REVERSES OF OPERATOR HERMITE-HADAMARD INEQUALITIES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, the convex set of selfadjoint operators with spectra in I. If $A \neq B$ and f, as an operator function, is Gâteaux differentiable on $[A,B] := \{(1-t)A + tB \mid t \in [0,1]\}$, then

$$0 \leq \int_0^1 f((1-t)A + tB)\,dt - f\left(\frac{A+B}{2}\right) \leq \frac{1}{8}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right]$$

and

$$0 \leq \frac{f(A)+f(B)}{2} - \int_0^1 f((1-t)A + tB)\,dt \leq \frac{1}{8}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right].$$

Two particular examples of interest are also given.

1. Introduction

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq)\,(1-\lambda) f(A) + \lambda f(B) \tag{1.1}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [7] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0,\infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0,\infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0,\infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t) = -t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

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In [5] we obtained among others the following Hermite-Hadamard type inequalities for operator convex functions $f : I \to \mathbb{R}$

$$f\left(\frac{A+B}{2}\right) \leq \int_0^1 f\left((1-s)A + sB\right) ds \leq \frac{f(A)+f(B)}{2}, \tag{1.2}$$

where A, B are selfadjoint operators with spectra included in I.

For recent inequalities for operator convex functions see [1]-[6] and [8]-[17].

Motivated by the above results, in this paper we show among others that if $A \neq B$ and f is Gâteaux differentiable on $[A,B] := \{(1-t)A + tB \mid t \in [0,1]\}$, then

$$\begin{aligned} 0 &\leq \int_0^1 f\left((1-t)A + tB\right) dt - f\left(\frac{A+B}{2}\right) \\ &\leq \frac{1}{8}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right] \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \frac{f(A)+f(B)}{2} - \int_0^1 f\left((1-t)A + tB\right) dt \\ &\leq \frac{1}{8}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right]. \end{aligned}$$

Two particular examples of interest for $f(x) = -\ln x$ and $f(x) = x^{-1}$ are also given.

2. Some Preliminary Facts

Let f be an operator convex function on I. For $(A,B) \in \mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in I, we consider the auxiliary function $\varphi_{(A,B)} : [0,1] \to \mathcal{SA}_I(H)$ defined by

$$\varphi_{(A,B)}(t) := f\left((1-t)A + tB\right). \tag{2.1}$$

For $x \in H$ we can also consider the auxiliary function $\varphi_{(A,B);x} : [0,1] \to \mathbb{R}$ defined by

$$\varphi_{(A,B);x}(t) := \left\langle \varphi_{(A,B)}(t)x, x \right\rangle = \left\langle f\left((1-t)A + tB\right)x, x \right\rangle. \tag{2.2}$$

We have the following basic fact:

Lemma 1. *Let f be an operator convex function on I. For any $(A,B) \in \mathcal{SA}_I(H)$, $\varphi_{(A,B)}$ is well defined and convex in the operator order. For any $(A,B) \in \mathcal{SA}_I(H)$ and $x \in H$ the function $\varphi_{(A,B);x}$ is convex in the usual sense on $[0,1]$.*

Proof. If $(A,B) \in \mathcal{SA}_I(H)$ and $t \in [0,1]$ the convex combination $(1-t)A + tB$ is a selfadjoint operator with the spectrum in I showing that $\mathcal{SA}_I(H)$ in the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators on H. By the continuous functional calculus of selfadjoint operator we also conclude that $f\left((1-t)A + tB\right)$ is a selfadjoint operator with spectrum in I.

Let $(A,B) \in \mathcal{SA}_I(H)$ and $t_1, t_2 \in [0,1]$. If $\alpha,\ \beta > 0$ with $\alpha + \beta = 1$, then

$$\begin{aligned}\varphi_{(A,B)}(\alpha t_1 + \beta t_2) &:= f((1-\alpha t_1 - \beta t_2) A + (\alpha t_1 + \beta t_2) B) \\ &= f((\alpha + \beta - \alpha t_1 - \beta t_2) A + (\alpha t_1 + \beta t_2) B) \\ &= f(\alpha [(1-t_1) A + t_1 B] + \beta [(1-t_2) A + t_2 B]) \\ &\leq \alpha f((1-t_1) A + t_1 B) + \beta f((1-t_2) A + t_2 B) \\ &= \alpha \varphi_{(A,B)}(t_1) + \beta \varphi_{(A,B)}(t_2),\end{aligned}$$

which proves the convexity $\varphi_{(A,B)}$ in the operator order.

Ley $(A,B) \in \mathcal{SA}_I(H)$ and $x \in H$. If $t_1, t_2 \in [0,1]$ and $\alpha,\ \beta > 0$ with $\alpha + \beta = 1$, then

$$\begin{aligned}\varphi_{(A,B);x}(\alpha t_1 + \beta t_2) &= \left\langle \varphi_{(A,B)}(\alpha t_1 + \beta t_2) x, x \right\rangle \\ &\leq \left\langle \left[\alpha \varphi_{(A,B)}(t_1) + \beta \varphi_{(A,B)}(t_2) \right] x, x \right\rangle \\ &= \alpha \left\langle \varphi_{(A,B)}(t_1) x, x \right\rangle + \beta \left\langle \varphi_{(A,B)}(t_2) x, x \right\rangle \\ &= \alpha \varphi_{(A,B);x}(t_1) + \beta \varphi_{(A,B);x}(t_2),\end{aligned}$$

which proves the convexity of $\varphi_{(A,B);x}$ on $[0,1]$. $\square$

A continuous function $g : \mathcal{SA}_I(H) \to \mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A \in \mathcal{SA}_I(H)$ along the direction $B \in \mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla g_A(B) := \lim_{s \to 0} \frac{g(A + sB) - g(A)}{s} \in \mathcal{B}(H). \tag{2.3}$$

If the limit (2.3) exists for all $B \in \mathcal{B}(H)$, then we say that f is *Gâteaux differentiable* in A and we can write $g \in \mathcal{G}(A)$. If this is true for any A in an open set $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $g \in \mathcal{G}(\mathcal{S})$.

If g is a continuous function on I, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators $A, B \in \mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators

$$[A,B] := \{(1-t) A + tB \mid t \in [0,1]\}.$$

We observe that $A,\ B \in [A,B]$ and $[A,B] \subset \mathcal{SA}_I(H)$.

Lemma 2. *Let f be an operator convex function on I and $(A,B) \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then the auxiliary function $\varphi_{(A,B)}$ is differentiable on $(0,1)$ and*

$$\varphi'_{(A,B)}(t) = \nabla f_{(1-t)A + tB}(B - A). \tag{2.4}$$

Also we have for the lateral derivative that

$$\varphi'_{(A,B)}(0+) = \nabla f_A(B - A) \tag{2.5}$$

and

$$\varphi'_{(A,B)}(1-) = \nabla f_B(B - A). \tag{2.6}$$

Proof. Let $t \in (0,1)$ and $h \neq 0$ small enough such that $t+h \in (0,1)$. Then

$$\begin{aligned} &\frac{\varphi_{(A,B)}(t+h)-\varphi_{(A,B)}(t)}{h} \\ &= \frac{f((1-t-h)A+(t+h)B)-f((1-t)A+tB)}{h} \\ &= \frac{f((1-t)A+tB+h(B-A))-f((1-t)A+tB)}{h}. \end{aligned} \tag{2.7}$$

Since $f \in \mathcal{G}([A,B])$, hence by taking the limit over $h \to 0$ in (2.7) we get

$$\begin{aligned} \varphi'_{(A,B)}(t) &= \lim_{h\to 0}\frac{\varphi_{(A,B)}(t+h)-\varphi_{(A,B)}(t)}{h} \\ &= \lim_{h\to 0}\frac{f((1-t)A+tB+h(B-A))-f((1-t)A+tB)}{h} \\ &= \nabla g_{(1-t)A+tB}(B-A), \end{aligned}$$

which proves (2.8).

Also, we have

$$\begin{aligned} \varphi'_{(A,B)}(0+) &= \lim_{h\to 0+}\frac{\varphi_{(A,B)}(h)-\varphi_{(A,B)}(0)}{h} \\ &= \lim_{h\to 0+}\frac{f((1-h)A+hB)-f(A)}{h} \\ &= \lim_{h\to 0+}\frac{f(A+h(B-A))-f(A)}{h} \\ &= \nabla f_A(B-A) \end{aligned}$$

since f is assumed to be Gâteaux differentiable in A. This proves (2.5).

The equality (2.6) follows in a similar way. □

Lemma 3. *Let f be an operator convex function on I and $(A,B) \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then for $0 < t_1 < t_2 < 1$ we have*

$$\nabla g_{(1-t_1)A+t_1B}(B-A) \le \nabla g_{(1-t_2)A+t_2B}(B-A) \tag{2.8}$$

in the operator order.

We also have

$$\nabla f_A(B-A) \le \nabla g_{(1-t_1)A+t_1B}(B-A) \tag{2.9}$$

and

$$\nabla g_{(1-t_2)A+t_2B}(B-A) \le \nabla f_B(B-A). \tag{2.10}$$

Proof. Let $x \in H$. The auxiliary function $\varphi_{(A,B);x}$ is convex in the usual sense on $[0,1]$ and differentiable on $(0,1)$ and for $t \in (0,1)$

$$\begin{aligned} \varphi'_{(A,B),x}(t) &= \lim_{h\to 0}\frac{\varphi_{(A,B),x}(t+h)-\varphi_{(A,B),x}(t)}{h} \\ &= \lim_{h\to 0}\left\langle \frac{\varphi_{(A,B)}(t+h)-\varphi_{(A,B)}(t)}{h}x, x\right\rangle \\ &= \left\langle \lim_{h\to 0}\frac{\varphi_{(A,B)}(t+h)-\varphi_{(A,B)}(t)}{h}x, x\right\rangle \\ &= \left\langle \nabla g_{(1-t)A+tB}(B-A)x, x\right\rangle. \end{aligned}$$

Since for $0 < t_1 < t_2 < 1$ we have by the gadient inequality for scalar convex functions that

$$\varphi'_{(A,B),x}(t_1) \le \varphi'_{(A,B),x}(t_2)$$

then we get

$$\left\langle \nabla g_{(1-t_1)A+t_1B}(B-A)x, x\right\rangle \le \left\langle \nabla g_{(1-t_2)A+t_2B}(B-A)x, x\right\rangle \tag{2.11}$$

for all $x \in H$, which is equivalent to the inequality (2.8) in the operator order.

Let $0 < t_1 < 1$. By the gadient inequality for scalar convex functions we also have

$$\varphi'_{(A,B),x}(0+) \le \varphi'_{(A,B),x}(t_1),$$

which, as above implies that

$$\langle \nabla f_A(B-A)x, x\rangle \le \left\langle \nabla g_{(1-t_1)A+t_1B}(B-A)x, x\right\rangle$$

for all $x \le H$, that is equivalent to the operator inequality (2.9).

The inequality (2.10) follows in a similar way. □

Corollary 1. *Let f be an operator convex function on I and $(A,B) \in \mathcal{SA}_I(H)$, with $A \ne B$. If $f \in \mathcal{G}([A,B])$, then for all $t \in (0,1)$ we have*

$$\nabla f_A(B-A) \le \nabla f_{(1-t)A+tB}(B-A) \le \nabla f_B(B-A). \tag{2.12}$$

3. Reverses of Operator Hermite-Hadamard Inequalities

It is well known that, if E is a Banach space and $f:[0,1] \to E$ is a continuous function, then f is Bochner integrable, and its Bochner integral coincides with its Riemann integral. We denote this integral as usual by $\int_0^1 f(t)\,dt$.

We have the following reverse of the first operator Hermite-Hadamard inequality:

Theorem 1. *Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \ne B$. If $f \in \mathcal{G}([A,B])$, then*

$$\begin{aligned} 0 &\le \int_0^1 f((1-t)A+tB)\,dt - f\left(\frac{A+B}{2}\right) \\ &\le \frac{1}{8}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right]. \end{aligned} \tag{3.1}$$

Proof. Using integration by parts formula for the Bochner integral, we have

$$\begin{aligned} \int_0^{1/2} t\varphi'_{(A,B)}(t)\,dt &= \frac{1}{2}\varphi_{(A,B)}\left(\frac{1}{2}\right) - \int_0^{1/2} \varphi_{(A,B)}(t)\,dt \\ &= \frac{1}{2}f\left(\frac{A+B}{2}\right) - \int_0^{1/2} f((1-t)A+tB)\,dt \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \int_{1/2}^1 (t-1)\varphi'_{(A,B)}(t)\,dt &= \frac{1}{2}\varphi_{(A,B)}\left(\frac{1}{2}\right) - \int_{1/2}^1 f((1-t)A+tB)\,dt \\ &= \frac{1}{2}f\left(\frac{A+B}{2}\right) - \int_{1/2}^1 f((1-t)A+tB)\,dt. \end{aligned} \tag{3.3}$$

If we add these two equalities, we get the following identity of interest

$$\int_0^1 f((1-t)A+tB)\,dt - f\left(\frac{A+B}{2}\right) = \int_{1/2}^1 (1-t)\,\varphi'_{(A,B)}(t)\,dt - \int_0^{1/2} t\varphi'_{(A,B)}(t)\,dt. \tag{3.4}$$

From Lemma 3 we have

$$\varphi'_{(A,B)}(1/2) \le \varphi'_{(A,B)}(t) \le \varphi'_{(A,B)}(1-) = \nabla f_B(B-A),\ t \in [1/2,1) \tag{3.5}$$

and

$$\nabla f_A(B-A) = \varphi'_{(A,B)}(0+) \le \varphi'_{(A,B)}(t) \le \varphi'_{(A,B)}(1/2),\ t \in (0,1/2], \tag{3.6}$$

This implies that

$$(1-t)\,\varphi'_{(A,B)}(1/2) \le (1-t)\,\varphi'_{(A,B)}(t) \le (1-t)\,\nabla f_B(B-A)$$

for $t \in [1/2,1)$ and

$$-t\varphi'_{(A,B)}(1/2) \le -t\varphi'_{(A,B)}(t) \le -t\nabla f_A(B-A)$$

for $t \in (0,1/2]$.

By integrating these inequalities on the corresponding intervals, we get

$$\frac{1}{8}\varphi'_{(A,B)}(1/2) \le \int_{1/2}^1 (1-t)\,\varphi'_{(A,B)}(t)\,dt \le \frac{1}{8}\nabla f_B(B-A)$$

and

$$-\frac{1}{8}\varphi'_{(A,B)}(1/2) \le -\int_0^{1/2} t\varphi'_{(A,B)}(t)\,dt \le -\frac{1}{8}\nabla f_A(B-A).$$

By addition, we deduce that

$$0 \le \int_{1/2}^1 (1-t)\,\varphi'_{(A,B)}(t) - \int_0^{1/2} t\varphi'_{(A,B)}(t)\,dt \le \frac{1}{8}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right]$$

and by the identity (3.4) we get (3.1). □

We have the following reverse of the second operator Hermite-Hadamard inequality:

Theorem 2. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then*

$$0 \le \frac{f(A)+f(B)}{2} - \int_0^1 f((1-t)A+tB)\,dt \le \frac{1}{8}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right]. \tag{3.7}$$

Proof. Using integration by parts formula for the Bochner integral, we have

$$\begin{aligned}\int_0^1 \left(t-\frac{1}{2}\right)\varphi'_{(A,B)}(t)\,dt &= \left(t-\frac{1}{2}\right)\varphi_{(A,B)}(t)\Big]_0^1 - \int_0^1 \varphi_{(A,B)}(t) \\ &= \frac{\varphi_{(A,B)}(1)+\varphi_{(A,B)}(0)}{2} - \int_0^1 \varphi_{(A,B)}(t) \\ &= \frac{f(B)+f(A)}{2} - \int_0^1 f((1-t)A+tB)\,dt.\end{aligned} \tag{3.8}$$

Observe that

$$\begin{aligned}&\int_0^1 \left(t-\frac{1}{2}\right)\varphi'_{(A,B)}(t)\,dt \\ &= \int_{1/2}^1 \left(t-\frac{1}{2}\right)\varphi'_{(A,B)}(t)\,dt - \int_0^{1/2}\left(\frac{1}{2}-t\right)\varphi'_{(A,B)}(t)\,dt.\end{aligned} \tag{3.9}$$

Therefore, we have the following identity of interest

$$\begin{aligned}&\frac{f(B)+f(A)}{2} - \int_0^1 f((1-t)A+tB)\,dt \\ &= \int_{1/2}^1 \left(t-\frac{1}{2}\right)\varphi'_{(A,B)}(t)\,dt - \int_0^{1/2}\left(\frac{1}{2}-t\right)\varphi'_{(A,B)}(t)\,dt.\end{aligned}$$

From the inequality (3.5) we obtain

$$\begin{aligned}\left(t-\frac{1}{2}\right)\varphi'_{(A,B)}(1/2) &\le \left(t-\frac{1}{2}\right)\varphi'_{(A,B)}(t) \\ &\le \left(t-\frac{1}{2}\right)\nabla f_B(B-A),\ t\in[1/2,1)\end{aligned}$$

and from (3.6)

$$\begin{aligned}\left(\frac{1}{2}-t\right)\nabla f_A(B-A) &\le \left(\frac{1}{2}-t\right)\varphi'_{(A,B)}(t) \\ &\le \left(\frac{1}{2}-t\right)\varphi'_{(A,B)}(1/2),\ t\in(0,1/2],\end{aligned}$$

namely

$$\begin{aligned}-\left(\frac{1}{2}-t\right)\varphi'_{(A,B)}(1/2) &\le -\left(\frac{1}{2}-t\right)\varphi'_{(A,B)}(t) \\ &\le -\left(\frac{1}{2}-t\right)\nabla f_A(B-A),\ t\in(0,1/2].\end{aligned}$$

Integrating these inequalities on the corresponding intervals, we get

$$\frac{1}{8}\varphi'_{(A,B)}(1/2) \le \int_{1/2}^1 \left(t-\frac{1}{2}\right)\varphi'_{(A,B)}(t)\,dt \le \frac{1}{8}\nabla f_B(B-A),$$

and

$$-\frac{1}{8}\varphi'_{(A,B)}(1/2) \le -\int_0^{1/2}\left(\frac{1}{2}-t\right)\varphi'_{(A,B)}(t)\,dt \le -\frac{1}{8}\nabla f_A(B-A).$$

If we add these two inequalities, we obtain

$$0 \le \int_{1/2}^{1} \left(t - \frac{1}{2}\right) \varphi'_{(A,B)}(t)\, dt - \int_{0}^{1/2} \left(\frac{1}{2} - t\right) \varphi'_{(A,B)}(t)\, dt$$
$$\le \frac{1}{8} \left[\nabla f_B (B - A) - \nabla f_A (B - A)\right],$$

which, by the use of identity (3.9) produces the desired result (3.7). □

Remark 1. *It is well known that, if h is a C^1-function defined on an open interval, then the operator function $h(X)$ is Fréchet differentiable and the derivative $Dh(A)(B)$ equals the Gâteaux derivative $\nabla f_A(B)$. So for operator convex functions f that are of class C^1 on I we have the inequalities*

$$0 \le \int_0^1 f((1-t)A + tB)\, dt - f\left(\frac{A+B}{2}\right) \le \frac{1}{8} \left[Df(B)(B - A) - Df(A)(B - A)\right] \tag{3.10}$$

and

$$0 \le \frac{f(A) + f(B)}{2} - \int_0^1 f((1-t)A + tB)\, dt \le \frac{1}{8} \left[Df(B)(B - A) - Df(A)(B - A)\right] \tag{3.11}$$

for all A, $B \in \mathcal{SA}_I(H)$.

4. Some Examples

We note that the function $f(x) = -\ln x$ is operator convex on $(0, \infty)$. The ln function is operator Gâteaux differentiable with the following explicit formula for the derivative (cf. Pedersen [12, p. 155]):

$$\nabla \ln_T(S) = \int_0^\infty (s1_H + T)^{-1} S (s1_H + T)^{-1}\, ds \tag{4.1}$$

for T, $S > 0$

If we write the inequalities (3.1) and (3.7) for $-\ln$ we get

$$\begin{aligned} 0 &\le \ln\left(\frac{A+B}{2}\right) - \int_0^1 \ln((1-t)A + tB)\, dt \\ &\le \frac{1}{8} \left[\int_0^\infty (s1_H + A)^{-1} (B - A) (s1_H + A)^{-1}\, ds \right. \\ &\quad \left. - \int_0^\infty (s1_H + B)^{-1} (B - A) (s1_H + B)^{-1}\, ds\right] \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} 0 &\le \int_0^1 \ln((1-t)A + tB)\, dt - \frac{\ln A + \ln B}{2} \\ &\le \frac{1}{8} \left[\int_0^\infty (s1_H + A)^{-1} (B - A) (s1_H + A)^{-1}\, ds \right. \\ &\quad \left. - \int_0^\infty (s1_H + B)^{-1} (B - A) (s1_H + B)^{-1}\, ds\right] \end{aligned} \tag{4.3}$$

for all $A,\, B>0$.

The function $f(x)=x^{-1}$ is also operator convex on $(0,\infty)$, operator Gâteaux differentiable and

$$\nabla f_T(S)=-T^{-1}ST^{-1}$$

for $T,\, S>0$.

If we write the inequalities (3.1) and (3.7) for this function, then we get

$$\begin{aligned}(4.4)\qquad 0 &\le \int_0^1 ((1-t)A+tB)^{-1}\,dt-\left(\frac{A+B}{2}\right)^{-1}\\ &\le \frac{1}{8}\left[A^{-1}(B-A)A^{-1}-B^{-1}(B-A)B^{-1}\right]\end{aligned}$$

and

$$\begin{aligned}(4.5)\qquad 0 &\le \frac{A^{-1}+B^{-1}}{2}-\int_0^1 ((1-t)A+tB)^{-1}\,dt\\ &\le \frac{1}{8}\left[A^{-1}(B-A)A^{-1}-B^{-1}(B-A)B^{-1}\right]\end{aligned}$$

for all $A,\, B>0$.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa