hadamard inequality for operator convex functions. Some particular examples of interest are also given.

2. Operator Schur Convexity of Some Functions

For I an interval, we consider the set $\mathcal{SA}_I\left(H\right)$ of all selfadjoint operators with spectra in I. $\mathcal{SA}_I\left(H\right)$ is a convex set in $\mathcal{B}\left(H\right)$ since for A, B selfadjoints with $\operatorname{Sp}\left(A\right),\operatorname{Sp}\left(B\right)\subset I$, $\alpha A+\beta B$ is selfadjoint with $\operatorname{Sp}\left(\alpha A+\beta B\right)\subset I$, where $\alpha,\beta\geq0$ and $\alpha+\beta=1$. Motivated by the Stępniak's result for functions of real variables, we can introduce the following concept:

Definition 1. *We say that the function $f:I\times I\to\mathbb{R}$ is called operator Schur convex, if f is symmetric, namely $f\left(x,y\right)=f\left(y,x\right)$ for all x, $y\in I$ and*

$$f\left(tA+\left(1-t\right)B,tB+\left(1-t\right)A\right)\leq f\left(A,B\right)$$

or, equivalently,

$$f\left(t\left(A,B\right)+\left(1-t\right)\left(B,A\right)\right)\leq f\left(A,B\right)$$

in the operator order, for all $\left(A,B\right)\in\mathcal{SA}_I\left(H\right)\times\mathcal{SA}_I\left(H\right)$ and $t\in[0,1]$. The function f is called operator Schur concave if $-f$ is operator Schur convex.

For $\left(A,B\right)\in\mathcal{SA}_I\left(H\right)\times\mathcal{SA}_I\left(H\right)$, let us define the following auxiliary function $\varphi_{\left(A,B\right)}:[0,1]\to\mathcal{SA}\left(H\otimes H\right)$, the set of all selfadjoint operators on $H\otimes H$, by

$$\begin{aligned}\varphi_{f,\left(A,B\right)}\left(t\right)&=f\left(t\left(A,B\right)+\left(1-t\right)\left(B,A\right)\right)\\&=f\left(tA+\left(1-t\right)B,tB+\left(1-t\right)A\right).\end{aligned} \tag{2.1}$$

A function $f:J\to\mathcal{SA}\left(K\right)$ defined of an interval of real numbers J with self adjoint operator values on a Hilbert space K is called *operator monotone increasing* on J if

$$f\left(t\right)\leq f\left(s\right)\text{ in the operator order}$$

for all s, $t\in J$ with $t<s$.

The following characterization of operator Schur convexity holds, see the recent paper [11]:

Theorem 4. *Let $f:I\times I\to\mathbb{R}$ be a continuous symmetric function on $I\times I$. Then f is operator Schur convex on $I\times I$ if and only if for all arbitrarily fixed $\left(A,B\right)\in\mathcal{SA}_I\left(H\right)\times\mathcal{SA}_I\left(H\right)$ the function $\varphi_{f,\left(A,B\right)}$ is operator monotone decreasing on $[0,1/2)$, operator monotone increasing on $(1/2,1]$, and $\varphi_{f,\left(A,B\right)}$ has a global minimum at $1/2$ in the operator order.*

Now, for an operator convex function $f : I \to \mathbb{R}$ and a $t \in [0,1]$ define the functions M_t, $T_t : I^2 \to \mathbb{R}$

$$M_t(x,y) := \frac{1}{2}\left[f((1-t)x+ty)+f((1-t)y+tx)\right]-f\left(\frac{x+y}{2}\right)\geq 0$$

and

$$T_t(x,y) := \frac{f(x)+f(y)}{2}-\frac{1}{2}\left[f((1-t)x+ty)+f((1-t)y+tx)\right]\geq 0.$$

The positivity of these functions follows by the fact that f is convex on I.

We have the following result concerning the Schur convexity of M_t.

Theorem 5. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. For all $t \in [0,1]$, $t \neq \frac{1}{2}$ the function M_t is operator Schur convex on I^2.*

Proof. Let $(A,B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$ and $s \in [0,1]$. Then

$$\begin{aligned}
&M_t(s(A,B)+(1-s)(B,A)) \\
&= M_t(sA+(1-s)B, sB+(1-s)A) \\
&= \frac{1}{2}f((1-t)(sA+(1-s)B)+t(sB+(1-s)A)) \\
&+ \frac{1}{2}f((1-t)(sB+(1-s)A)+t(sA+(1-s)B)) \\
&- f\left(\frac{sA+(1-s)B+sB+(1-s)A}{2}\right) \\
&= \frac{1}{2}f(s((1-t)A+tB)+(1-s)((1-t)B+tA)) \\
&+ \frac{1}{2}f(s((1-t)B+tA)+(1-s)((1-t)A+tB))-f\left(\frac{A+B}{2}\right).
\end{aligned}$$

By the operator convexity of f we have

$$\begin{aligned}
&f(s((1-t)A+tB)+(1-s)((1-t)B+tA)) \\
&\leq sf((1-t)A+tB)+(1-s)f((1-t)B+tA)
\end{aligned}$$

and

$$\begin{aligned}
&f(s((1-t)B+tA)+(1-s)((1-t)A+tB)) \\
&\leq sf((1-t)B+tA)+(1-s)f((1-t)A+tB).
\end{aligned}$$

for all $(A,B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$ and $s \in [0,1]$.

If we add these two inequalities and divide by 2 we get

$$\begin{aligned}
&\frac{1}{2}f(s((1-t)A+tB)+(1-s)((1-t)B+tA)) \\
&+ \frac{1}{2}f(s((1-t)B+tA)+(1-s)((1-t)A+tB)) \\
&\leq \frac{1}{2}\left[f((1-t)B+tA)+f((1-t)A+tB)\right]
\end{aligned}$$

for all $(A,B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$ and $s \in [0,1]$.

Therefore

$$\begin{aligned}
& M_t\left(s(A,B)+(1-s)(B,A)\right) \\
& \leq \frac{1}{2}\left[f\left((1-t)B+tA\right)+f\left((1-t)A+tB\right)\right]-f\left(\frac{A+B}{2}\right) \\
& = M_t(A,B)
\end{aligned}$$

for all $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$ and $s\in[0,1]$, which shows that M_t is Schur convex on I^2. □

For a convex function $f:I\to\mathbb{R}$ and $q:[0,1]\to[0,\infty)$ a Lebesgue integrable function we consider the function $M_{\breve{q}}:I^2\to[0,\infty)$ defined by

$$\begin{aligned}
M_{\breve{q}}(x,y) &:= \int_0^1 M_t(x,y)\,q(t)\,dt \\
&= \frac{1}{2}\int_0^1\left[f\left((1-t)x+ty\right)+f\left((1-t)y+tx\right)\right]q(t)\,dt \\
&\quad - f\left(\frac{x+y}{2}\right)\int_0^1 q(t)\,dt \\
&= \int_0^1 f\left((1-t)x+ty\right)\breve{q}(t)\,dt - f\left(\frac{x+y}{2}\right)\int_0^1 q(t)\,dt,
\end{aligned}$$

where

$$\breve{q}(t):=\frac{1}{2}\left[q(t)+q(1-t)\right],\ t\in[0,1].$$

Corollary 1. *Let $f:I\to\mathbb{R}$ be an operator convex function on I and $q:[0,1]\to[0,\infty)$ a Lebesgue integrable function on $[0,1]$, then $M_{\breve{q}}$ is operator Schur convex on I^2.*

Proof. Let $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$ and $s\in[0,1]$. By the operator Schur convexity of M_t for all $t\in[0,1]$, we have

$$\begin{aligned}
M_{\breve{q}}\left(s(A,B)+(1-s)(B,A)\right) &= \int_0^1 M_t\left(s(A,B)+(1-s)(B,A)\right)q(t)\,dt \\
&\leq \int_0^1 M_t(A,B)\,q(t)\,dt = M_{\breve{q}}(A,B),
\end{aligned}$$

which proves the Schur convexity of $M_{\breve{q}}$. □

Corollary 2. *Let $f:I\to\mathbb{R}$ be an operator convex function on I and $p:[0,1]\to[0,\infty)$ a Lebesgue integrable symmetric function on $[0,1]$, then M_p is operator Schur convex on I^2.*

We denote by $[A,B]$ the closed segment defined by $\{(1-s)A+sB,\ s\in[0,1]\}$. We also define the functional

$$\Psi_{f,t}(A,B):=(1-t)f(A)+tf(B)-f\left((1-t)A+tB\right)\geq 0,$$

where $A,\ B\in I$ and $t\in[0,1]$.

In [7] we obtained among others the following result :

Lemma 1. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for each $A,\ B \in \mathcal{SA}_I(H)$ and $C \in [A, B]$ we have*

$$(0 \le)\, \Psi_{f,t}(A, C) + \Psi_{f,t}(C, B) \le \Psi_{f,t}(A, B) \tag{2.2}$$

for each $t \in [0, 1]$, i.e., the functional $\Psi_{f,t}(\cdot, \cdot)$ is superadditive as a function of interval.

If $C,\ D \in [A, B]$, then

$$(0 \le)\, \Psi_{f,t}(C, D) \le \Psi_{f,t}(A, B) \tag{2.3}$$

for each $t \in [0, 1]$, i.e., the functional $\Psi_f(\cdot, \cdot)$ is nondecreasing as a function of interval.

By utilising this lemma we can prove the following result as well:

Theorem 6. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I in $\mathbb{R}$. For all $t \in (0, 1)$, the function T_t is Schur convex on I^2.*

Proof. Let $(A, B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$ with $A \ne B$ and $s \in [0, 1]$. Then

$$\begin{aligned}
&T_t(s(A, B) + (1 - s)(B, A)) \\
&= T_t(sA + (1 - s)B, sB + (1 - s)A) \\
&= \frac{f(sA + (1 - s)B) + f(sB + (1 - s)A)}{2} \\
&- \frac{1}{2} f((1 - t)(sA + (1 - s)B) + t(sB + (1 - s)A)) \\
&- \frac{1}{2} f((1 - t)(sB + (1 - s)A) + t(sA + (1 - s)B)).
\end{aligned}$$

From (2.3) we have for $C,\ D \in [A, B]$

$$\Psi_{f,t}(C, D) \le \Psi_{f,t}(A, B) \text{ and } \Psi_{f,1-t}(C, D) \le \Psi_{f,1-t}(A, B),$$

which, by addition gives that

$$\Psi_{f,t}(C, D) + \Psi_{f,1-t}(C, D) \le \Psi_{f,t}(A, B) + \Psi_{f,1-t}(A, B)$$

namely

$$\begin{aligned}
&(1 - t) f(C) + tf(D) - f((1 - t)C + tD) \\
&+ tf(C) + (1 - t) f(D) - f(tC + (1 - t)D) \\
&\le (1 - t) f(A) + tf(B) - f((1 - t)A + tB) \\
&+ tf(A) + (1 - t) f(B) - f(tA + (1 - t)B),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&f(C) + f(D) - f((1 - t)C + tD) - f(tC + (1 - t)uD) \\
&\le f(A) + f(B) - f((1 - t)A + tB) - f(tA + (1 - t)B)
\end{aligned} \tag{2.4}$$

for all $C,\ D \in [A, B]$.

If we take $C = sA + (1-s)B$ and $D = sB + (1-s)A$, with $s \in [0,1]$ then C, $D \in [A,B]$ and by (2.4) we get

$$\begin{aligned}&f(sA+(1-s)B)+f(sB+(1-s)A)\\&-f((1-t)(sA+(1-s)B)+t(sB+(1-s)A))\\&-f((1-t)(sB+(1-s)A)+t(sA+(1-s)B))\\&\leq f(A)+f(B)-f((1-t)A+tB)-f(tA+(1-t)B).\end{aligned}$$

This inequality is equivalent to

$$T_t(s(A,B)+(1-s)(B,A)) \leq T_t(A,B)$$

for all $(A,B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$ and $s \in [0,1]$. This proves the operator Schur convexity of T_t. □

Remark 1. *Since both M_t and T_t are operator Schur convex when f is operator convex on I it follows that the sum, namely the Jensen's functional*

$$J(A,B) := \frac{f(A)+f(B)}{2} - f\left(\frac{A+B}{2}\right)$$

is also operator Schur convex on I^2.

For a convex function $f : I \to \mathbb{R}$ and $q : [0,1] \to [0,\infty)$ a Lebesgue integrable function we consider the function $T_{\breve{q}} : I^2 \to [0,\infty)$ defined by

$$\begin{aligned}T_{\breve{q}}(x,y) &:= \int_0^1 T_t(x,y)\,q(t)\,dt\\&= \frac{f(x)+f(y)}{2}\int_0^1 q(t)\,dt\\&-\frac{1}{2}\int_0^1 [f((1-t)x+ty)+f((1-t)y+tx)]\,q(t)\,dt\\&= \frac{f(x)+f(y)}{2}\int_0^1 q(t)\,dt - \int_0^1 f((1-t)x+ty)\,\breve{q}(t)\,dt.\end{aligned}$$

Corollary 3. *Let $f : I \to \mathbb{R}$ be an operator convex function on I and $q : [0,1] \to [0,\infty)$ a Lebesgue integrable function on $[0,1]$, then $T_{\breve{q}}$ is operator Schur convex on I^2. In particular, if $p : [0,1] \to [0,\infty)$ is a Lebesgue integrable symmetric function on $[0,1]$, then T_p is operator Schur convex on I^2.*

If we take $p \equiv 1$ and consider the functions

$$M(x,y) := \int_0^1 f((1-t)x+ty)\,dt - f\left(\frac{x+y}{2}\right)$$

and

$$T(x,y) := \frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)y+ty)\,dt$$

then we conclude that M and T are *operator Schur convex functions on* I^2 if f is operator convex on I.

Also, if we consider the symmetric weights $p_1(t) = \left|t-\frac{1}{2}\right|$ and $p_2(t) = t(1-t)$, $t \in [0,1]$, then

$$M_{\left|\cdot-\frac{1}{2}\right|}(x,y) := \int_0^1 f((1-t)x+ty)\left|t-\frac{1}{2}\right|dt - \frac{1}{4}f\left(\frac{x+y}{2}\right)$$

and

$$M_{\cdot(1-\cdot)}(x,y) := \int_0^1 f((1-t)x+ty)\,t(1-t)\,dt - \frac{1}{6}f\left(\frac{A+B}{2}\right)$$

are Schur convex on I^2 if f is convex on I.

The trapezoid functions

$$T_{\left|\cdot-\frac{1}{2}\right|}(x,y) := \frac{f(x)+f(y)}{8} - \int_0^1 f((1-t)x+ty)\left|t-\frac{1}{2}\right|dt$$

and

$$T_{\cdot(1-\cdot)}(x,y) := \frac{f(x)+f(y)}{12} - \int_0^1 f((1-t)x+ty)\,t(1-t)\,dt$$

are also *operator Schur convex on* I^2 if f is operator convex on I.

3. Some Examples

Assume that f is a continuous function on the interval I and $x, y \in I$. Also, let $p : [0,1] \to [0,\infty)$ be a Lebesgue integrable symmetric function on $[0,1]$. If we consider the functions

$$M_p(x,y) := \int_0^1 f((1-t)x+ty)\,p(t)\,dt - f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt$$

and

$$T_p(x,y) := \frac{f(x)+f(y)}{2}\int_0^1 p(t)\,dt - \int_0^1 f((1-t)x+ty)\,p(t)\,dt$$

then

$$M_p(x,x) = T_p(x,x) = 0 \text{ for } x \in I.$$

If $x \neq y$, then by the change of the variable $u = (1-t)x+ty$, we have $du = (y-x)\,dt$, $t = \frac{u-x}{y-x}$, and we can consider the functions of two variables M_p, $T_p : I^2 \to \mathbb{R}$ defined by

$$M_p(x,y) := \begin{cases} \frac{1}{y-x}\int_x^y f(u)\,p\left(\frac{u-x}{y-x}\right)du - f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt, \\ (x,y) \in I^2,\ x \neq y, \\ \\ 0,\ \ (x,y) \in I^2,\ x \neq y \end{cases} \tag{3.1}$$

and

$$T_p(x,y) := \begin{cases} \frac{f(x)+f(y)}{2}\int_0^1 p(t)\,dt - \frac{1}{y-x}\int_x^y f(u)\,p\left(\frac{u-x}{y-x}\right)du, \\ (x,y) \in I^2,\ x \neq y, \\ \\ 0,\ \ (x,y) \in I^2,\ x \neq y. \end{cases} \tag{3.2}$$

In particular, we have the functions $M, T : I^2 \to \mathbb{R}$ introduced in [4] and defined by

$$M(x,y) := \begin{cases} \frac{1}{y-x}\int_x^y f(u)\,du - f\left(\frac{x+y}{2}\right),\ \ (x,y) \in I^2,\ x \neq y, \\ \\ 0,\ \ (x,y) \in I^2,\ x \neq y, \end{cases}$$

and

$$T(x,y) := \begin{cases} \frac{f(x)+f(y)}{2} - \frac{1}{y-x}\int_x^y f(u)\,du,\ \ (x,y) \in I^2,\ x \neq y, \\ \\ 0,\ \ (x,y) \in I^2,\ x \neq y. \end{cases}$$

We can also consider the weighted functions defined on I^2

$$M_{\left|\cdot-\frac{1}{2}\right|}(x,y):=\begin{cases}\frac{1}{(y-x)^2}\int_x^y f(u)\left|u-\frac{x+y}{2}\right|du-\frac{1}{4}f\left(\frac{x+y}{2}\right),\\(x,y)\in I^2,\ x\neq y,\\ \\0,\ \ (x,y)\in I^2,\ x\neq y,\end{cases}$$

$$T_{\left|\cdot-\frac{1}{2}\right|}(x,y):=\begin{cases}\frac{f(x)+f(y)}{8}-\frac{1}{(y-x)^2}\int_x^y f(u)\left|u-\frac{x+y}{2}\right|du,\\(x,y)\in I^2,\ x\neq y,\\ \\0,\ \ (x,y)\in I^2,\ x\neq y,\end{cases}$$

$$M_{\cdot(1-\cdot)}(x,y):=\begin{cases}\frac{1}{(y-x)^3}\int_x^y f(u)(u-x)(y-u)\,du-\frac{1}{6}f\left(\frac{x+y}{2}\right),\\(x,y)\in I^2,\ x\neq y,\\ \\0,\ \ (x,y)\in I^2,\ x\neq y,\end{cases}$$

and

$$T_{\cdot(1-\cdot)}(x,y):=\begin{cases}\frac{f(x)+f(y)}{12}-\frac{1}{(y-x)^3}\int_x^y f(u)(u-x)(y-u)\,du,\\(x,y)\in I^2,\ x\neq y,\\ \\0,\ \ (x,y)\in I^2,\ x\neq y.\end{cases}$$

By utilising Corollary 2 and Corollary 3 we can state the following Schur convexity result:

Proposition 1. *Assume that f is an operator convex function on the interval I and let $p:[0,1]\to[0,\infty)$ be a Lebesgue integrable symmetric function on $[0,1]$. Then the functions M_p and T_p are operator Schur convex on I^2.*

Since the function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\le r\le 2$ or $-1\le r\le 0$ and is operator concave on $(0,\infty)$ if $0\le r\le 1$, hence for $p:[0,1]\to[0,\infty)$ a Lebesgue integrable symmetric function on $[0,1]$,

$$M_{p,r}(x,y):=\begin{cases}\frac{1}{y-x}\int_x^y u^r p\left(\frac{u-x}{y-x}\right)du-\left(\frac{x+y}{2}\right)^r\int_0^1 p(t)\,dt,\\(x,y)\in(0,\infty)\times(0,\infty),\ x\neq y,\\ \\0,\ \ (x,y)\in(0,\infty)\times(0,\infty),\ x\neq y\end{cases}\tag{3.3}$$

and

$$T_{p,r}(x,y):=\begin{cases}\frac{x^r+y^r}{2}\int_0^1 p(t)\,dt-\frac{1}{y-x}\int_x^y u^r p\left(\frac{u-x}{y-x}\right)du\\(x,y)\in(0,\infty)\times(0,\infty),\ x\neq y,\\ \\0,\ \ (x,y)\in(0,\infty)\times(0,\infty),\ x\neq y\end{cases}\tag{3.4}$$

are *operator Schur convex* on $(0,\infty)\times(0,\infty)$ if either $1\le r\le 2$ or $-1\le r\le 0$ and are *operator Schur concave* on $(0,\infty)\times(0,\infty)$ if $0\le r\le 1$.

In particular,

$$M_r(x,y):=\begin{cases}\frac{y^{r+1}-y^{r+1}}{(r+1)(y-x)}-\left(\frac{x+y}{2}\right)^r,\ \ (x,y)\in(0,\infty)\times(0,\infty),\ x\neq y,\\ \\0,\ \ (x,y)\in(0,\infty)\times(0,\infty),\ x\neq y\end{cases}\tag{3.5}$$

and

$$T_r(x,y) := \begin{cases} \frac{x^r+y^r}{2} - \frac{y^{r+1}-y^{r+1}}{(r+1)(y-x)}, \ (x,y) \in (0,\infty)\times(0,\infty), \ x \neq y, \\ 0, \ (x,y) \in (0,\infty)\times(0,\infty), \ x \neq y. \end{cases} \tag{3.6}$$

are *operator Schur convex* on $(0,\infty)\times(0,\infty)$ if either $1 \le r \le 2$ or $-1 < r \le 0$ and are *operator Schur concave* on $(0,\infty)\times(0,\infty)$ if $0 \le r \le 1$.

For $r = -1$, if we put

$$M_{-1}(x,y) := \begin{cases} \frac{\ln y - \ln x}{y-x} - \left(\frac{x+y}{2}\right)^{-1}, \ (x,y) \in (0,\infty)\times(0,\infty), \ x \neq y, \\ 0, \ (x,y) \in (0,\infty)\times(0,\infty), \ x = y, \end{cases} \tag{3.7}$$

and

$$T_{-1}(x,y) := \begin{cases} \frac{x^{-1}+y^{-1}}{2} - \frac{\ln y - \ln x}{y-x}, \ (x,y) \in (0,\infty)\times(0,\infty), \ x \neq y, \\ 0, \ (x,y) \in (0,\infty)\times(0,\infty), \ x \neq y, \end{cases} \tag{3.8}$$

then we conclude that M_{-1} and T_{-1} are *operator Schur convex* on $(0,\infty)\times(0,\infty)$.

The logarithmic function $f(t) = \ln t$ is operator concave on $(0,\infty)$. For $p : [0,1] \to [0,\infty)$ a Lebesgue integrable symmetric function on $[0,1]$,

$$M_{p,\ln}(x,y) := \begin{cases} \frac{1}{y-x}\int_x^y p\left(\frac{u-x}{y-x}\right)\ln u du - \ln\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt, \\ (x,y) \in (0,\infty)\times(0,\infty), \ x \neq y, \\ 0, \ (x,y) \in (0,\infty)\times(0,\infty), \ x \neq y \end{cases} \tag{3.9}$$

and

$$T_{p,\ln}(x,y) := \begin{cases} \frac{\ln x + \ln y}{2}\int_0^1 p(t)\,dt - \frac{1}{y-x}\int_x^y p\left(\frac{u-x}{y-x}\right)\ln u du \\ (x,y) \in (0,\infty)\times(0,\infty), \ x \neq y, \\ 0, \ (x,y) \in (0,\infty)\times(0,\infty), \ x \neq y \end{cases} \tag{3.10}$$

are *operator Schur concave* on $(0,\infty)\times(0,\infty)$.

In particular,

$$M_{\ln}(x,y) := \begin{cases} \frac{y\ln y - x\ln x}{y-x} - 1 - \ln\left(\frac{x+y}{2}\right), \\ (x,y) \in (0,\infty)\times(0,\infty), \ x \neq y, \\ 0, \ (x,y) \in (0,\infty)\times(0,\infty), \ x \neq y \end{cases} \tag{3.11}$$

and

$$T_{\ln}(x,y) := \begin{cases} \frac{\ln x+\ln y}{2} - \frac{y\ln y - x\ln x}{y-x} + 1 \\ (x,y)\in(0,\infty)\times(0,\infty),\ x\neq y, \\ \\ 0,\ \ (x,y)\in(0,\infty)\times(0,\infty),\ x\neq y \end{cases} \tag{3.12}$$

$$= \begin{cases} 1 - \frac{x+y}{2}\frac{\ln y - \ln x}{y-x}, \\ (x,y)\in(0,\infty)\times(0,\infty),\ x\neq y, \\ \\ 0,\ \ (x,y)\in(0,\infty)\times(0,\infty),\ x\neq y \end{cases} \tag{3.13}$$

are *operator Schur concave* on $(0,\infty)\times(0,\infty)$.

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