REVERSES OF FÉJER'S INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. Let f be a convex function on I and $a,\ b\in I$ with $a<b$. If $p:[a,b]\to[a,\infty)$ is Lebesgue integrable and symmetric, namely $p(b+a-t)=p(t)$ for all $t\in[a,b]$, then we show in this paper that

$$\begin{aligned}0&\le\frac{1}{2}\int_a^b\left|t-\frac{a+b}{2}\right|p(t)\,dt\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right]\\&\le\int_a^b p(t)\,f(t)\,dt-\left(\int_a^b p(t)\,dt\right)f\left(\frac{a+b}{2}\right)\\&\le\frac{1}{2}\int_a^b\left|t-\frac{a+b}{2}\right|p(t)\,dt\left[f'_-(b)-f'_+(a)\right]\end{aligned}$$

and

$$\begin{aligned}0&\le\frac{1}{2}\int_a^b\left[\frac{1}{2}(b-a)-\left|t-\frac{a+b}{2}\right|\right]p(t)\,dt\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right]\\&\le\left(\int_a^b p(t)\,dt\right)\frac{f(a)+f(b)}{2}-\int_a^b p(t)\,f(t)\,dt\\&\le\frac{1}{2}\int_a^b\left[\frac{1}{2}(b-a)-\left|t-\frac{a+b}{2}\right|\right]p(t)\,dt\left[f'_-(b)-f'_+(a)\right].\end{aligned}$$

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(t)dt\le\frac{f(a)+f(b)}{2},\quad a,\ b\in\mathbb{R},\ a<b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [7]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [1]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [7]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the *Hermite-Hadamard inequality*. For a monograph devoted to this result see [5]. The recent survey paper [4] provides other related results.

Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ and assume that $f'_+(a)$ and $f'_-(b)$ are finite. We recall the following improvement and reverse inequality for

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the first Hermite-Hadamard result that has been established in [2]

$$0 \leq \frac{1}{8}\left[f'_{+}\left(\frac{a+b}{2}\right) - f'_{-}\left(\frac{a+b}{2}\right)\right](b-a) \tag{1.2}$$
$$\leq \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \leq \frac{1}{8}(b-a)\left[f'_{-}(b) - f'_{+}(a)\right].$$

The following inequality that provides a reverse and improvement of the second Hermite-Hadamard result has been obtained in [3]

$$0 \leq \frac{1}{8}\left[f'_{+}\left(\frac{a+b}{2}\right) - f'_{-}\left(\frac{a+b}{2}\right)\right](b-a) \tag{1.3}$$
$$\leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{8}(b-a)\left[f'_{-}(b) - f'_{+}(a)\right].$$

The constant $\frac{1}{8}$ is best possible in both (1.2) and (1.3).

In 1906, Fejér [6], while studying trigonometric polynomials, obtained inequalities which generalize that of Hermite & Hadamard:

Theorem 1. *Consider the integral $\int_a^b f(t)\,p(t)\,dt$, where f is a convex function in the interval (a,b) and p is a positive function in the same interval such that*

$$p(a+t) = p(b-t),\ \ 0 \leq t \leq \frac{1}{2}(b-a),$$

i.e., $y = p(t)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b), 0\right)$ and is normal to the t-axis. Under those conditions the following inequalities are valid:

$$f\left(\frac{a+b}{2}\right)\int_a^b p(t)\,dt \leq \int_a^b f(t)\,p(t)\,dt \leq \frac{f(a)+f(b)}{2}\int_a^b p(t)\,dt. \tag{1.4}$$

If f is concave on (a,b), then the inequalities reverse in (1.4).

Clearly, for $p(t) \equiv 1$ on $[a,b]$ we get 1.1.

If we take $p(t) = \left|t - \frac{a+b}{2}\right|$, $t \in [a,b]$ in Theorem 1, then we have

$$\frac{1}{4}f\left(\frac{a+b}{2}\right)(b-a)^2 \leq \int_a^b \left|t - \frac{a+b}{2}\right| f(t)\,dt \leq \frac{f(a)+f(b)}{8}(b-a)^2, \tag{1.5}$$

for any convex function $f : [a,b] \to \mathbb{R}$.

We observe that, if we take $p(t) = (b-t)(t-a)$, $t \in [a,b]$, then p satisfies the conditions in Theorem 1, and by (1.4) we have the following inequality as well

$$\frac{1}{6}f\left(\frac{a+b}{2}\right)(b-a)^3 \leq \int_a^b (b-t)(t-a)\,f(t)\,dt \leq \frac{f(a)+f(b)}{12}(b-a)^3, \tag{1.6}$$

for any convex function $f : [a,b] \to \mathbb{R}$.

Motivated by the above results, in this paper we obtain an improvement and a reverse for each inequality in (1.4) and therefore generalize the Hermite-Hadamard inequalities (1.2) and (1.3).

2. Improvements and Reverse of Féjer Inequalities

Following Roberts and Varberg [8, p. 5], we recall that if $f : I \to \mathbb{R}$ is a convex function, then for any $x_0 \in \mathring{I}$ (the interior of the interval I) the limits

$$f'_{-}(x_0) := \lim_{x \to x_0 -} \frac{f(x) - f(x_0)}{x - x_0} \text{ and } f'_{+}(x_0) := \lim_{x \to x_0 +} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and $f'_{-}(x_0) \le f'_{+}(x_0)$. The functions f'_{-} and f'_{+} are monotonic nondecreasing on $\mathring{I}$ and this property can be extended to the whole interval I (see [8, p. 7]).

From the monotonicity of the lateral derivatives f'_{-} and f'_{+} we also have *the gradient inequality*

$$f'_{-}(x)(x - y) \ge f(x) - f(y) \ge f'_{+}(y)(x - y)$$

for any $x, y \in \mathring{I}$.

If $I = [a, b]$, then at the end points we also have the inequalities

$$f(x) - f(a) \ge f'_{+}(a)(x - a)$$

for any $x \in (a, b]$ and

$$f(y) - f(b) \ge f'_{-}(b)(y - b)$$

for any $y \in [a, b)$.

We have the following refinement and reverse of Fejer's first inequality:

Theorem 2. *Let f be a convex function on I and $a, b \in I$, with $a < b$. If $p : [a, b] \to [0, \infty)$ is Lebesgue integrable and symmetric, namely $p(b + a - t) = p(t)$ for all $t \in [a, b]$, then*

$$\begin{aligned} 0 &\le \frac{1}{2} \int_a^b \left| t - \frac{a+b}{2} \right| p(t)\, dt \left[f'_{+}\left(\frac{a+b}{2}\right) - f'_{-}\left(\frac{a+b}{2}\right) \right] \\ &\le \int_a^b p(t) f(t)\, dt - \left(\int_a^b p(t)\, dt \right) f\left(\frac{a+b}{2}\right) \\ &\le \frac{1}{2} \int_a^b \left| t - \frac{a+b}{2} \right| p(t)\, dt \left[f'_{-}(b) - f'_{+}(a) \right]. \end{aligned} \tag{2.1}$$

Proof. Let $a, b \in I$, with $a < b$. Using the integration by parts formula for Lebesgue integral, we have

$$\begin{aligned} &\int_{\frac{a+b}{2}}^b \left(\int_t^b p(s)\, ds \right) f'(t)\, dt \\ &= \left(\int_t^b p(s)\, ds \right) f(t) \Bigg]_{\frac{a+b}{2}}^b + \int_{\frac{a+b}{2}}^b p(t) f(t)\, dt \\ &= - \left(\int_{\frac{a+b}{2}}^b p(s)\, ds \right) f\left(\frac{a+b}{2}\right) + \int_{\frac{a+b}{2}}^b p(t) f(t)\, dt \end{aligned}$$

and

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} \left(\int_a^t p(s)\,ds\right) f'(t)\,dt \\
&= \left(\int_a^t p(s)\,ds\right) f(t)\Big]_a^{\frac{a+b}{2}} - \int_a^{\frac{a+b}{2}} p(t) f(t)\,dt \\
&= \left(\int_a^{\frac{a+b}{2}} p(s)\,ds\right) f\left(\frac{a+b}{2}\right) - \int_a^{\frac{a+b}{2}} p(t) f(t)\,dt.
\end{aligned}$$

By subtracting the second identity from the first, we get

$$\begin{aligned}
&\int_{\frac{a+b}{2}}^b \left(\int_t^b p(s)\,ds\right) f'(t)\,dt - \int_a^{\frac{a+b}{2}} \left(\int_a^t p(s)\,ds\right) f'(t)\,dt \\
&= \int_{\frac{a+b}{2}}^b p(t) f(t)\,dt + \int_a^{\frac{a+b}{2}} p(t) f(t)\,dt \\
&- \left(\int_{\frac{a+b}{2}}^b p(s)\,ds\right) f\left(\frac{a+b}{2}\right) - \left(\int_a^{\frac{a+b}{2}} p(s)\,ds\right) f\left(\frac{a+b}{2}\right).
\end{aligned}$$

By the symmetry of p we get

$$\int_{\frac{a+b}{2}}^b p(s)\,ds = \int_a^{\frac{a+b}{2}} p(s)\,ds = \frac{1}{2}\int_a^b p(s)\,ds$$

and then we can state the following identity of interest in itself

$$\begin{aligned}
&\int_a^b p(t) f(t)\,dt - f\left(\frac{a+b}{2}\right)\int_a^b p(s)\,ds \\
&= \int_{\frac{a+b}{2}}^b \left(\int_t^b p(s)\,ds\right) f'(t)\,dt - \int_a^{\frac{a+b}{2}} \left(\int_a^t p(s)\,ds\right) f'(t)\,dt.
\end{aligned} \tag{2.2}$$

By the monotonicity of the derivative we have

$$f'_+(a) \le f'(t) \le f'_-\left(\frac{a+b}{2}\right), \text{ for almost every } t \in \left(a, \frac{a+b}{2}\right)$$

and

$$f'_+\left(\frac{a+b}{2}\right) \le f'(t) \le f'_-(b), \text{ for almost every } t \in \left(\frac{a+b}{2}, b\right).$$

This implies

$$\begin{aligned}
f'_+(a)\left(\int_a^t p(s)\,ds\right) &\le f'(t)\left(\int_a^t p(s)\,ds\right) \\
&\le f'_-\left(\frac{a+b}{2}\right)\left(\int_a^t p(s)\,ds\right), \ t \in \left[a, \frac{a+b}{2}\right]
\end{aligned}$$

and

$$f'_{+}\left(\frac{a+b}{2}\right)\left(\int_{t}^{b}p\left(s\right)ds\right)\leq f'\left(t\right)\left(\int_{t}^{b}p\left(s\right)ds\right)$$
$$\leq f'_{-}\left(b\right)\left(\int_{t}^{b}p\left(s\right)ds\right),\ t\in\left[\frac{a+b}{2},b\right],$$

and by integration

$$f'_{+}\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^{b}\left(\int_{t}^{b}p\left(s\right)ds\right)dt\leq\int_{\frac{a+b}{2}}^{b}\left(\int_{t}^{b}p\left(s\right)ds\right)f'\left(t\right)dt$$
$$\leq f'_{-}\left(b\right)\int_{\frac{a+b}{2}}^{b}\left(\int_{t}^{b}p\left(s\right)ds\right)dt$$

and

$$-f'_{-}\left(\frac{a+b}{2}\right)\int_{a}^{\frac{a+b}{2}}\left(\int_{a}^{t}p\left(s\right)ds\right)dt\leq-\int_{a}^{\frac{a+b}{2}}\left(\int_{a}^{t}p\left(s\right)ds\right)f'\left(t\right)dt$$
$$\leq-f'_{+}\left(a\right)\int_{a}^{\frac{a+b}{2}}\left(\int_{a}^{t}p\left(s\right)ds\right)dt.$$

If we add these inequalities, then we get

$$\begin{aligned}(2.3)\quad & f'_{+}\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^{b}\left(\int_{t}^{b}p\left(s\right)ds\right)dt-f'_{-}\left(\frac{a+b}{2}\right)\int_{a}^{\frac{a+b}{2}}\left(\int_{a}^{t}p\left(s\right)ds\right)dt\\ &\leq\int_{\frac{a+b}{2}}^{b}\left(\int_{t}^{b}p\left(s\right)ds\right)f\left(t\right)dt-\int_{a}^{\frac{a+b}{2}}\left(\int_{a}^{t}p\left(s\right)ds\right)f\left(t\right)dt\\ &\leq f'_{-}\left(b\right)\int_{\frac{a+b}{2}}^{b}\left(\int_{t}^{b}p\left(s\right)ds\right)dt-f'_{+}\left(a\right)\int_{a}^{\frac{a+b}{2}}\left(\int_{a}^{t}p\left(s\right)ds\right)dt.\end{aligned}$$

Integrating by parts in the Lebesgue integral, we have

$$\begin{aligned}\int_{\frac{a+b}{2}}^{b}\left(\int_{t}^{b}p\left(s\right)ds\right)dt&=\left.\left(\int_{t}^{b}p\left(s\right)ds\right)t\right]_{\frac{a+b}{2}}^{b}+\int_{\frac{a+b}{2}}^{b}tp\left(t\right)dt\\ &=\int_{\frac{a+b}{2}}^{b}tp\left(t\right)dt-\frac{a+b}{2}\int_{\frac{a+b}{2}}^{b}p\left(s\right)ds\\ &=\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)p\left(t\right)dt=\frac{1}{2}\int_{a}^{b}\left|t-\frac{a+b}{2}\right|p\left(t\right)dt,\end{aligned}$$

where for the last equality we used the symmetry of p.

Similarly,

$$\begin{aligned}\int_a^{\frac{a+b}{2}}\left(\int_a^t p(s)\,ds\right)dt &= \left(\int_a^t p(s)\,ds\right)t\Bigg]_a^{\frac{a+b}{2}} - \int_a^{\frac{a+b}{2}} p(t)\,tdt \\ &= \frac{a+b}{2}\int_a^{\frac{a+b}{2}} p(s)\,ds - \int_a^{\frac{a+b}{2}} p(t)\,tdt \\ &= \int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)p(t)\,dt = \frac{1}{2}\int_a^b \left|t-\frac{a+b}{2}\right| p(t)\,dt.\end{aligned}$$

Then by (2.3) we obtain the desired result (2.1). $\square$

Remark 1. *If we take $p \equiv 1$ in (2.1) and since $\int_a^b \left|t-\frac{a+b}{2}\right| = \frac{1}{4}(b-a)^2$, hence by (2.1) we recapture the inequalities (1.2) from Introduction.*

We also have the following refinement and reverse of Fejer's second inequality:

Theorem 3. *Let f be a convex function on I and $a,\ b \in I$, with $a < b$. If $p : [a,b] \to [0,\infty)$ is Lebesgue integrable and symmetric, namely $p(b+a-t) = p(t)$ for all $t \in [a,b]$, then*

$$\begin{aligned}(2.4)\qquad 0 &\le \frac{1}{2}\int_a^b \left[\frac{1}{2}(b-a) - \left|t-\frac{a+b}{2}\right|\right] p(t)\,dt \left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right] \\ &\le \left(\int_a^b p(t)\,dt\right)\frac{f(a)+f(b)}{2} - \int_a^b p(t) f(t)\,dt \\ &\le \frac{1}{2}\int_a^b \left[\frac{1}{2}(b-a) - \left|t-\frac{a+b}{2}\right|\right] p(t)\,dt \left[f'_-(b) - f'_+(a)\right].\end{aligned}$$

Proof. Using the integration by parts for Lebesgue integral, we have

$$\begin{aligned}&\int_a^b \left(\int_a^t p(s)\,ds - \frac{1}{2}\int_a^b p(s)\,ds\right) f'(t)\,dt \\ &= \left(\int_a^t p(s)\,ds - \frac{1}{2}\int_a^b p(s)\,ds\right) f(t)\Bigg]_a^b - \int_a^b p(t) f(t)\,dt \\ &= \left(\int_a^b p(s)\,ds - \frac{1}{2}\int_a^b p(s)\,ds\right) f(b) + \left(\frac{1}{2}\int_a^b p(s)\,ds\right) f(a) \\ &- \int_a^b p(t) f(t)\,dt \\ &= \left(\int_a^b p(t)\,dt\right)\frac{f(a)+f(b)}{2} - \int_a^b p(t) f(t)\,dt.\end{aligned}$$

We also have

$$\begin{aligned}
&\int_a^b \left(\int_a^t p(s)\,ds - \frac{1}{2}\int_a^b p(s)\,ds\right) f'(t)\,dt \\
&= \int_a^b \left(\int_a^t p(s)\,ds - \int_a^{\frac{a+b}{2}} p(s)\,ds\right) f(t)\,dt \\
&= \int_a^{\frac{a+b}{2}} \left(\int_a^t p(s)\,ds - \int_a^{\frac{a+b}{2}} p(s)\,ds\right) f(t)\,dt \\
&+ \int_{\frac{a+b}{2}}^b \left(\int_a^t p(s)\,ds - \int_a^{\frac{a+b}{2}} p(s)\,ds\right) f(t)\,dt \\
&= \int_{\frac{a+b}{2}}^b \left(\int_a^t p(s)\,ds - \int_a^{\frac{a+b}{2}} p(s)\,ds\right) f(t)\,dt \\
&- \int_a^{\frac{a+b}{2}} \left(\int_a^{\frac{a+b}{2}} p(s)\,ds - \int_a^t p(s)\,ds\right) f(t)\,dt.
\end{aligned}$$

Observe that

$$\int_a^t p(s)\,ds - \int_a^{\frac{a+b}{2}} p(s)\,ds \geq 0 \text{ for } t \in \left[\frac{a+b}{2}, b\right]$$

and

$$\int_a^{\frac{a+b}{2}} p(s)\,ds - \int_a^t p(s)\,ds \geq 0 \text{ for } t \in \left[a, \frac{a+b}{2}\right].$$

By the monotonicity of the derivative we have

$$\begin{aligned}
&f'_+\left(\frac{a+b}{2}\right) \int_{\frac{a+b}{2}}^b \left(\int_a^t p(s)\,ds - \int_a^{\frac{a+b}{2}} p(s)\,ds\right) dt \\
&\leq \int_{\frac{a+b}{2}}^b \left(\int_a^t p(s)\,ds - \int_a^{\frac{a+b}{2}} p(s)\,ds\right) f'(t)\,dt \\
&\leq f'_-(b) \int_{\frac{a+b}{2}}^b \left(\int_a^t p(s)\,ds - \int_a^{\frac{a+b}{2}} p(s)\,ds\right) dt
\end{aligned}$$

and

$$\begin{aligned}
&- f'_-\left(\frac{a+b}{2}\right) \int_a^{\frac{a+b}{2}} \left(\int_a^{\frac{a+b}{2}} p(s)\,ds - \int_a^t p(s)\,ds\right) dt \\
&\leq - \int_a^{\frac{a+b}{2}} \left(\int_a^{\frac{a+b}{2}} p(s)\,ds - \int_a^t p(s)\,ds\right) f'(t)\,dt \\
&\leq -f'_+(a) \int_a^{\frac{a+b}{2}} \left(\int_a^{\frac{a+b}{2}} p(s)\,ds - \int_a^t p(s)\,ds\right) dt.
\end{aligned}$$

If we add these inequalities, then we get

$$\begin{aligned}
(2.5)\qquad & \left[f'_{+}\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^{b}\left(\int_{a}^{t}p(s)\,ds-\int_{a}^{\frac{a+b}{2}}p(s)\,ds\right)dt\right.\\
& \left.-f'_{-}\left(\frac{a+b}{2}\right)\int_{a}^{\frac{a+b}{2}}\left(\int_{a}^{\frac{a+b}{2}}p(s)\,ds-\int_{a}^{t}p(s)\,ds\right)dt\right]\\
& \leq\int_{\frac{a+b}{2}}^{b}\left(\int_{a}^{t}p(s)\,ds-\int_{a}^{\frac{a+b}{2}}p(s)\,ds\right)f'(t)\,dt\\
& -\int_{\frac{a+b}{2}}^{b}\left(\int_{a}^{t}p(s)\,ds-\int_{a}^{\frac{a+b}{2}}p(s)\,ds\right)f'(t)\,dt\\
& \leq f'_{-}(b)\int_{\frac{a+b}{2}}^{b}\left(\int_{a}^{t}p(s)\,ds-\int_{a}^{\frac{a+b}{2}}p(s)\,ds\right)dt\\
& -f'_{+}(a)\int_{a}^{\frac{a+b}{2}}\left(\int_{a}^{\frac{a+b}{2}}p(s)\,ds-\int_{a}^{t}p(s)\,ds\right)dt.
\end{aligned}$$

Observe that

$$\begin{aligned}
& \int_{\frac{a+b}{2}}^{b}\left(\int_{a}^{t}p(s)\,ds-\int_{a}^{\frac{a+b}{2}}p(s)\,ds\right)dt\\
& =\int_{\frac{a+b}{2}}^{b}\left(\int_{a}^{t}p(s)\,ds\right)dt-\frac{b-a}{2}\int_{a}^{\frac{a+b}{2}}p(s)\,ds\\
& =\left.\left(\int_{a}^{t}p(s)\,ds\right)t\right]_{\frac{a+b}{2}}^{b}-\int_{\frac{a+b}{2}}^{b}tp(t)\,dt-\frac{b-a}{2}\int_{a}^{\frac{a+b}{2}}p(s)\,ds\\
& =b\int_{a}^{b}p(s)\,ds-\frac{a+b}{2}\int_{a}^{\frac{a+b}{2}}p(s)\,ds-\int_{\frac{a+b}{2}}^{b}tp(t)\,dt-\frac{b-a}{2}\int_{a}^{\frac{a+b}{2}}p(s)\,ds\\
& =b\int_{a}^{b}p(s)\,ds-b\int_{a}^{\frac{a+b}{2}}p(s)\,ds-\int_{\frac{a+b}{2}}^{b}tp(t)\,dt\\
& =b\int_{\frac{a+b}{2}}^{b}p(s)\,ds-\int_{\frac{a+b}{2}}^{b}tp(t)\,dt=\int_{\frac{a+b}{2}}^{b}(b-t)\,p(t)\,dt
\end{aligned}$$

and

$$\begin{aligned}
& \int_{a}^{\frac{a+b}{2}}\left(\int_{a}^{\frac{a+b}{2}}p(s)\,ds-\int_{a}^{t}p(s)\,ds\right)dt\\
& =\frac{b-a}{2}\int_{a}^{\frac{a+b}{2}}p(s)\,ds-\int_{a}^{\frac{a+b}{2}}\left(\int_{a}^{t}p(s)\,ds\right)dt\\
& =\frac{b-a}{2}\int_{a}^{\frac{a+b}{2}}p(s)\,ds-\left(\left.\left(\int_{a}^{t}p(s)\,ds\right)t\right]_{a}^{\frac{a+b}{2}}-\int_{a}^{\frac{a+b}{2}}tp(t)\,dt\right)
\end{aligned}$$

$$= \frac{b-a}{2} \int_a^{\frac{a+b}{2}} p(s)\,ds - \frac{a+b}{2} \int_a^{\frac{a+b}{2}} p(s)\,ds + \int_a^{\frac{a+b}{2}} tp(t)\,dt$$
$$= \int_a^{\frac{a+b}{2}} tp(t)\,dt - a \int_a^{\frac{a+b}{2}} p(s)\,ds = \int_a^{\frac{a+b}{2}} (t-a)\,p(t)\,dt.$$

If we change the variable $s = b + a - t$, then

$$\int_a^{\frac{a+b}{2}} (t-a)\,p(t)\,dt = \int_{\frac{a+b}{2}}^b (b-s)\,p(b+a-s)\,ds = \int_{\frac{a+b}{2}}^b (b-s)\,p(s)\,ds.$$

Finally, observe that

$$\begin{aligned}
&\frac{1}{2}\int_a^b \left[\frac{1}{2}(b-a) - \left|t - \frac{a+b}{2}\right|\right] p(t)\,dt \\
&= \frac{1}{2}\int_a^{\frac{a+b}{2}} \left[\frac{1}{2}(b-a) - \left|t - \frac{a+b}{2}\right|\right] p(t)\,dt \\
&+ \frac{1}{2}\int_{\frac{a+b}{2}}^b \left[\frac{1}{2}(b-a) - \left|t - \frac{a+b}{2}\right|\right] p(t)\,dt \\
&= \frac{1}{2}\int_a^{\frac{a+b}{2}} \left[\frac{1}{2}(b-a) - \frac{a+b}{2} + t\right] p(t)\,dt \\
&+ \frac{1}{2}\int_{\frac{a+b}{2}}^b \left[\frac{1}{2}(b-a) - t + \frac{a+b}{2}\right] p(t)\,dt \\
&= \frac{1}{2}\int_a^{\frac{a+b}{2}} (t-a)\,p(t)\,dt + \frac{1}{2}\int_{\frac{a+b}{2}}^b (b-t)\,p(t)\,dt \\
&= \frac{1}{2}\int_a^{\frac{a+b}{2}} (t-a)\,p(t)\,dt + \frac{1}{2}\int_a^{\frac{a+b}{2}} (t-a)\,p(t)\,dt = \int_a^{\frac{a+b}{2}} (t-a)\,p(t)\,dt
\end{aligned}$$

and by (2.5) we get (2.4). □

Remark 2. *Observe that for $p \equiv 1$ we recapture the inequalities (1.3) from Introduction.*

If we consider the symmetric weight $p(t) = \left|t - \frac{a+b}{2}\right|$, $t \in [a,b]$ we obtain from Theorem 2 that

$$\begin{aligned}
0 &\le \frac{1}{24}(b-a)^3 \left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right] \\
&\le \int_a^b \left|t - \frac{a+b}{2}\right| f(t)\,dt - \frac{1}{4}(b-a)^2 f\left(\frac{a+b}{2}\right) \\
&\le \frac{1}{24}(b-a)^3 \left[f'_-(b) - f'_+(a)\right]
\end{aligned} \tag{2.6}$$

and from Theorem 3 that

$$0 \le \frac{1}{48}(b-a)^3\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right] \le (b-a)^2\frac{f(a)+f(b)}{8} - \int_a^b \left|t-\frac{a+b}{2}\right| f(t)\,dt \le \frac{1}{48}(b-a)^3\left[f'_-(b) - f'_+(a)\right], \tag{2.7}$$

where f is convex on $[a,b]$. These provide refinements and reverses of the inequalities (1.5).

If we consider the symmetric weight $p(t) = (t-a)(b-t)$, $t \in [a,b]$ we obtain from Theorem 2 that

$$0 \le \frac{1}{64}(b-a)^4\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right] \le \int_a^b (t-a)(b-t) f(t)\,dt - \frac{1}{6}(b-a)^3 f\left(\frac{a+b}{2}\right) \le \frac{1}{64}(b-a)^4\left[f'_-(b) - f'_+(a)\right] \tag{2.8}$$

and from Theorem 3 that

$$0 \le \frac{5}{192}(b-a)^4\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right] \le (b-a)^3\frac{f(a)+f(b)}{12} - \int_a^b (t-a)(b-t) f(t)\,dt \le \frac{5}{192}(b-a)^4\left[f'_-(b) - f'_+(a)\right], \tag{2.9}$$

where f is convex on $[a,b]$. These provide refinements and reverses of the inequalities (1.6).

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