REVERSES OF OPERATOR HERMITE-HADAMARD INEQUALITIES

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Abstract. Let $f$ be an operator convex function on $I$ and $A, B \in \mathcal{SA}_I(H)$, the convex set of selfadjoint operators with spectra in $I$. If $A \neq B$ and $f$, as an operator function, is Gâteaux differentiable on $[A, B] := \{(1-t)A + tB \mid t \in [0,1]\}$, then

$$0 \leq \int_0^1 f((1-t)A + tB) \, dt - f\left(\frac{A + B}{2}\right)$$

and

$$0 \leq \frac{f(A) + f(B)}{2} - \int_0^1 f((1-t)A + tB) \, dt$$

and

$$0 \leq \frac{f(A) + f(B)}{2} - \int_0^1 f((1-t)A + tB) \, dt$$

The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0,1)$. The entropy function $f(t) = t \ln t$ is operator concave on $(0,1)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

1. Introduction

A real valued continuous function $f$ on an interval $I$ is said to be operator convex (operator concave) on $I$ if

\begin{equation}
 f((1-\lambda)A + \lambda B) \leq (\geq) (1-\lambda)f(A) + \lambda f(B)
\end{equation}

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator $A$ and $B$ on a Hilbert space $H$ whose spectra are contained in $I$. Notice that a function $f$ is operator concave if $-f$ is operator convex.

A real valued continuous function $f$ on an interval $I$ is said to be operator monotone if it is monotone with respect to the operator order, i.e., $A \leq B$ with $\text{Sp}(A), \text{Sp}(B) \subseteq I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [7] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0,\infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0,\infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0,\infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0, \infty)$. The entropy function $f(t) = -t \ln t$ is operator concave on $(0, \infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

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In [5] we obtained among others the following Hermite-Hadamard type inequalities for operator convex functions \( f : I \rightarrow \mathbb{R} \)

\[
(1.2) \quad f \left( \frac{A + B}{2} \right) \leq \int_0^1 f \left( (1-s)A + sB \right) ds \leq \frac{f(A) + f(B)}{2},
\]

where \( A, B \) are selfadjoint operators with spectra included in \( I \).

For recent inequalities for operator convex functions see [1]-[6] and [8]-[17].

Motivated by the above results, in this paper we show among others that if \( A \neq B \) and \( f \) is Gâteaux differentiable on \( [A, B] := \{(1-t)A + tB \mid t \in [0,1]\} \), then

\[
0 \leq \int_0^1 f \left( (1-t)A + tB \right) dt - f \left( \frac{A + B}{2} \right) \leq \frac{1}{8} \left| \nabla f_B (B - A) - \nabla f_A (B - A) \right|
\]

and

\[
0 \leq \frac{f(A) + f(B)}{2} - \int_0^1 f \left( (1-t)A + tB \right) dt \leq \frac{1}{8} \left| \nabla f_B (B - A) - \nabla f_A (B - A) \right|.
\]

Two particular examples of interest for \( f(x) = -\ln x \) and \( f(x) = x^{-1} \) are also given.

2. Some Preliminary Facts

Let \( f \) be an operator convex function on \( I \). For \( (A, B) \in \mathcal{S}A_I (H) \), the class of all selfadjoint operators with spectra in \( I \), we consider the auxiliary function \( \varphi_{(A,B)} : [0,1] \rightarrow \mathcal{S}A_I (H) \) defined by

\[
(2.1) \quad \varphi_{(A,B)} (t) := f \left( (1-t)A + tB \right).
\]

For \( x \in H \) we can also consider the auxiliary function \( \varphi_{(A,B);x} : [0,1] \rightarrow \mathbb{R} \) defined by

\[
(2.2) \quad \varphi_{(A,B);x} (t) := \left\langle \varphi_{(A,B)} (t) x, x \right\rangle = \left( f \left( (1-t)A + tB \right) x \right) x.
\]

We have the following basic fact:

**Lemma 1.** Let \( f \) be an operator convex function on \( I \). For any \( (A, B) \in \mathcal{S}A_I (H) \), \( \varphi_{(A,B)} \) is well defined and convex in the operator order. For any \( (A, B) \in \mathcal{S}A_I (H) \) and \( x \in H \) the function \( \varphi_{(A,B);x} \) is convex in the usual sense on \( [0,1] \).

**Proof.** If \( (A, B) \in \mathcal{S}A_I (H) \) and \( t \in [0,1] \) the convex combination \( (1-t)A + tB \) is a selfadjoint operator with the spectrum in \( I \) showing that \( \mathcal{S}A_I (H) \) in the Banach algebra \( B(H) \) of all bounded linear operators on \( H \). By the continuous functional calculus of selfadjoint operator we also conclude that \( f \left( (1-t)A + tB \right) \) is a selfadjoint operator with spectrum in \( I \).
Let \((A, B) \in \mathcal{SA}_I(H)\) and \(t_1, t_2 \in [0, 1]\). If \(\alpha, \beta > 0\) with \(\alpha + \beta = 1\), then
\[
\varphi_{(A,B)}(\alpha t_1 + \beta t_2) := f((1 - \alpha t_1 - \beta t_2) A + (\alpha t_1 + \beta t_2) B)
\]
\[
= f((\alpha + \beta - \alpha t_1 - \beta t_2) A + (\alpha t_1 + \beta t_2) B)
\]
\[
= f(\alpha [(1 - t_1) A + t_1 B] + \beta [(1 - t_2) A + t_2 B])
\]
\[
\leq \alpha f((1 - t_1) A + t_1 B) + \beta f((1 - t_2) A + t_2 B)
\]
\[
= \alpha \varphi_{(A,B)}(t_1) + \beta \varphi_{(A,B)}(t_2),
\]
which proves the convexity of \(\varphi_{(A,B)}\) in the operator order. Let \((A, B) \in \mathcal{SA}_I(H)\) and \(x \in H\). If \(t_1, t_2 \in [0, 1]\) and \(\alpha, \beta > 0\) with \(\alpha + \beta = 1\), then
\[
\varphi_{(A,B);x}(\alpha t_1 + \beta t_2) = \left\langle \varphi_{(A,B)}(\alpha t_1 + \beta t_2), x \right\rangle
\]
\[
\leq \left\langle \alpha \varphi_{(A,B)}(t_1), x \right\rangle + \beta \left\langle \varphi_{(A,B)}(t_2), x \right\rangle
\]
\[
= \alpha \varphi_{(A,B);x}(t_1) + \beta \varphi_{(A,B);x}(t_2),
\]
which proves the convexity of \(\varphi_{(A,B);x}\) on \([0, 1]\).

A continuous function \(g : \mathcal{SA}_I(H) \to \mathcal{B}(H)\) is said to be Gâteaux differentiable in \(A \in \mathcal{SA}_I(H)\) along the direction \(B \in \mathcal{B}(H)\) if the following limit exists in the strong topology of \(\mathcal{B}(H)\)
\[
(2.3) \quad \nabla g_A(B) := \lim_{s \to 0} \frac{g(A + sB) - g(A)}{s} \in \mathcal{B}(H).
\]
If the limit (2.3) exists for all \(B \in \mathcal{B}(H)\), then we say that \(f\) is Gâteaux differentiable in \(A\) and we can write \(g \in \mathcal{G}(A)\). If this is true for any \(A\) in an open set \(S\) from \(\mathcal{SA}_I(H)\) we write that \(g \in \mathcal{G}(S)\).

If \(g\) is a continuous function on \(I\), by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators \(A, B \in \mathcal{SA}_I(H)\) we consider the segment of selfadjoint operators
\[
[A, B] := \{(1 - t) A + tB \mid t \in [0, 1]\}.
\]
We observe that \(A, B \in [A, B]\) and \([A, B] \subset \mathcal{SA}_I(H)\).

**Lemma 2.** Let \(f\) be an operator convex function on \(I\) and \((A, B) \in \mathcal{SA}_I(H)\), with \(A \neq B\). If \(f \in \mathcal{G}([A, B])\), then the auxiliary function \(\varphi_{(A,B)}\) is differentiable on \((0, 1)\) and
\[
(2.4) \quad \varphi'_{(A,B)}(t) = \nabla f_{(1-t)A+tB}(B - A).
\]
Also we have for the lateral derivative that
\[
(2.5) \quad \varphi'_{(A,B)}(0+) = \nabla f_A(B - A)
\]
and
\[
(2.6) \quad \varphi'_{(A,B)}(1-) = \nabla f_B(B - A).
\]
Proof. Let $t \in (0, 1)$ and $h \neq 0$ small enough such that $t + h \in (0, 1)$. Then
\begin{equation}
\frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h} = \frac{f((1-t-t)A + (t+h)B) - f((1-t)A + tB)}{h} = \frac{f((1-t)A + tB + h(B-A)) - f((1-t)A + tB)}{h}.
\end{equation}

Since $f \in G([A,B])$, hence by taking the limit over $h \to 0$ in (2.7) we get
\begin{align*}
\varphi'_{(A,B)}(t) &= \lim_{h \to 0} \frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h} \\
&= \lim_{h \to 0} \frac{f((1-t)A + tB + h(B-A)) - f((1-t)A + tB)}{h} \\
&= \nabla g_{(1-t)A+tB}(B-A),
\end{align*}
which proves (2.8).

Also, we have
\begin{align*}
\varphi'_{(A,B)}(0+) &= \lim_{h \to 0^+} \frac{\varphi_{(A,B)}(h) - \varphi_{(A,B)}(0)}{h} \\
&= \lim_{h \to 0^+} \frac{f((1-h)A + hB) - f(A)}{h} \\
&= \lim_{h \to 0^+} \frac{f(A + h(B-A)) - f(A)}{h} \\
&= \nabla f_A(B-A)
\end{align*}
since $f$ is assumed to be Gâteaux differentiable in $A$. This proves (2.5).

The equality (2.6) follows in a similar way.

\textbf{Lemma 3.} Let $f$ be an operator convex function on $I$ and $(A,B) \in SA_I(H)$, with $A \neq B$. If $f \in G([A,B])$, then for $0 < t_1 < t_2 < 1$ we have
\begin{equation}
\nabla g_{(1-t_1)A+t_1B}(B-A) \leq \nabla g_{(1-t_2)A+t_2B}(B-A)
\end{equation}
in the operator order.

We also have
\begin{equation}
\nabla f_A(B-A) \leq \nabla g_{(1-t_1)A+t_1B}(B-A)
\end{equation}
and
\begin{equation}
\nabla g_{(1-t_2)A+t_2B}(B-A) \leq \nabla f_B(B-A).
\end{equation}

\textbf{Proof.} Let $x \in H$. The auxiliary function $\varphi_{(A,B);x}$ is convex in the usual sense on $[0,1]$ and differentiable on $(0,1)$ and for $t \in (0,1)$
\begin{align*}
\varphi'_{(A,B);x}(t) &= \lim_{h \to 0} \frac{\varphi_{(A,B);x}(t+h) - \varphi_{(A,B);x}(t)}{h} \\
&= \lim_{h \to 0} \left\langle \frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h} , x \right\rangle \\
&= \left\langle \lim_{h \to 0} \frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h} , x \right\rangle \\
&= \left\langle \nabla g_{(1-t)A+tB}(B-A)x , x \right\rangle.
\end{align*}
Since for $0 < t_1 < t_2 < 1$ we have by the gradient inequality for scalar convex functions that
\[
\varphi'_{(A,B),x}(t_1) \leq \varphi'_{(A,B),x}(t_2)
\]
then we get
\[
(2.11) \quad \langle \nabla g_{(1-t_1)A+t_1B} (B-A) x,x \rangle \leq \langle \nabla g_{(1-t_2)A+t_2B} (B-A) x,x \rangle
\]
for all $x \in H$, which is equivalent to the inequality (2.8) in the operator order.

Let $0 < t_1 < 1$. By the gradient inequality for scalar convex functions we also have
\[
\varphi'_{(A,B),x}(0+) \leq \varphi'_{(A,B),x}(t_1),
\]
which, as above implies that
\[
\langle \nabla f_{A} (B-A) x,x \rangle \leq \langle \nabla g_{(1-t_1)A+t_1B} (B-A) x,x \rangle
\]
for all $x \in H$, that is equivalent to the operator inequality (2.9).

The inequality (2.10) follows in a similar way.

\[\square\]

**Corollary 1.** Let $f$ be an operator convex function on $I$ and $(A, B) \in \mathcal{S}A_{f} (H)$, with $A \neq B$. If $f \in \mathcal{G} ([A, B])$, then for all $t \in (0, 1)$ we have
\[
(2.12) \quad \nabla f_{A} (B-A) \leq \nabla f_{(1-t)A+tB} (B-A) \leq \nabla f_{B} (B-A).
\]

### 3. Reverses of Operator Hermite-Hadamard Inequalities

It is well known that, if $E$ is a Banach space and $f : [0, 1] \to E$ is a continuous function, then $f$ is Bochner integrable, and its Bochner integral coincides with its Riemann integral. We denote this integral as usual by $\int_{0}^{1} f (t) \, dt$.

We have the following reverse of the first operator Hermite-Hadamard inequality:

**Theorem 1.** Let $f$ be an operator convex function on $I$ and $A, B \in \mathcal{S}A_{f} (H)$, with $A \neq B$. If $f \in \mathcal{G} ([A, B])$, then
\[
(3.1) \quad 0 \leq \int_{0}^{1} f ((1-t) A + tB) \, dt - f \left( \frac{A + B}{2} \right)
\]
\[
\leq \frac{1}{8} [\nabla f_{B} (B-A) - \nabla f_{A} (B-A)].
\]

**Proof.** Using integration by parts formula for the Bochner integral, we have
\[
(3.2) \quad \int_{0}^{1/2} t \varphi'_{(A,B)} (t) \, dt = \frac{1}{2} \varphi_{(A,B)} \left( \frac{1}{2} \right) - \int_{0}^{1/2} \varphi_{(A,B)} (t) \, dt
\]
\[
= \frac{1}{2} f \left( \frac{A + B}{2} \right) - \int_{0}^{1/2} f ((1-t) A + tB) \, dt
\]
and
\[
(3.3) \quad \int_{1/2}^{1} (t-1) \varphi'_{(A,B)} (t) \, dt = \frac{1}{2} \varphi_{(A,B)} \left( \frac{1}{2} \right) - \int_{1/2}^{1} f ((1-t) A + tB) \, dt
\]
\[
= \frac{1}{2} f \left( \frac{A + B}{2} \right) - \int_{1/2}^{1} f ((1-t) A + tB) \, dt.
\]
If we add these two equalities, we get the following identity of interest

\[
\int_0^1 f \left( (1-t)A + tB \right) dt - f \left( \frac{A+B}{2} \right) = \int_{1/2}^1 (1-t) \varphi'_{(A,B)} (t) dt - \int_0^{1/2} t \varphi'_{(A,B)} (t) dt.
\]

From Lemma 3 we have

\[
\varphi'_{(A,B)} (1/2) \leq \varphi'_{(A,B)} (t) \leq \varphi'_{(A,B)} (1-) = \nabla f_B (B - A), \; t \in [1/2, 1)
\]

and

\[
\nabla f_A (B - A) = \varphi'_{(A,B)} (0+) \leq \varphi'_{(A,B)} (t) \leq \varphi'_{(A,B)} (1/2), \; t \in (0, 1/2],
\]

This implies that

\[
(1-t) \varphi'_{(A,B)} (1/2) \leq (1-t) \varphi'_{(A,B)} (t) \leq (1-t) \nabla f_B (B - A)
\]

for \( t \in [1/2, 1) \) and

\[
-t \varphi'_{(A,B)} (1/2) \leq -t \varphi'_{(A,B)} (t) \leq -t \nabla f_A (B - A)
\]

for \( t \in (0, 1/2] \).

By integrating these inequalities on the corresponding intervals, we get

\[
\frac{1}{8} \varphi'_{(A,B)} (1/2) \leq \int_{1/2}^1 (1-t) \varphi'_{(A,B)} (t) dt \leq \frac{1}{8} \nabla f_B (B - A)
\]

and

\[
-\frac{1}{8} \varphi'_{(A,B)} (1/2) \leq - \int_0^{1/2} t \varphi'_{(A,B)} (t) dt \leq - \frac{1}{8} \nabla f_A (B - A).
\]

By addition, we deduce that

\[
0 \leq \int_{1/2}^1 (1-t) \varphi'_{(A,B)} (t) dt - \int_0^{1/2} t \varphi'_{(A,B)} (t) dt \leq \frac{1}{8} [\nabla f_B (B - A) - \nabla f_A (B - A)]
\]

and by the identity (3.4) we get (3.1).

We have the following reverse of the second operator Hermite-Hadamard inequality:

**Theorem 2.** Let \( f \) be an operator convex function on \( I \) and \( A, B \in \mathcal{S}_A I (H) \), with \( A \neq B \). If \( f \in \mathcal{G} ([A, B]) \), then

\[
0 \leq \frac{f(A) + f(B)}{2} - \int_0^1 f ((1-t)A + tB) dt \leq \frac{1}{8} [\nabla f_B (B - A) - \nabla f_A (B - A)].
\]

\[
(3.7)
\]
Proof. Using integration by parts formula for the Bochner integral, we have

\begin{equation}
\int_0^1 \left( t - \frac{1}{2} \right) \varphi'_{(A,B)} (t) \, dt = \left( t - \frac{1}{2} \right) \varphi_{(A,B)} (t) \bigg|_0^1 - \int_0^1 \varphi_{(A,B)} (t) \, dt \\
= \frac{\varphi_{(A,B)} (1) + \varphi_{(A,B)} (0)}{2} - \int_0^1 \varphi_{(A,B)} (t) \, dt \\
= \frac{f (B) + f (A)}{2} - \int_0^1 f ((1 - t) A + tB) \, dt.
\end{equation}

Observe that

\begin{equation}
\int_0^1 \left( t - \frac{1}{2} \right) \varphi'_{(A,B)} (t) \, dt \\
= \int_{1/2}^1 \left( t - \frac{1}{2} \right) \varphi'_{(A,B)} (t) \, dt - \int_0^{1/2} \left( \frac{1}{2} - t \right) \varphi'_{(A,B)} (t) \, dt.
\end{equation}

Therefore, we have the following identity of interest

\begin{align*}
\frac{f (B) + f (A)}{2} - \int_0^1 f ((1 - t) A + tB) \, dt \\
= \int_{1/2}^1 \left( t - \frac{1}{2} \right) \varphi'_{(A,B)} (t) \, dt - \int_0^{1/2} \left( \frac{1}{2} - t \right) \varphi'_{(A,B)} (t) \, dt.
\end{align*}

From the inequality (3.5) we obtain

\begin{align*}
\left( t - \frac{1}{2} \right) \varphi'_{(A,B)} (1/2) &\leq \left( t - \frac{1}{2} \right) \varphi_{(A,B)} (t) \\
&\leq \left( t - \frac{1}{2} \right) \nabla f_B (B - A), \ t \in [1/2,1)
\end{align*}

and from (3.6)

\begin{align*}
\left( \frac{1}{2} - t \right) \nabla f_A (B - A) &\leq \left( \frac{1}{2} - t \right) \varphi'_{(A,B)} (t) \\
&\leq \left( \frac{1}{2} - t \right) \varphi_{(A,B)} (1/2), \ t \in (0,1/2],
\end{align*}

namely

\begin{align*}
- \left( \frac{1}{2} - t \right) \varphi'_{(A,B)} (1/2) &\leq - \left( \frac{1}{2} - t \right) \varphi_{(A,B)} (t) \\
&\leq - \left( \frac{1}{2} - t \right) \nabla f_A (B - A), \ t \in (0,1/2].
\end{align*}

Integrating these inequalities on the corresponding intervals, we get

\begin{align*}
\frac{1}{8} \varphi_{(A,B)} (1/2) &\leq \int_{1/2}^1 \left( t - \frac{1}{2} \right) \varphi_{(A,B)} (t) \, dt \leq \frac{1}{8} \nabla f_B (B - A),
\end{align*}

and

\begin{align*}
- \frac{1}{8} \varphi_{(A,B)} (1/2) &\leq - \int_0^{1/2} \left( \frac{1}{2} - t \right) \varphi_{(A,B)} (t) \, dt \leq - \frac{1}{8} \nabla f_A (B - A).
\end{align*}
If we add these two inequalities, we obtain
\begin{align*}
0 & \leq \int_{1/2}^{1} \left( t - \frac{1}{2} \right) \varphi'_{(A,B)}(t) \, dt - \int_{0}^{1/2} \left( \frac{1}{2} - t \right) \varphi'_{(A,B)}(t) \, dt \\
& \leq \frac{1}{8} \left[ \nabla f_B (B - A) - \nabla f_A (B - A) \right],
\end{align*}
which, by the use of identity (3.9) produces the desired result (3.7).

\textbf{Remark 1.} It is well known that, if \( h \) is a \( C^1 \)-function defined on an open interval, then the operator function \( h(X) \) is Fréchet differentiable and the derivative \( Dh(A)(B) \) equals the Gâteaux derivative \( \nabla f_A (B) \). So for operator convex functions \( f \) that are of class \( C^1 \) on \( I \) we have the inequalities
\begin{align*}
0 & \leq \int_{0}^{1} f ((1 - t) A + tB) \, dt - f \left( \frac{A + B}{2} \right) \\
& \leq \frac{1}{8} \left[ Df (B) (B - A) - Df (A) (B - A) \right]
\end{align*}
and
\begin{align*}
0 & \leq \frac{f (A) + f (B)}{2} - \int_{0}^{1} f ((1 - t) A + tB) \, dt \\
& \leq \frac{1}{8} \left[ Df (B) (B - A) - Df (A) (B - A) \right]
\end{align*}
for all \( A, B \in SA_I (H) \).

4. Some Examples

We note that the function \( f(x) = - \ln x \) is operator convex on \( (0, \infty) \). The \( \ln \) function is operator Gâteaux differentiable with the following explicit formula for the derivative (cf. Pedersen [12, p. 155]):
\begin{align*}
\nabla \ln_T (S) &= \int_{0}^{\infty} (s1_H + T)^{-1} S (s1_H + T)^{-1} \, ds
\end{align*}
for \( T, S > 0 \).

If we write the inequalities (3.1) and (3.7) for \( - \ln \) we get
\begin{align*}
0 & \leq \int_{0}^{1} \ln \left( \frac{A + B}{2} \right) - \int_{0}^{1} \ln ((1 - t) A + tB) \, dt \\
& \leq \frac{1}{8} \left[ \int_{0}^{\infty} (s1_H + A)^{-1} (B - A) (s1_H + A)^{-1} \, ds \\
& \quad - \int_{0}^{\infty} (s1_H + B)^{-1} (B - A) (s1_H + B)^{-1} \, ds \right]
\end{align*}
and
\begin{align*}
0 & \leq \int_{0}^{1} \ln ((1 - t) A + tB) \, dt - \frac{\ln A + \ln B}{2} \\
& \leq \frac{1}{8} \left[ \int_{0}^{\infty} (s1_H + A)^{-1} (B - A) (s1_H + A)^{-1} \, ds \\
& \quad - \int_{0}^{\infty} (s1_H + B)^{-1} (B - A) (s1_H + B)^{-1} \, ds \right]
\end{align*}
for all $A, B > 0$.

The function $f(x) = x^{-1}$ is also operator convex on $(0, \infty)$, operator Gâteaux differentiable and

$$
\nabla f_T(S) = -T^{-1}ST^{-1}
$$

for $T, S > 0$.

If we write the inequalities (3.1) and (3.7) for this function, then we get

$$
0 \leq \int_0^1 ((1 - t) A + tB)^{-1} dt - \left( \frac{A + B}{2} \right)^{-1}
$$

$$
\leq \frac{1}{8} [A^{-1} (B - A) A^{-1} - B^{-1} (B - A) B^{-1}]
$$

and

$$
0 \leq \frac{A^{-1} + B^{-1}}{2} - \int_0^1 ((1 - t) A + tB)^{-1} dt
$$

$$
\leq \frac{1}{8} [A^{-1} (B - A) A^{-1} - B^{-1} (B - A) B^{-1}]
$$

for all $A, B > 0$.

References


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