OPERATOR SCHUR CONVEXITY OF INTEGRAL MEANS

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Abstract. For a Lebesgue integrable function $p:[0,1]\to[0,\infty)$ we consider the function $S_{f,p}$, $M_{f,p}:I\times I\to\mathbb{R}$ defined by

$$S_{f,p}(x,y)=\int_0^1 f(tx+(1-t)y,ty+(1-t)x)\,p(t)\,dt$$

and

$$M_{f,p}(x,y)=\int_0^1 f(tx+(1-t)y,ty+(1-t)x)\,p(t)\,dt-f\left(\frac{x+y}{2},\frac{x+y}{2}\right)\int_0^1 p(t)\,dt,$$

where $f:I\times I\to\mathbb{R}$ is an operator Schur convex function on $I\times I$. In this paper we show among others that $S_{f,p}$ and $M_{f,p}$ preserve the operator Schur convexity of f. We also provide some applications for powers and logarithms.

1. Introduction

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{1.1}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A\leq B$ with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset I$ imply $f(A)\leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [15] and the references therein.

As examples of such functions, we note that $f(t)=t^r$ is operator monotone on $[0,\infty)$ if and only if $0\leq r\leq 1$. The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\leq r\leq 2$ or $-1\leq r\leq 0$ and is operator concave on $(0,\infty)$ if $0\leq r\leq 1$. The logarithmic function $f(t)=\ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t)=-t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t)=e^t$ is neither operator convex nor operator monotone. For recent inequalities for operator convex functions see [1], [3], [6], [7], [8], [10]-[19] and [26]-[30].

Let $I_1,...,I_k$ be intervals from $\mathbb{R}$ and let $f:I_1\times...\times I_k\to\mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A=(A_1,...,A_n)$

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be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i E_i (d\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2] we define

$$f(A) = f(A_1, ..., A_n) = \int_{I_1 \times ... \times I_k} f(\lambda_1, ..., \lambda_1) E_1(d\lambda_1) \otimes ... \otimes E_k(d\lambda_k) \tag{1.2}$$

as a bounded selfadjoint operator on $H_1 \otimes ... \otimes H_k$.

The above function $f : I_1 \times ... \times I_k \to \mathbb{R}$ is said to be operator convex, if the operator inequality

$$f((1-\alpha) A + \alpha B) \le (1-\alpha) f(A) + \alpha f(B) \tag{1.3}$$

for all $\alpha \in [0, 1]$, for any Hilbert spaces $H_1, ..., H_k$ and any k-tuples of of selfadjoint operators $A = (A_1, ..., A_n)$, $B = (B_1, ..., B_n)$ on $H_1 \otimes ... \otimes H_k$ contained in the domain of f. The definition is meaningful since also the spectrum of $\alpha A_i + (1-\alpha) B_i$ is contained in the interval I_i for each $i = 1, ..., k$.

In the following we restrict ourself to the case $k = 1$, $I_1 = I_2 = I$ and $H_1 = H_1 = H$. The operator convexity of $f : I \times I \to \mathbb{R}$ in this case means, for instance,

$$f((1-\alpha) A_1 + \alpha B_1, (1-\alpha) A_2 + \alpha B_2) \le (1-\alpha) f(A_1, A_2) + \alpha f(B_1, B_2) \tag{1.4}$$

or, equivalently,

$$f((1-\alpha)(A_1, A_2) + \alpha (B_1, B_2)) \le (1-\alpha) f(A_1, A_2) + \alpha f(B_1, B_2) \tag{1.5}$$

for all selfadjoint operators A_1, A_2, B_1, B_2 with spectra in I and for all $\alpha \in [0, 1]$.

For I an interval, we consider the set $\mathcal{SA}_I(H)$ of all selfadjoint operators with spectra in I. $\mathcal{SA}_I(H)$ is a convex set in $\mathcal{B}(H)$ since for A, B selfadjoints with $\mathrm{Sp}(A)$, $\mathrm{Sp}(B) \subset I$, $\alpha A + \beta B$ is selfadjoint with $\mathrm{Sp}(\alpha A + \beta B) \subset I$, where α, $\beta \ge 0$ and $\alpha + \beta = 1$.We can introduce the following concept [11]:

Definition 1. *We say that the function $f : I \times I \to \mathbb{R}$ is called operator Schur convex, if f is symmetric, namely $f(x, y) = f(y, x)$ for all x, $y \in I$ and*

$$f(tA + (1-t) B, tB + (1-t) A) \le f(A, B)$$

or, equivalently,

$$f(t(A, B) + (1-t)(B, A)) \le f(A, B)$$

in the operator order, for all $(A, B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$ and $t \in [0, 1]$. The function f is called operator Schur concave if $-f$ is operator Schur convex.

For $(A, B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$, let us define the following auxiliary function $\varphi_{(A,B)} : [0, 1] \to \mathcal{SA}(H \otimes H)$, the set of all selfadjoint operators on $H \otimes H$, by

$$\begin{aligned} \varphi_{f,(A,B)}(t) &= f(t(A, B) + (1-t)(B, A)) \\ &= f(tA + (1-t) B, tB + (1-t) A). \end{aligned} \tag{1.6}$$

A function $f : J \to \mathcal{SA}(K)$ defined of an interval of real numbers J with self adjoint operator values on a Hilbert space K is called *operator monotone increasing* on J if

$$f(t) \le f(s) \text{ in the operator order}$$

for all s, $t \in J$ with $t < s$.

The following characterization of operator Schur convexity holds [11]:

Theorem 1. *Let $f : I \times I \to \mathbb{R}$ be a continuous symmetric function on $I \times I$. Then f is operator Schur convex on $I \times I$ if and only if for all arbitrarily fixed $(A, B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$ the function $\varphi_{f,(A,B)}$ is operator monotone decreasing on $[0, 1/2)$, operator monotone increasing on $(1/2, 1]$, and $\varphi_{f,(A,B)}$ has a global minimum at $1/2$ in the operator order.*

We have the following integral inequality in the operator order [11]:

Theorem 2. *Assume that the function $f : I \times I \to \mathbb{R}$ is operator Schur convex on $I \times I$. Then for any Lebesgue integrable function $p : [0, 1] \to [0, \infty)$ with $\int_0^1 p(t)\,dt = 1$ we have*

$$f\left(\frac{A+B}{2}, \frac{A+B}{2}\right) dt \le \int_0^1 f(tA + (1-t)B, tB + (1-t)A)\,p(t)\,dt \le f(A, B) \tag{1.7}$$

for all $(A, B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$.

In particular, we have

$$f\left(\frac{A+B}{2}, \frac{A+B}{2}\right) \le \int_0^1 f(tA + (1-t)B, tB + (1-t)A)\,dt \le f(A, B) \tag{1.8}$$

for all $(A, B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$.

For a Lebesgue integrable function $p : [0, 1] \to [0, \infty)$ we consider the function $S_{f,p}$, $M_{f,p} : I \times I \to \mathbb{R}$ defined by

$$S_{f,p}(x, y) = \int_0^1 f(tx + (1-t)y, ty + (1-t)x)\,p(t)\,dt$$

and

$$M_{f,p}(x, y) = \int_0^1 f(tx + (1-t)y, ty + (1-t)x)\,p(t)\,dt - f\left(\frac{x+y}{2}, \frac{x+y}{2}\right)\int_0^1 p(t)\,dt,$$

where $f : I \times I \to \mathbb{R}$ is an operator Schur convex function on $I \times I$. In this paper we show among others that $S_{f,p}$ and $M_{f,p}$ preserve the operator Schur convexity of f. We also provide some applications for powers and logarithms.

2. Operator Schur Convexity for Functions of Composite Arguments

Assume that the function $f : I \times I \to \mathbb{R}$ is *Schur convex* on the convex and symmetric set $I \times I \subset \mathbb{R}^2$. For $t \in [0, 1]$, we define the function $S_{f,t} : I \times I \to \mathbb{R}$ defined by

$$S_{f,t}(x, y) := f(t(x, y) + (1-t)(y, x)) = f(tx + (1-t)y, ty + (1-t)x). \tag{2.1}$$

In the case when $t = 0$ or $t = 1$ the definition (2.1) becomes, by the symmetry of f in $I \times I$, that

$$S_{f,0}(x, y) = S_{f,1}(x, y) = f(x, y), \quad (x, y) \in I \times I.$$

We have:

Theorem 3. *Assume that the function $f : I \times I \to \mathbb{R}$ is operator Schur convex on $I \times I$ then $S_{f,t}$ is operator Schur convex on $I \times I$ for all $t \in (0, 1)$.*

Proof. Let $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$ and $s\in[0,1]$, $t\in(0,1)$. Observe that

$$\begin{aligned}&t(sA+(1-s)B,sB+(1-s)A)+(1-t)(sB+(1-s)A,sA+(1-s)B)\\&=t(s(A,B)+(1-s)(B,A))+(1-t)(s(B,A)+(1-s)(A,B))\\&=s[t(A,B)+(1-t)(B,A)]+(1-s)[t(B,A)+(1-t)(A,B)]\\&=s(tA+(1-t)B,tB+(1-t)A)+(1-s)[(tB+(1-t)A,tA+(1-t)B)]\\&=s(C,D)+(1-s)(D,C),\end{aligned}$$

where $C:=tA+(1-t)B$ and $D:=tB+(1-t)A$ for all $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$ and $s,\ t\in[0,1]$.

By Schur convexity of f on $I\times I$ we get

$$f(s(C,D)+(1-s)(D,C))\le f(C,D)$$

for all $s\in[0,1]$.

Therefore

$$\begin{aligned}&S_{f,t}(s(A,B)+(1-s)(B,A))\\&=f[t(sA+(1-s)B,sB+(1-s)A)+(1-t)(sB+(1-s)A,sA+(1-s)B)]\\&\le f(tA+(1-t)B,tB+(1-t)A)=S_{f,t}(A,B)\end{aligned}\tag{2.2}$$

for $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$ and $s,\ t\in[0,1]$.

This proves the operator Schur convexity of $S_{f,t}$ on $I\times I$. □

We define for $t\in[0,1]$, $t\ne\frac{1}{2}$ the function $M_{f,t}$ on $I\times I$ by

$$\begin{aligned}M_{f,t}(x,y)&:=f(t(x,y)+(1-t)(y,x))-f\left(\frac{x+y}{2},\frac{x+y}{2}\right)\\&=f(tx+(1-t)y,ty+(1-t)x)-f\left(\frac{x+y}{2},\frac{x+y}{2}\right)\\&=S_{f,t}(x,y)-f\left(\frac{x+y}{2},\frac{x+y}{2}\right),\end{aligned}$$

where $f:I\times I\to\mathbb{R}$ is operator Schur convex on the convex and symmetric subset $I\times I\subset\mathbb{R}^2$.

We have the following result.

Corollary 1. *Let f be an operator Schur convex function on $I\times I$ and $t\in[0,1]$, $t\ne\frac{1}{2}$. Then the function $M_{f,t}$ is operator Schur convex on $I\times I$.*

Proof. Let $s\in[0,1]$ and $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$. Then

$$\begin{aligned}&M_{f,t}(s(A,B)+(1-s)(B,A))\\&=S_{f,t}(s(A,B)+(1-s)(B,A))\\&-f\left(\frac{sA+(1-s)B+sB+(1-s)A}{2},\frac{sA+(1-s)B+sB+(1-s)A}{2}\right)\\&=M_{f,t}(s(A,B)+(1-s)(B,A))-f\left(\frac{A+B}{2},\frac{A+B}{2}\right)\\&\le S_{f,t}(A,B)-f\left(\frac{A+B}{2},\frac{A+B}{2}\right)=M_{f,t}(A,B),\end{aligned}$$

which proves the operator Schur convexity of $M_{f,t}$ on $I\times I$. □

Assume that the function $f : I \times I \to \mathbb{R}$ is continuous. For $(t,s) \in [0,1]^2$ we consider the function $P_{f,(t,s)} : I \times I \to \mathbb{R}$ defined by

$$\begin{aligned} &P_{f,(t,s)}(x,y) \\ &\qquad := \frac{1}{2}\left[f\left(tx + (1-t)y, sx + (1-s)y\right) + f\left((1-t)x + ty, sy + (1-s)x\right)\right], \end{aligned}$$

where $(x,y) \in I \times I$.

Theorem 4. *Assume that $f : I \times I \to R$ is operator convex on $I \times I$ and $(t,s) \in [0,1]^2$. Then the function $P_{f,(t,s)}$ is operator Schur convex on $I \times I$.*

Proof. Let α, $\beta \geq 0$ with $\alpha + \beta = 1$, $(A,B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$ and consider

$$\begin{aligned} &2P_{(t,s)}\left(\alpha(A,B) + \beta(B,A)\right) \\ &= P_{(t,s)}\left(\alpha A + \beta B, \alpha B + \beta A\right) \\ &= f\left(t(\alpha A + \beta B) + (1-t)(\alpha B + \beta A), s(\alpha A + \beta B) + (1-s)(\alpha B + \beta A)\right) \\ &+ f\left((1-t)(\alpha A + \beta B) + t(\alpha B + \beta A), s(\alpha B + \beta A) + (1-s)(\alpha A + \beta B)\right). \end{aligned}$$

Observe that

$$\begin{aligned} &\left(t(\alpha A + \beta B) + (1-t)(\alpha B + \beta A), s(\alpha A + \beta B) + (1-s)(\alpha B + \beta A)\right) \\ &= \alpha\left(tA + (1-t)B, sA + (1-s)B\right) + \beta\left(tB + (1-t)A, sB + (1-s)A\right) \end{aligned}$$

and

$$\begin{aligned} &\left((1-t)(\alpha A + \beta B) + t(\alpha B + \beta A), s(\alpha B + \beta A) + (1-s)(\alpha A + \beta B)\right) \\ &= \alpha\left((1-t)A + tB, sB + (1-s)A\right) + \beta\left((1-t)B + tA, sA + (1-s)B\right). \end{aligned}$$

Since f is operator convex on $I \times I$, hence

$$\begin{aligned} &f\left[\alpha\left(tA + (1-t)B, sA + (1-s)B\right) + \beta\left(tB + (1-t)A, sB + (1-s)A\right)\right] \\ &\leq \alpha f\left(tA + (1-t)B, sA + (1-s)B\right) + \beta f\left(tB + (1-t)A, sB + (1-s)A\right) \end{aligned}$$

and

$$\begin{aligned} &f\left[\alpha\left((1-t)A + tB, sB + (1-s)A\right) + \beta\left((1-t)B + tA, sA + (1-s)B\right)\right] \\ &\leq \alpha f\left((1-t)A + tB, sB + (1-s)A\right) + \beta f\left((1-t)B + tA, sA + (1-s)B\right). \end{aligned}$$

If we add these two inequalities, we get

$$\begin{aligned} 2P_{(t,s)}\left(\alpha(A,B) + \beta(B,A)\right) &\leq \alpha f\left(tA + (1-t)B, sA + (1-s)B\right) \\ &+ \beta f\left((1-t)B + tA, sA + (1-s)B\right) \\ &+ \beta f\left(tB + (1-t)A, sB + (1-s)A\right) \\ &+ \alpha f\left((1-t)A + tB, sB + (1-s)A\right) \\ &= f\left(tA + (1-t)B, sA + (1-s)B\right) \\ &+ f\left(tB + (1-t)A, sB + (1-s)A\right) = 2P_{(t,s)}(A,B), \end{aligned}$$

which shows that $P_{(t,s)}$ is Schur convex on $I \times I$. $\square$

For $(t,s)\in[0,1]^2$ we also consider the function $Q_{f,(t,s)}:I\times I\to\mathbb{R}$ defined by

$$\begin{aligned}
&Q_{f,(t,s)}(x,y)\\
&:=P_{f,(t,s)}(x,y)-P_{f,(t,s)}\left(\frac{x+y}{2},\frac{x+y}{2}\right)\\
&=\frac{1}{2}\left[f(tx+(1-t)y,sx+(1-s)y)+f((1-t)x+ty,sy+(1-s)x)\right]\\
&-f\left(\frac{x+y}{2},\frac{x+y}{2}\right).
\end{aligned}$$

Corollary 2. *Assume that $f:I\times I\to R$ is operator convex on $I\times I$ and $(t,s)\in[0,1]^2$. Then the function $Q_{(t,s)}$ is operator Schur convex on $I\times I$.*

3. Operator Schur Convexity of Integral Mean

For a Lebesgue integrable function $p:[0,1]\to[0,\infty)$ and an operator Schur convex function $f:I\times I\to\mathbb{R}$ on the convex and symmetric set $I\times I\subset\mathbb{R}^2$ we define the functions $S_{f,p}$ and $M_{f,p}$ on $I\times I$ by

$$\begin{aligned}
S_{f,p}(x,y)&:=\int_0^1 S_{f,t}(x,y)\,p(t)\,dt\\
&=\int_0^1 f(tx+(1-t)y,ty+(1-t)x)\,p(t)\,dt
\end{aligned}$$

and

$$\begin{aligned}
M_{f,p}(x,y)&:=\int_0^1 f(tx+(1-t)y,ty+(1-t)x)\,p(t)\,dt\\
&-f\left(\frac{x+y}{2},\frac{x+y}{2}\right)\int_0^1 p(t)\,dt.
\end{aligned}$$

In particular, if $p\equiv 1$, then we also consider the functions

$$S_f(x,y):=\int_0^1 f(tx+(1-t)y,ty+(1-t)x)\,dt$$

and

$$M_f(x,y):=\int_0^1 f(tx+(1-t)y,ty+(1-t)x)\,dt-f\left(\frac{x+y}{2},\frac{x+y}{2}\right).$$

We have:

Theorem 5. *Assume that the function $f:I\times I\to\mathbb{R}$ is operator Schur convex on $I\times I$ and $p:[0,1]\to[0,\infty)$ is a Lebesgue integrable function on $[0,1]$, then the functions $S_{f,p}$ and $M_{f,p}$ are operator Schur convex on $I\times I$.*

Proof. Let $s\in[0,1]$ and $(A,B)\in\mathcal{SA}_I(H)\times\mathcal{SA}_I(H)$. Then, by the operator Schur convexity of $S_{f,t}$ for $t\in[0,1]$, we have

$$\begin{aligned}
S_{f,p}(s(A,B)+(1-s)(B,A))&=\int_0^1 S_{f,t}(s(A,B)+(1-s)(B,A))\,p(t)\,dt\\
&\le\int_0^1 S_{f,t}(A,B)\,p(t)\,dt=S_{f,p}(A,B),
\end{aligned}$$

which proves the operator Schur convexity of $S_{f,p}$.

The proof for $M_{f,p}$ is similar. □

Corollary 3. *Assume that the function $f : I \times I \to \mathbb{R}$ is operator Schur convex on $I \times I$, then the functions S_f and M_f are operator Schur convex on $I \times I$.*

We also have the following double integral inequalities:

Corollary 4. *Assume that the function $f : I \times I \to \mathbb{R}$ is operator Schur convex on the convex and symmetric set $I \times I \subset \mathbb{R}^2$. Then for any Lebesgue integrable functions w, $p : [0,1] \to [0,\infty)$ we have*

$$\begin{aligned}
&f\left(\frac{A+B}{2}, \frac{A+B}{2}\right) \int_0^1 p(t)\,dt \int_0^1 w(s)\,ds \\
&\leq \int_0^1 \int_0^1 f\left[t(sA+(1-s)B)+(1-t)(sB+(1-s)A),\right. \\
&\left. t(sB+(1-s)A)+(1-t)(sA+(1-s)B)\right] p(t)\,w(s)\,dtds \\
&\leq \int_0^1 f(tA+(1-t)B, tB+(1-t)A)\,p(t)\,dt \int_0^1 w(s)\,ds \\
&\left(\leq f(A,B)\int_0^1 p(t)\,dt \int_0^1 w(s)\,ds\right)
\end{aligned} \tag{3.1}$$

for all $(A,B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$.

The proof follows by Theorem **??** applied for the function $S_{f,p}$. This is a refinement of the inequality (1.7) from Introduction.

For p, $w \equiv 1$ we get for $(A,B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$ that

$$\begin{aligned}
&f\left(\frac{A+B}{2}, \frac{A+B}{2}\right) \\
&\leq \int_0^1 \int_0^1 f\left[t(sA+(1-s)B)+(1-t)(sB+(1-s)A),\right. \\
&\left. t(sB+(1-s)A)+(1-t)(sA+(1-s)B)\right] dtds \\
&\leq \int_0^1 f(tA+(1-t)B, tB+(1-t)A)\,dt \ \ (\leq f(A,B)),
\end{aligned} \tag{3.2}$$

where $f : I \times I \to \mathbb{R}$ is *operator Schur convex* on the convex and symmetric set $I \times I \subset \mathbb{R}^2$. This is a refinement of the inequality (1.8) from Introduction.

Consider the two variable weight $W : [0,1]^2 \to [0,\infty)$ that is Lebesgue integrable on $[0,1]^2$ and define

$$\begin{aligned}
P_{f,W}(x,y) &:= \int_0^1 \int_0^1 P_{f,(t,s)}(x,y)\,W(t,s)\,dtds \\
&= \frac{1}{2}\int_0^1 \int_0^1 f(tx+(1-t)y, sx+(1-s)y)\,W(t,s)\,dtds \\
&+ \frac{1}{2}\int_0^1 \int_0^1 f((1-t)x+ty, sy+(1-s)x)\,W(t,s)\,dtds.
\end{aligned}$$

If W is symmetric on $[0,1]^2$ in the sense that $W(t,s) = W(s,t)$ for all $(t,s) \in [0,1]^2$, then

$$P_{f,W}(x,y) = \int_0^1 \int_0^1 f(tx+(1-t)y, sx+(1-s)y)\,W(t,s)\,dtds.$$

In particular, if $w : [0,1] \to [0,\infty)$ is Lebesgue integrable on $[0,1]$, then by taking $W(t,s) = w(t)w(s)$, $(t,s) \in [0,1]^2$ we can also consider the function

$$P_{f,w}(x,y) = \int_0^1 \int_0^1 f(tx + (1-t)y, sx + (1-s)y) w(t) w(s) \, dtds$$

and the unweighted function

$$P_f(x,y) = \int_0^1 \int_0^1 f(tx + (1-t)y, sx + (1-s)y) \, dtds.$$

In a similar way, we can consider

$$Q_{f,W}(x,y) := P_{f,W}(x,y) - f\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \int_0^1 \int_0^1 W(t,s) \, dtds,$$

$$Q_{f,w}(x,y) := P_{f,w}(x,y) - f\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \left(\int_0^1 w(t) \, dt\right)^2,$$

and

$$Q_f(x,y) := P_f(x,y) - f\left(\frac{x+y}{2}, \frac{x+y}{2}\right).$$

Theorem 6. *Assume that the function $f : I \times I \to \mathbb{R}$ is operator Schur convex on $I \times I$ and $W : [0,1]^2 \to [0,\infty)$ is Lebesgue integrable on $[0,1]^2$, then $P_{f,W}$ and $Q_{f,W}$ are operator Schur convex on $I \times I$.*

Proof. Let $\alpha \in [0,1]$ and $(A,B) \in \mathcal{SA}_I(H) \times \mathcal{SA}_I(H)$. Then, by the operator Schur convexity of $P_{f,(t,s)}$ for $(t,s) \in [0,1]^2$, we have

$$\begin{aligned} & P_{f,W}(\alpha(A,B) + (1-\alpha)(B,A)) \\ & = \int_0^1 \int_0^1 P_{f,(t,s)}(\alpha(A,B) + (1-\alpha)(B,A)) W(t,s) \, dtds \\ & \leq \int_0^1 \int_0^1 P_{f,(t,s)}(A,B) W(t,s) \, dtds = P_{f,W}(A,B), \end{aligned}$$

which proves the operator Schur convexity of $P_{f,W}$.

The operator Schur convexity of $Q_{f,W}$ goes in a similar way. □

Corollary 5. *Assume that $f : I \times I \to \mathbb{R}$ is operator convex on $I \times I$ and $w : [0,1] \to [0,\infty)$ is Lebesgue integrable on $[0,1]$, then $P_{f,w}$ and $Q_{f,w}$ are operator Schur convex on $I \times I$. In particular, P_f and Q_f are operator Schur convex on $I \times I$.*

4. Some Examples

For a Lebesgue integrable function $p : [0,1] \to [0,\infty)$ and an operator Schur convex function $f : I^2 \to \mathbb{R}$ where I is an interval of real numbers, by changing the variable

$$u = (1-t)x + ty, \ t \in [0,1] \text{ with } (x,y) \in I^2 \text{ and } x \neq y$$

we can express the functions $S_{f,p}$ and $M_{f,p}$ on I^2 by

$$\begin{aligned} S_{f,p}(x,y) &= \int_0^1 f(tx + (1-t)y, ty + (1-t)x) p(t) \, dt \qquad (4.1) \\ &= \frac{1}{y-x} \int_x^y f(u, x+y-u) p\left(\frac{u-x}{y-x}\right) du \end{aligned}$$

and

$$M_{f,p}(x,y)=\int_0^1 f(tx+(1-t)y,ty+(1-t)x)\,p(t)\,dt -f\left(\frac{x+y}{2},\frac{x+y}{2}\right)\int_0^1 p(t)\,dt = \frac{1}{y-x}\int_x^y f(u,x+y-u)\,p\left(\frac{u-x}{y-x}\right)du -f\left(\frac{x+y}{2},\frac{x+y}{2}\right)\int_0^1 p(t)\,dt. \tag{4.2}$$

For $(x,y)\in I^2$ with $x=y$ we have

$$S_{f,p}(x,x)=f(x,x)\int_0^1 p(t)\,dt \text{ and } M_{f,p}(x,x)=0. \tag{4.3}$$

In particular, if $p\equiv 1$, then we also consider the functions

$$S_f(x,y):=\begin{cases}\frac{1}{y-x}\int_x^y f(u,x+y-u)\,du \text{ for } (x,y)\in I^2 \text{ with } x\neq y,\\ \\ f(x,x) \text{ for } (x,y)\in I^2 \text{ with } x=y\end{cases} \tag{4.4}$$

and

$$M_f(x,y)=\begin{cases}\frac{1}{y-x}\int_x^y f(u,x+y-u)\,du-f\left(\frac{x+y}{2},\frac{x+y}{2}\right)\\ \text{for } (x,y)\in I^2 \text{ with } x\neq y,\\ \\ 0 \text{ for } (x,y)\in I^2 \text{ with } x=y.\end{cases} \tag{4.5}$$

Proposition 1. *Assume that $f:I^2\to\mathbb{R}$ is operator Schur convex on I^2 and $p:[0,1]\to[0,\infty)$ is Lebesgue integrable on $[0,1]$, then $S_{f,p}$ and $M_{f,p}$ defined by (4.1)-(4.3) are operator Schur convex on I^2. In particular, the functions S_f and M_f defined by (4.4) and (4.5) are operator Schur convex on I^2.*

If $w:[0,1]\to[0,\infty)$ is Lebesgue integrable on $[0,1]$ and $f:I^2\to\mathbb{R}$ is convex on I^2, then by changing the variables $ty+(1-t)x=u$ and $sy+(1-s)x=v$ and we can also consider the function

$$P_{f,w}(x,y):=\frac{1}{(y-x)^2}\int_x^y\int_x^y f(u,v)\,w\left(\frac{u-x}{y-x}\right)w\left(\frac{v-x}{y-x}\right)dudv \tag{4.6}$$

if $(x,y)\in I^2$ with $x\neq y$ and

$$P_{f,w}(x,x):=f(x,x)\left(\int_0^1 w(t)\,dt\right)^2. \tag{4.7}$$

We also can consider

$$Q_{f,w}(x,y):=\frac{1}{(y-x)^2}\int_x^y\int_x^y f(u,v)\,w\left(\frac{u-x}{y-x}\right)w\left(\frac{v-x}{y-x}\right)dudv -f\left(\frac{x+y}{2},\frac{x+y}{2}\right)\left(\int_0^1 w(t)\,dt\right)^2 \tag{4.8}$$

if $(x,y)\in I^2$ with $x\neq y$ and

$$Q_{f,w}(x,x):=0. \tag{4.9}$$

In particular, we have

$$P_f(x,y) := \begin{cases} \frac{1}{(y-x)^2}\int_x^y \int_x^y f(u,v)\,dudv \text{ if } (x,y)\in I^2 \text{ with } x\neq y, \\ \\ f(x,x) \text{ if } (x,y)\in I^2 \text{ with } x\neq y \end{cases} \tag{4.10}$$

and

$$Q_f(x,y) := \begin{cases} \frac{1}{(y-x)^2}\int_x^y \int_x^y f(u,v)\,dudv - f\left(\frac{x+y}{2},\frac{x+y}{2}\right) \\ \text{if } (x,y)\in I^2 \text{ with } x\neq y, \\ \\ 0 \text{ if } (x,y)\in I^2 \text{ with } x\neq y. \end{cases} \tag{4.11}$$

Proposition 2. *Assume that $f : I^2 \to \mathbb{R}$ is operator convex on I^2 and $w : [0,1] \to [0,\infty)$ is Lebesgue integrable on $[0,1]$, then $P_{f,w}$ and $Q_{f,w}$ defined by (4.6)-(4.9) are operator Schur convex on I^2. In particular, the functions S_f and M_f defined by (4.10) and (4.11) are operator Schur convex on I^2.*

In the recent paper [11], we gave several examples of operator Schur convex and concave functions as follows.

The two variables function

$$f_r(x,y) := \begin{cases} \frac{y^{r+1}-y^{r+1}}{(r+1)(y-x)}, & (x,y)\in(0,\infty)\times(0,\infty),\ x\neq y, \\ \\ x^r, & (x,y)\in(0,\infty)\times(0,\infty),\ x=y. \end{cases} \tag{4.12}$$

is *operator Schur convex* on $(0,\infty)\times(0,\infty)$ if either $1\le r\le 2$ or $-1<r\le 0$ and is *operator Schur concave* on $(0,\infty)\times(0,\infty)$ if $0\le r\le 1$.

For $r=-1$, if we put

$$f_{-1}(x,y) := \begin{cases} \frac{\ln y-\ln x}{y-x}, & (x,y)\in(0,\infty)\times(0,\infty),\ x\neq y, \\ \\ x^{-1}, & (x,y)\in(0,\infty)\times(0,\infty),\ x=y, \end{cases} \tag{4.13}$$

then we conclude that F_{-1} is *operator Schur convex* on $(0,\infty)\times(0,\infty)$.

Since $f(t)=\ln t$, $t\in(0,\infty)$ is operator concave, then

$$f_{\ln}(x,y) := \begin{cases} \frac{y\ln y-x\ln x}{y-x}-1, & (x,y)\in(0,\infty)\times(0,\infty),\ x\neq y, \\ \\ \ln x, & (x,y)\in(0,\infty)\times(0,\infty),\ x=y, \end{cases} \tag{4.14}$$

is $f_{\ln}$ is *operator Schur concave* on $(0,\infty)\times(0,\infty)$.

If we replace the function f in the general examples above by f_r, f_{-1} and $f_{\ln}$ we have more particular power and logarithmic examples of operator Schur convex functions. The details are omitted.

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