REVERSES AND REFINEMENTS OF FÉJER'S FIRST INEQUALITY FOR RIEMANN-STIELTJES INTEGRAL OF CONVEX FUNCTIONS

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Abstract. Let f be a continuous convex function on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ a function of bounded variation with the property that

$$g(a)\le g(t)\le g(b)\ \text{ for all } t\in[a,b],$$

then we have the following refinement and reverse of Féjer's first inequality

$$\begin{aligned}0&\le\frac{1}{2}\left[f_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right]\int_a^b\left|t-\frac{a+b}{2}\right|dg(t)\\&\le\int_a^b\breve{f}(t)\,dg(t)-f\left(\frac{a+b}{2}\right)[g(b)-g(a)]\\&\le\frac{1}{2}\left[f'_-(b)-f'_+(a)\right]\int_a^b\left|t-\frac{a+b}{2}\right|dg(t),\end{aligned}$$

where $\breve{f}(t):=\frac{1}{2}[f(t)+f(a+b-t)]$. Applications for functions of selfadjoint operators in Hilbert spaces with examples for power function and logarithm are also provided.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(t)dt\le\frac{f(a)+f(b)}{2},\quad a,\ b\in\mathbb{R},\ a<b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [9]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [2]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [9]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the *Hermite-Hadamard inequality*. For a monograph devoted to this result see [6]. The recent survey paper [5] provides other related results.

Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ and assume that $f'_+(a)$ and $f'_-(b)$ are finite. We recall the following improvement and reverse inequality for

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the first Hermite-Hadamard result that has been established in [3]

$$0 \leq \frac{1}{8}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right](b-a) \tag{1.2}$$
$$\leq \frac{1}{b-a}\int_{a}^{b} f(t)\,dt - f\left(\frac{a+b}{2}\right) \leq \frac{1}{8}(b-a)\left[f'_{-}(b)-f'_{+}(a)\right].$$

The constant $\frac{1}{8}$ is best possible in both sides of (1.2).

By the convexity of $f:[a,b]\to\mathbb{R}$ we have

$$f\left(\frac{a+b}{2}\right) \leq \breve{f}(t) := \frac{1}{2}\left[f(t)+f(a+b-t)\right] \leq \frac{1}{2}\left[f(a)+f(b)\right] \tag{1.3}$$

for all $t\in[a,b]$.

If $g:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b \breve{f}(t)\,dg(t)$ exists and by using the properties of Riemann-Stieltjes integral for monotonic nondreasing integrators, we deduce from (1.3) that the following Féjer's type inequalities for Riemann-Stieltjes integral

$$f\left(\frac{a+b}{2}\right)\left[g(b)-g(a)\right] \leq \int_a^b \breve{f}(t)\,dg(t) \leq \frac{1}{2}\left[f(a)+f(b)\right]\left[g(b)-g(a)\right]. \tag{1.4}$$

If g is expressed by a Riemann-Stieltjes integral $g(t)=\int_a^t p(s)\,dv(s)$, with g is monotonic nondreasing, then (1.4) becomes

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b p(s)\,dv(s) &\leq \int_a^b \breve{f}(t)\,p(t)\,dv(t) \\ &\leq \frac{1}{2}\left[f(a)+f(b)\right]\int_a^b p(s)\,dv(s). \end{aligned} \tag{1.5}$$

If, for instance, p is continuous and nonnegative on $[a,b]$ and v is monotonic nondreasing on $[a,b]$, then the inequality (1.5) holds true.

Motivated by the above results, in this paper we establish some refinements and reverses of the first Féjer's inequality. Applications for functions of selfadjoint operators in Hilbert spaces with examples for power function and logarithm are also provided.

2. The Main Results

Following Roberts and Varberg [10, p. 5], we recall that if $f:I\to\mathbb{R}$ is a convex function, then for any $x_0\in\mathring{I}$ (the interior of the interval I) the limits

$$f'_{-}(x_0) := \lim_{x\to x_0-}\frac{f(x)-f(x_0)}{x-x_0} \text{ and } f'_{+}(x_0) := \lim_{x\to x_0+}\frac{f(x)-f(x_0)}{x-x_0}$$

exists and $f'_{-}(x_0)\leq f'_{+}(x_0)$. The functions f'_{-} and f'_{+} are monotonic nondecreasing on $\mathring{I}$ and this property can be extended to the whole interval I (see [10, p. 7]).

From the monotonicity of the lateral derivatives f'_{-} and f'_{+} we also have *the gradient inequality*

$$f'_{-}(x)(x-y) \geq f(x)-f(y) \geq f'_{+}(y)(x-y)$$

for any $x,\,y\in\mathring{I}$.

If $I=[a,b]$, then at the end points we also have the inequalities

$$f(x)-f(a) \geq f'_{+}(a)(x-a)$$

for any $x \in (a, b]$ and

$$f(y) - f(b) \geq f'_-(b)(y - b)$$

for any $y \in [a, b)$.

We have:

Theorem 1. *Let f be a continuous convex function on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ a function of bounded variation with the property that*

$$g(a) \leq g(t) \leq g(b) \ \text{ for all } t \in [a, b], \tag{2.1}$$

then

$$\begin{aligned} & f'_+\left(\frac{a+b}{2}\right)\left[\frac{1}{2}(b-a)g(b) - \int_{\frac{a+b}{2}}^{b} g(t)\,dt\right] \\ & - f'_-\left(\frac{a+b}{2}\right)\left[\int_a^{\frac{a+b}{2}} g(t)\,dt - \frac{1}{2}(b-a)g(a)\right] \\ & \leq \int_a^b f(t)\,dg(t) - f\left(\frac{a+b}{2}\right)[g(b) - g(a)] \\ & \leq f'_-(b)\left[\frac{1}{2}(b-a)g(b) - \int_{\frac{a+b}{2}}^{b} g(t)\,dt\right] \\ & - f'_+(a)\left[\int_a^{\frac{a+b}{2}} g(t)\,dt - \frac{1}{2}(b-a)g(a)\right] \end{aligned} \tag{2.2}$$

or, equivalently,

$$\begin{aligned} & f'_+\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^{b}\left(t - \frac{a+b}{2}\right)dg(t) \\ & - f'_-\left(\frac{a+b}{2}\right)\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2} - t\right)dg(t) \\ & \leq \int_a^b f(t)\,dg(t) - f\left(\frac{a+b}{2}\right)[g(b) - g(a)] \\ & \leq f'_-(b)\int_{\frac{a+b}{2}}^{b}\left(t - \frac{a+b}{2}\right)dg(t) \\ & - f'_+(a)\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2} - t\right)dg(t). \end{aligned} \tag{2.3}$$

Proof. Using the integration by parts formula for Riemann-Stieltjes integral, we have

$$\begin{aligned} & \int_{\frac{a+b}{2}}^{b} f'(t)[g(b) - g(t)]\,dt \\ & = f(t)[g(b) - g(t)]\Big|_{\frac{a+b}{2}}^{b} - \int_{\frac{a+b}{2}}^{b} f(t)\,d[g(b) - g(t)] \end{aligned}$$

$$= -f\left(\frac{a+b}{2}\right)\left[g(b) - g\left(\frac{a+b}{2}\right)\right] + \int_{\frac{a+b}{2}}^{b} f(t)\, dg(t)$$
$$= \int_{\frac{a+b}{2}}^{b} f(t)\, dg(t) - f\left(\frac{a+b}{2}\right)\left[g(b) - g\left(\frac{a+b}{2}\right)\right]$$

and

$$\int_a^{\frac{a+b}{2}} f'(t)\,[g(t) - g(a)]\, dt$$
$$= f(t)\,[g(t) - g(a)]\Big|_a^{\frac{a+b}{2}} - \int_a^{\frac{a+b}{2}} f(t)\, d\,[g(t) - g(a)]\, dt$$
$$= f\left(\frac{a+b}{2}\right)\left[g\left(\frac{a+b}{2}\right) - g(a)\right] - \int_a^{\frac{a+b}{2}} f(t)\, dg(t).$$

If we subtract the second equality from the first, then we get

$$\int_{\frac{a+b}{2}}^{b} f'(t)\,[g(b) - g(t)]\, dt - \int_a^{\frac{a+b}{2}} f'(t)\,[g(t) - g(a)]\, dt$$
$$= \int_{\frac{a+b}{2}}^{b} f(t)\, dg(t) - f\left(\frac{a+b}{2}\right)\left[g(b) - g\left(\frac{a+b}{2}\right)\right]$$
$$- f\left(\frac{a+b}{2}\right)\left[g\left(\frac{a+b}{2}\right) - g(a)\right] + \int_a^{\frac{a+b}{2}} f(t)\, dg(t)$$
$$= \int_a^b f(t)\, dg(t) - f\left(\frac{a+b}{2}\right)[g(b) - g(a)],$$

namely, the following equality of interest

$$\int_a^b f(t)\, dg(t) - f\left(\frac{a+b}{2}\right)[g(b) - g(a)] \tag{2.4}$$
$$= \int_{\frac{a+b}{2}}^{b} f'(t)\,[g(b) - g(t)]\, dt - \int_a^{\frac{a+b}{2}} f'(t)\,[g(t) - g(a)]\, dt.$$

By the monotonicity of the derivative we have

$$f'_+(a) \le f'(t) \le f'_-\left(\frac{a+b}{2}\right),$$

for almost every $t \in \left(a, \frac{a+b}{2}\right)$ and

$$f'_+\left(\frac{a+b}{2}\right) \le f'(t) \le f'_-(b),$$

for almost every $t \in \left(\frac{a+b}{2}, b\right)$.

Since $g(a) \le g(t) \le g(b)$ for all $t \in [a,b]$, then

$$f'_+(a)\,[g(t) - g(a)] \le f'(t)\,[g(t) - g(a)] \le f'_-\left(\frac{a+b}{2}\right)[g(t) - g(a)],$$

for almost every $t \in \left(a, \frac{a+b}{2}\right)$ and

$$f'_+\left(\frac{a+b}{2}\right)[g(b) - g(t)] \le f'(t)\,[g(b) - g(t)] \le f'_-(b)\,[g(b) - g(t)],$$

for almost every $t \in \left(\frac{a+b}{2}, b\right)$.

These imply that

$$f'_+(a) \int_a^{\frac{a+b}{2}} [g(t) - g(a)]\, dt \le \int_a^{\frac{a+b}{2}} f'(t) [g(t) - g(a)]\, dt$$
$$\le f'_-\left(\frac{a+b}{2}\right) \int_a^{\frac{a+b}{2}} [g(t) - g(a)]\, dt,$$

and

$$f'_+\left(\frac{a+b}{2}\right) \int_{\frac{a+b}{2}}^b [g(b) - g(t)]\, dt \le \int_{\frac{a+b}{2}}^b f'(t) [g(b) - g(t)]\, dt$$
$$\le f'_-(b) \int_{\frac{a+b}{2}}^b [g(b) - g(t)]\, dt.$$

Therefore

$$f'_+\left(\frac{a+b}{2}\right) \int_{\frac{a+b}{2}}^b [g(b) - g(t)]\, dt - f'_-\left(\frac{a+b}{2}\right) \int_a^{\frac{a+b}{2}} [g(t) - g(a)]\, dt$$
$$\le \int_{\frac{a+b}{2}}^b f'(t) [g(b) - g(t)]\, dt - \int_a^{\frac{a+b}{2}} f'(t) [g(t) - g(a)]\, dt$$
$$\le f'_-(b) \int_{\frac{a+b}{2}}^b [g(b) - g(t)]\, dt - f'_+(a) \int_a^{\frac{a+b}{2}} [g(t) - g(a)]\, dt,$$

which is equivalent to (2.2).

Observe that

$$\int_{\frac{a+b}{2}}^b g(t)\, dt = g(t)\, t\big|_{\frac{a+b}{2}}^b - \int_{\frac{a+b}{2}}^b t dg(t)$$
$$= g(b)\, b - g\left(\frac{a+b}{2}\right) \frac{a+b}{2} - \int_{\frac{a+b}{2}}^b t dg(t).$$

Then

$$\frac{1}{2}(b-a)\, g(b) - \int_{\frac{a+b}{2}}^b g(t)\, dt$$
$$= \frac{1}{2}(b-a)\, g(b) - g(b)\, b + g\left(\frac{a+b}{2}\right) \frac{a+b}{2} + \int_{\frac{a+b}{2}}^b t dg(t)$$
$$= g\left(\frac{a+b}{2}\right) \frac{a+b}{2} - g(b) \frac{a+b}{2} + \int_{\frac{a+b}{2}}^b t dg(t)$$
$$= \int_{\frac{a+b}{2}}^b t dg(t) - \left[g(b) - g\left(\frac{a+b}{2}\right)\right] \frac{a+b}{2}$$
$$= \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2}\right) dg(t).$$

Also,

$$\int_a^{\frac{a+b}{2}} g(t)\,dt = g(t)\,t\big|_a^{\frac{a+b}{2}} - \int_a^{\frac{a+b}{2}} t dg(t)$$
$$= g\left(\frac{a+b}{2}\right)\frac{a+b}{2} - g(a)\,a - \int_a^{\frac{a+b}{2}} t dg(t),$$

which implies that

$$\int_a^{\frac{a+b}{2}} g(t)\,dt - \frac{1}{2}(b-a)\,g(a)$$
$$= g\left(\frac{a+b}{2}\right)\frac{a+b}{2} - g(a)\,a - \int_a^{\frac{a+b}{2}} t dg(t) - \frac{1}{2}(b-a)\,g(a)$$
$$= \frac{a+b}{2}\left[g\left(\frac{a+b}{2}\right) - g(a)\right] - \int_a^{\frac{a+b}{2}} t dg(t)$$
$$= \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right) dg(t).$$

By using (2.2) we get (2.3). □

Remark 1. *If f is convex and differentiable at $\frac{a+b}{2}$, then the first inequality in (2.3) becomes*

$$f'\left(\frac{a+b}{2}\right)\int_a^b \left(t - \frac{a+b}{2}\right) dg(t) \leq \int_a^b f(t)\,dg(t) - f\left(\frac{a+b}{2}\right)[g(b) - g(a)], \tag{2.5}$$

provided g satisfies the condition (2.1).

Corollary 1. *Let $p : [a,b] \to \mathbb{R}$ be continuous and v of bounded variation on $[a,b]$ and such that*

$$0 \leq \int_a^t p(s)\,dv(s) \leq \int_a^b p(s)\,dv(s) \text{ for all } t \in [a,b], \tag{2.6}$$

then for any f a continuous convex function on $[a,b]$, we have

$$\begin{aligned} &f'_+\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2}\right) p(t)\,dv(t) \\ &- f'_-\left(\frac{a+b}{2}\right)\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right) p(t)\,dv(t) \\ &\leq \int_a^b f(t)\,p(t)\,dv(t) - f\left(\frac{a+b}{2}\right)\int_a^b p(s)\,dv(s) \\ &\leq f'_-(b)\int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2}\right) p(t)\,dv(t) \\ &- f'_+(a)\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right) p(t)\,dv(t). \end{aligned} \tag{2.7}$$

If f is differentiable in $\frac{a+b}{2}$, then

$$\begin{aligned} &f'\left(\frac{a+b}{2}\right)\int_a^b\left(t-\frac{a+b}{2}\right)p(t)\,dv(t) \\ &\leq \int_a^b f(t)\,p(t)\,dv(t) - f\left(\frac{a+b}{2}\right)\int_a^b p(s)\,dv(s) \\ &\leq f'_-(b)\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)p(t)\,dv(t) \\ &- f'_+(a)\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)p(t)\,dv(t). \end{aligned} \tag{2.8}$$

The proof follows by (2.3) for $g(t)=\int_a^t p(s)\,dv(s)$ and on observing that, by the properties of the Riemann-Stieltjes integral [1, p. 158-159], we have

$$\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)dg(t)=\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)p(t)\,dv(t),$$
$$\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)dg(t)=\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)p(t)\,dv(t)$$

and

$$\int_a^b f(t)\,dg(t)=\int_a^b f(t)\,p(t)\,dv(t).$$

Corollary 2. *Let $p:[a,b]\to[0,\infty)$ be continuous and v monotonic nondecreasing on $[a,b]$, then the inequalities (2.7) and (2.8) hold true.*

We provide now some Riemann-Stieltjes integral inequalities for the symmetric transform of a convex function $f:[a,b]\to\mathbb{R}$ defined by

$$\breve{f}(t):=\frac{1}{2}\left[f(t)+f(a+b-t)\right].$$

Theorem 2. *Let f be a continuous convex function on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ a function of bounded variation with the property (2.1), then*

$$\begin{aligned} 0 &\leq \frac{1}{2}\left[f_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right]\int_a^b\left|t-\frac{a+b}{2}\right|dg(t) \\ &\leq \int_a^b \breve{f}(t)\,dg(t)-f\left(\frac{a+b}{2}\right)[g(b)-g(a)] \\ &\leq \frac{1}{2}\left[f'_-(b)-f'_+(a)\right]\int_a^b\left|t-\frac{a+b}{2}\right|dg(t). \end{aligned} \tag{2.9}$$

Proof. The function $h:[a,b]\to\mathbb{R}$ defined by $h(t)=f(a+b-t)$ is convex and

$$\begin{aligned} h_+(a) &= \lim_{s\to 0+}\frac{h(a+s)-h(a)}{s}=\lim_{s\to 0+}\frac{f(b-s)-f(b)}{s} \\ &= -\lim_{s\to 0+}\frac{f(b-s)-f(b)}{-s}=-\lim_{u\to 0-}\frac{f(b+u)-f(b)}{u} \\ &= -f'_-(b). \end{aligned}$$

Similarly

$$h_-\left(\frac{a+b}{2}\right)=-f_+\left(\frac{a+b}{2}\right),\ h_+\left(\frac{a+b}{2}\right)=-f_-\left(\frac{a+b}{2}\right),$$

and

$$h_-(b)=-f'_+(a).$$

By writing the inequality (2.3) for the function h we have

$$\begin{aligned}&h'_+\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)dg(t)\\&-h'_-\left(\frac{a+b}{2}\right)\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)dg(t)\\&\leq\int_a^b h(t)\,dg(t)-h\left(\frac{a+b}{2}\right)[g(b)-g(a)]\\&\leq h'_-(b)\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)dg(t)\\&-h'_+(a)\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)dg(t),\end{aligned}$$

namely

$$\begin{aligned}&f_+\left(\frac{a+b}{2}\right)\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)dg(t)\\&-f_-\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)dg(t)\\&\leq\int_a^b f(a+b-t)\,dg(t)-f\left(\frac{a+b}{2}\right)[g(b)-g(a)]\\&\leq f'_-(b)\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)dg(t)\\&-f'_+(a)\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)dg(t).\end{aligned}$$

If we add this inequality and (2.3) and divide by 2, then we get

$$\begin{aligned}&\frac{1}{2}\left[f_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right]\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)dg(t)\\&+\frac{1}{2}\left[f'_+\left(\frac{a+b}{2}\right)-f_-\left(\frac{a+b}{2}\right)\right]\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)dg(t)\\&\leq\int_a^b\breve{f}(t)\,dg(t)-f\left(\frac{a+b}{2}\right)[g(b)-g(a)]\\&\leq\frac{1}{2}\left[f'_-(b)-f'_+(a)\right]\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)dg(t)\\&+\frac{1}{2}\left[f'_-(b)-f'_+(a)\right]\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)dg(t),\end{aligned}$$

that is equivalent to the second and third inequality in (2.9).

Since

$$0 \leq \frac{1}{2}(b-a)g(b) - \int_{\frac{a+b}{2}}^{b} g(t)\,dt = \int_{\frac{a+b}{2}}^{b} \left(t - \frac{a+b}{2}\right) dg(t).$$

and

$$0 \leq \int_{a}^{\frac{a+b}{2}} g(t)\,dt - \frac{1}{2}(b-a)g(a) = \int_{a}^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right) dg(t).$$

hence by addition we get

$$0 \leq \int_{\frac{a+b}{2}}^{b} \left(t - \frac{a+b}{2}\right) dg(t) + \int_{a}^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right) dg(t) = \int_{a}^{b} \left|t - \frac{a+b}{2}\right| dg(t)$$

which proves the first inequality in □

Corollary 3. *Let $p : [a,b] \to \mathbb{R}$ be continuous and v of bounded variation on $[a,b]$ and such that the condition (2.6) holds, then for any f a continuous convex function on $[a,b]$, we have*

$$0 \leq \frac{1}{2}\left[f_{+}\left(\frac{a+b}{2}\right) - f'_{-}\left(\frac{a+b}{2}\right)\right] \int_{a}^{b} \left|t - \frac{a+b}{2}\right| p(t)\,dv(t) \tag{2.10}$$
$$\leq \int_{a}^{b} \breve{f}(t)\,p(t)\,dv(t) - f\left(\frac{a+b}{2}\right) \int_{a}^{b} p(t)\,dv(t)$$
$$\leq \frac{1}{2}\left[f'_{-}(b) - f'_{+}(a)\right] \int_{a}^{b} \left|t - \frac{a+b}{2}\right| p(t)\,dv(t).$$

3. Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda \in \mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \leq \lambda, \\ \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$E_\lambda := \varphi_\lambda(A) \tag{3.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [8, p. 256]:

Theorem 3 (Spectral Representation Theorem)**.** *Let A be a bounded selfadjoint operator on the Hilbert space H and let $m = \min\{\lambda \,|\lambda \in \operatorname{Sp}(A)\} =: \min \operatorname{Sp}(A)$ and $M = \max\{\lambda \,|\lambda \in \operatorname{Sp}(A)\} =: \max \operatorname{Sp}(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda \in \mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_\lambda \leq E_{\lambda'}$ *for* $\lambda \leq \lambda'$;

b) $E_{m-0} = 0$, $E_M = 1_H$ *and* $E_{\lambda+0} = E_\lambda$ *for all* $\lambda \in \mathbb{R}$;

c) *We have the representation*

$$A = \int_{m-0}^{M} \lambda dE_\lambda .$$

More generally, for every continuous complex-valued function φ *defined on* $\mathbb{R}$ *there exists a unique operator* $\varphi(A) \in \mathcal{B}(H)$ *such that for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *satisfying the inequality*

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda_k') \left[E_{\lambda_k} - E_{\lambda_{k-1}} \right] \right\| \leq \varepsilon$$

whenever

$$\begin{cases} \lambda_0 < m = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = M, \\ \\ \lambda_k - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\ \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \leq k \leq n \end{cases}$$

this means that

$$\varphi(A) = \int_{m-0}^{M} \varphi(\lambda) \, dE_\lambda, \tag{3.2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 4. *With the assumptions of Theorem 3 for* A*,* E_λ *and* φ *we have the representations*

$$\varphi(A) x = \int_{m-0}^{M} \varphi(\lambda) \, dE_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi(A) x, y \rangle = \int_{m-0}^{M} \varphi(\lambda) \, d \langle E_\lambda x, y \rangle \quad \text{for all } x, \ y \in H. \tag{3.3}$$

In particular,

$$\langle \varphi(A) x, x \rangle = \int_{m-0}^{M} \varphi(\lambda) \, d \langle E_\lambda x, x \rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(A) x\|^2 = \int_{m-0}^{M} |\varphi(\lambda)|^2 \, d \|E_\lambda x\|^2 \quad \text{for all } x \in H.$$

We consider the continuous functions $\left(\ell - \frac{m+M}{2}\right)_+$ and $\left(\frac{m+M}{2} - \ell\right)_+$ defined by

$$\left(\ell - \frac{m+M}{2}\right)_+ (t) = \begin{cases} t - \frac{m+M}{2}, \ t \geq \frac{m+M}{2}, \\ \\ 0, \ t < \frac{m+M}{2} \end{cases}$$

and

$$\left(\frac{m+M}{2} - \ell\right)_+ (t) = \begin{cases} \frac{m+M}{2} - t, \ t \leq \frac{m+M}{2}, \\ \\ 0, \ t > \frac{m+M}{2}. \end{cases}$$

Let A be a bounded selfadjoint operator on the Hilbert space H and let $m = \min\{\lambda \,|\lambda \in \operatorname{Sp}(A)\} =: \min \operatorname{Sp}(A)$ and $M = \max\{\lambda \,|\lambda \in \operatorname{Sp}(A)\} =: \max \operatorname{Sp}(A)$.

Also, assume that $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A. Then we can define the operators

$$\left(A-\frac{m+M}{2}1_H\right)_+ := \int_{m-0}^{M}\left(\ell-\frac{m+M}{2}\right)_+(\lambda)\,dE_\lambda$$

and

$$\left(\frac{m+M}{2}1_H-A\right)_+ := \int_{m-0}^{M}\left(\frac{m+M}{2}-\ell\right)_+(\lambda)\,dE_\lambda.$$

where p is continuous on an open interval containing $[m,M]$.

We can state the following result for functions of selfadjoint operators:

Theorem 4. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $m=\min\{\lambda\,|\lambda\in\operatorname{Sp}(A)\}=:\min\operatorname{Sp}(A)$ and $M=\max\{\lambda\,|\lambda\in\operatorname{Sp}(A)\}=:\max\operatorname{Sp}(A)$. Also, assume that $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A and $f:I\to\mathbb{R}$ is continuous convex on I, $[m,M]\subset\mathring{I}$ (the interior of I), then*

$$\begin{aligned}(3.4)\quad & f'_+\left(\frac{m+M}{2}\right)\left(A-\frac{m+M}{2}1_H\right)_+ - f'_-\left(\frac{m+M}{2}\right)\left(\frac{m+M}{2}1_H-A\right)_+\\ &\le f(A)-f\left(\frac{m+M}{2}\right)1_H\\ &\le f'_-(M)\left(A-\frac{m+M}{2}1_H\right)_+ - f'_+(m)\left(\frac{m+M}{2}1_H-A\right)_+.\end{aligned}$$

If f is differentiable in $\frac{m+M}{2}$, then

$$\begin{aligned}(3.5)\quad & f'\left(\frac{m+M}{2}\right)\left(A-\frac{m+M}{2}1_H\right)\\ &\le f(A)-f\left(\frac{m+M}{2}\right)1_H\\ &\le f'_-(M)\left(A-\frac{m+M}{2}1_H\right)_+ - f'_+(m)\left(\frac{m+M}{2}1_H-A\right)_+.\end{aligned}$$

Proof. Using the inequality (2.3), we have

$$\begin{aligned}& f'_+\left(\frac{m-\varepsilon+M}{2}\right)\int_{\frac{m-\varepsilon+M}{2}}^{M}\left(t-\frac{m-\varepsilon+M}{2}\right)d\langle E_t x,x\rangle\\ &-f'_-\left(\frac{m-\varepsilon+M}{2}\right)\int_{m-\varepsilon}^{\frac{m-\varepsilon+M}{2}}\left(\frac{m-\varepsilon+M}{2}-t\right)d\langle E_t x,x\rangle\\ &\le\int_{m-\varepsilon}^{M}f(t)\,d\langle E_t x,x\rangle - f\left(\frac{m-\varepsilon+M}{2}\right)\int_{m-\varepsilon}^{M}d\langle E_t x,x\rangle\\ &\le f'_-(M)\int_{\frac{m-\varepsilon+M}{2}}^{M}\left(t-\frac{m-\varepsilon+M}{2}\right)d\langle E_t x,x\rangle\\ &-f'_+(m-\varepsilon)\int_{m-\varepsilon}^{\frac{m-\varepsilon+M}{2}}\left(\frac{m-\varepsilon+M}{2}-t\right)d\langle E_t x,x\rangle.\end{aligned}$$

for small $\varepsilon>0$ and for any $x\in H$.

Taking the limit over $\varepsilon \to 0+$, we get

$$\begin{aligned}
& f'_{+}\left(\frac{m+M}{2}\right)\int_{\frac{m+M}{2}}^{M}\left(t-\frac{m+M}{2}\right)d\left\langle E_{t}x,x\right\rangle \\
& - f'_{-}\left(\frac{m+M}{2}\right)\int_{m-0}^{\frac{m+M}{2}}\left(\frac{m+M}{2}-t\right)d\left\langle E_{t}x,x\right\rangle \\
& \leq \int_{m-0}^{M} f(t)\,d\left\langle E_{t}x,x\right\rangle - f\left(\frac{m+M}{2}\right)\int_{m-0}^{M} d\left\langle E_{t}x,x\right\rangle \\
& \leq f'_{-}(M)\int_{\frac{m+M}{2}}^{M}\left(t-\frac{m+M}{2}\right)d\left\langle E_{t}x,x\right\rangle \\
& - f'_{+}(m)\int_{m-0}^{\frac{m+M}{2}}\left(\frac{m+M}{2}-t\right)d\left\langle E_{t}x,x\right\rangle ,
\end{aligned}$$

which is equivalent to the operator inequality (3.4). □

By utilising the Riemann-Stieltjes integral inequality (2.9) we can also prove in a similar way the result:

Theorem 5. *With the assumptions of Theorem 4 we have*

$$\begin{aligned}
0 & \leq \frac{1}{2}\left[f_{+}\left(\frac{m+M}{2}\right)-f'_{-}\left(\frac{m+M}{2}\right)\right]\left|A-\frac{m+M}{2}1_{H}\right| \\
& \leq \frac{1}{2}\left[f(A)+f((m+M)1_{H}-A)\right]-f\left(\frac{m+M}{2}\right)1_{H} \\
& \leq \frac{1}{2}\left[f'_{-}(M)-f'_{+}(m)\right]\left|A-\frac{m+M}{2}1_{H}\right| .
\end{aligned} \tag{3.6}$$

If f is differentiable in $\frac{m+M}{2}$, then

$$\begin{aligned}
0 & \leq \frac{1}{2}\left[f(A)+f((m+M)1_{H}-A)\right]-f\left(\frac{m+M}{2}\right)1_{H} \\
& \leq \frac{1}{2}\left[f'_{-}(M)-f'_{+}(m)\right]\left|A-\frac{m+M}{2}1_{H}\right| .
\end{aligned} \tag{3.7}$$

Consider the convex function $f(t)=-\ln t$, $t>0$ and assume that $0<m1_{H}\leq A\leq M1_{H}<\infty$, then by (3.5) we have

$$\begin{aligned}
& \frac{2}{m+M}\left(\frac{m+M}{2}1_{H}-A\right) \\
& \leq \ln\left(\frac{m+M}{2}\right)1_{H}-\ln A \\
& \leq \frac{1}{m}\left(\frac{m+M}{2}1_{H}-A\right)_{+}-\frac{1}{M}\left(A-\frac{m+M}{2}1_{H}\right)_{+}
\end{aligned} \tag{3.8}$$

and by (3.7)

$$\begin{aligned}
0 & \leq \ln\left(\frac{m+M}{2}\right)1_{H}-\frac{1}{2}\ln\left[A\left((m+M)1_{H}-A\right)\right] \\
& \leq \frac{1}{2}\frac{M-m}{mM}\left|A-\frac{m+M}{2}1_{H}\right| .
\end{aligned} \tag{3.9}$$

Also, consider the convex function $f(t)=t^{\alpha}$, $t>0$ and $\alpha\in(-\infty,0)\cup[1,\infty)$, and assume that $0<m1_H\le A\le M1_H<\infty$, then by (3.5) we get

$$\begin{aligned}\alpha\left(\frac{m+M}{2}\right)^{\alpha-1}&\left(A-\frac{m+M}{2}1_H\right)\\&\le A^{\alpha}-\left(\frac{m+M}{2}\right)^{\alpha}1_H\\&\le\alpha\left[M^{\alpha-1}\left(A-\frac{m+M}{2}1_H\right)_+-m^{\alpha-1}\left(\frac{m+M}{2}1_H-A\right)_+\right]\end{aligned}\tag{3.10}$$

and by (3.7) we get

$$\begin{aligned}0&\le\frac{1}{2}\left[A^{\alpha}+((m+M)1_H-A)^{\alpha}\right]-\left(\frac{m+M}{2}\right)^{\alpha}1_H\\&\le\frac{1}{2}\alpha\left(M^{\alpha-1}-m^{\alpha-1}\right)\left|A-\frac{m+M}{2}1_H\right|.\end{aligned}\tag{3.11}$$

For $\alpha=-1$ we get

$$\begin{aligned}\left(\frac{m+M}{2}\right)^{-2}&\left(\frac{m+M}{2}1_H-A\right)\\&\le A^{-1}-\left(\frac{m+M}{2}\right)^{-1}1_H\\&\le m^{-2}\left(\frac{m+M}{2}1_H-A\right)_+-M^{-2}\left(A-\frac{m+M}{2}1_H\right)_+\end{aligned}\tag{3.12}$$

and by (3.7) we get

$$\begin{aligned}0&\le\frac{1}{2}\left[A^{-1}+((m+M)1_H-A)^{-1}\right]-\left(\frac{m+M}{2}\right)^{-1}1_H\\&\le\frac{1}{2}\frac{M^2-m^2}{m^2M^2}\left|A-\frac{m+M}{2}1_H\right|.\end{aligned}\tag{3.13}$$

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