REVERSES AND REFINEMENTS OF FÉJER'S SECOND INEQUALITY FOR RIEMANN-STIELTJES INTEGRAL OF CONVEX FUNCTIONS

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Abstract. Let f be a continuous convex function on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ a function of bounded variation with the property that

$$g(t)\geq\frac{g(a)+g(b)}{2}\text{ for }t\in\left[\frac{a+b}{2},b\right]$$

and

$$\frac{g(a)+g(b)}{2}\geq g(t)\text{ for }t\in\left[a,\frac{a+b}{2}\right],$$

then we have the following refinement and reverse of Féjer's second inequality

$$\begin{aligned}0&\leq\frac{1}{2}\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right]\\&\quad\times\int_a^b\left[\frac{1}{2}(b-a)-\left|\frac{a+b}{2}-t\right|\right]dg(t)\\&\leq\frac{f(b)+f(a)}{2}[g(b)-g(a)]-\int_a^b\breve{f}(t)\,dg(t)\\&\leq\frac{1}{2}\left[f'_-(b)-f'_+(a)\right]\\&\quad\times\int_a^b\left[\frac{1}{2}(b-a)-\left|\frac{a+b}{2}-t\right|\right]dg(t)\end{aligned}$$

where $\breve{f}(t):=\frac{1}{2}[f(t)+f(a+b-t)]$. Some similar results are also provided.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)dt\leq\frac{f(a)+f(b)}{2},\quad a,\ b\in\mathbb{R},\ a<b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [8]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [2]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [8]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the *Hermite-Hadamard inequality.* For a monograph devoted to this result see [6]. The recent survey paper [5] provides other related results.

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Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ and assume that $f'_{+}(a)$ and $f'_{-}(b)$ are finite. We recall the following improvement and reverse inequality for the first Hermite-Hadamard result that has been established in [3]

$$\begin{aligned}(1.2)\qquad 0 &\leq \frac{1}{8}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right](b-a)\\ &\leq \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\leq \frac{1}{8}(b-a)\left[f'_{-}(b)-f'_{+}(a)\right].\end{aligned}$$

The constant $\frac{1}{8}$ is best possible in both sides of (1.2).

By the convexity of $f:[a,b]\to\mathbb{R}$ we have

$$(1.3)\qquad f\left(\frac{a+b}{2}\right)\leq \breve{f}(t):=\frac{1}{2}\left[f(t)+f(a+b-t)\right]\leq\frac{1}{2}\left[f(a)+f(b)\right]$$

for all $t\in[a,b]$.

If $g:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b \breve{f}(t)\,dg(t)$ exists and by using the properties of Riemann-Stieltjes integral for monotonic nondreasing integrators, we deduce from (1.3) that the following Féjer's type inequalities for Riemann-Stieltjes integral

$$(1.4)\quad f\left(\frac{a+b}{2}\right)[g(b)-g(a)]\leq\int_a^b \breve{f}(t)\,dg(t)\leq\frac{1}{2}\left[f(a)+f(b)\right][g(b)-g(a)].$$

If g is expressed by a Riemann-Stieltjes integral $g(t)=\int_a^t p(s)\,dv(s)$, with g is monotonic nondreasing, then (1.4) becomes

$$\begin{aligned}(1.5)\qquad f\left(\frac{a+b}{2}\right)\int_a^b p(s)\,dv(s) &\leq \int_a^b \breve{f}(t)\,p(t)\,dv(t)\\ &\leq \frac{1}{2}\left[f(a)+f(b)\right]\int_a^b p(s)\,dv(s).\end{aligned}$$

If, for instance, p is continuous and nonnegative on $[a,b]$ and v is monotonic nondreasing on $[a,b]$, then the inequality (1.5) holds true.

Motivated by the above results, in this paper we establish some refinements and reverses of the second Féjer's inequality. Some related results are also provided.

2. The Results

Following Roberts and Varberg [9, p. 5], we recall that if $f:I\to\mathbb{R}$ is a convex function, then for any $x_0\in\mathring{I}$ (the interior of the interval I) the limits

$$f'_{-}(x_0):=\lim_{x\to x_0-}\frac{f(x)-f(x_0)}{x-x_0}\text{ and }f'_{+}(x_0):=\lim_{x\to x_0+}\frac{f(x)-f(x_0)}{x-x_0}$$

exists and $f'_{-}(x_0)\leq f'_{+}(x_0)$. The functions f'_{-} and f'_{+} are monotonic nondecreasing on $\mathring{I}$ and this property can be extended to the whole interval I (see [9, p. 7]).

From the monotonicity of the lateral derivatives f'_{-} and f'_{+} we also have *the gradient inequality*

$$f'_{-}(x)(x-y)\geq f(x)-f(y)\geq f'_{+}(y)(x-y)$$

for any $x,y\in\mathring{I}$.

If $I=[a,b]$, then at the end points we also have the inequalities

$$f(x)-f(a)\geq f'_{+}(a)(x-a)$$

for any $x \in (a,b]$ and

$$f(y) - f(b) \geq f'_{-}(b)(y-b)$$

for any $y \in [a,b)$.

We also have:

Theorem 1. *Let f be a continuous convex function on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ a function of bounded variation with the property that*

$$g(t) \geq \frac{g(a)+g(b)}{2} \text{ for } t \in \left[\frac{a+b}{2}, b\right] \tag{2.1}$$

and

$$\frac{g(a)+g(b)}{2} \geq g(t) \text{ for } t \in \left[a, \frac{a+b}{2}\right], \tag{2.2}$$

then

$$\begin{aligned} & f'_{+}\left(\frac{a+b}{2}\right) \int_{\frac{a+b}{2}}^{b} \left(g(t) - \frac{g(a)+g(b)}{2}\right) dt \\ & - f'_{-}\left(\frac{a+b}{2}\right) \int_{a}^{\frac{a+b}{2}} \left(\frac{g(a)+g(b)}{2} - g(t)\right) dt \\ & \leq \frac{f(b)+f(a)}{2} [g(b)-g(a)] - \int_{a}^{b} f(t)\, dg(t) \\ & \leq f'_{-}(b) \int_{\frac{a+b}{2}}^{b} \left(g(t) - \frac{g(a)+g(b)}{2}\right) dt \\ & - f'_{+}(a) \int_{a}^{\frac{a+b}{2}} \left(\frac{g(a)+g(b)}{2} - g(t)\right) dt \end{aligned} \tag{2.3}$$

or, equivalently

$$\begin{aligned} & f'_{+}\left(\frac{a+b}{2}\right) \left[\left(\frac{b-a}{2}\right) \frac{g(b)-g(a)}{2} - \int_{\frac{a+b}{2}}^{b} \left(t - \frac{a+b}{2}\right) dg(t)\right] \\ & - f'_{-}\left(\frac{a+b}{2}\right) \left[\left(\frac{b-a}{2}\right) \frac{g(b)-g(a)}{2} - \int_{a}^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right) dg(t)\right] \\ & \leq \frac{f(b)+f(a)}{2} [g(b)-g(a)] - \int_{a}^{b} f(t)\, dg(t) \\ & \leq f'_{-}(b) \left[\left(\frac{b-a}{2}\right) \frac{g(b)-g(a)}{2} - \int_{\frac{a+b}{2}}^{b} \left(t - \frac{a+b}{2}\right) dg(t)\right] \\ & - f'_{+}(a) \left[\left(\frac{b-a}{2}\right) \frac{g(b)-g(a)}{2} - \int_{a}^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right) dg(t)\right]. \end{aligned} \tag{2.4}$$

Proof. Using the integration by parts for the Riemann-Stieltjes integral, we have

$$\begin{aligned}
&\int_a^b \left(g(t)-\frac{g(a)+g(b)}{2}\right) f'(t)\,dt \\
&= \left(g(t)-\frac{g(a)+g(b)}{2}\right) f(t)\Big|_a^b - \int_a^b f(t)\,d\left(g(t)-\frac{g(a)+g(b)}{2}\right) \\
&= \frac{f(b)+f(a)}{2}\left[g(b)-g(a)\right] - \int_a^b f(t)\,dg(t),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{f(b)+f(a)}{2}\left[g(b)-g(a)\right] - \int_a^b f(t)\,dg(t) \\
&= \int_{\frac{a+b}{2}}^b \left(g(t)-\frac{g(a)+g(b)}{2}\right) f'(t)\,dt \\
&- \int_a^{\frac{a+b}{2}} \left(\frac{g(a)+g(b)}{2}-g(t)\right) f'(t)\,dt.
\end{aligned} \tag{2.5}$$

By the convexity of f and the conditions (2.1) and (2.2) we get

$$\begin{aligned}
&f'_+\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^b \left(g(t)-\frac{g(a)+g(b)}{2}\right) dt \\
&\le \int_{\frac{a+b}{2}}^b \left(g(t)-\frac{g(a)+g(b)}{2}\right) f'(t)\,dt \\
&\le f'_-(b)\int_{\frac{a+b}{2}}^b \left(g(t)-\frac{g(a)+g(b)}{2}\right) dt
\end{aligned}$$

and

$$\begin{aligned}
&f'_+(a)\int_a^{\frac{a+b}{2}} \left(\frac{g(a)+g(b)}{2}-g(t)\right) dt \\
&\le \int_a^{\frac{a+b}{2}} \left(\frac{g(a)+g(b)}{2}-g(t)\right) f'(t)\,dt \\
&\le f'_-\left(\frac{a+b}{2}\right)\int_a^{\frac{a+b}{2}} \left(\frac{g(a)+g(b)}{2}-g(t)\right) dt.
\end{aligned}$$

These imply that

$$
\begin{aligned}
& f_{+}^{\prime}\left(\frac{a+b}{2}\right) \int_{\frac{a+b}{2}}^{b}\left(g(t)-\frac{g(a)+g(b)}{2}\right) dt \\
& -f_{-}^{\prime}\left(\frac{a+b}{2}\right) \int_{a}^{\frac{a+b}{2}}\left(\frac{g(a)+g(b)}{2}-g(t)\right) dt \\
& \leq \int_{\frac{a+b}{2}}^{b}\left(g(t)-\frac{g(a)+g(b)}{2}\right) f^{\prime}(t)\, dt-\int_{a}^{\frac{a+b}{2}}\left(\frac{g(a)+g(b)}{2}-g(t)\right) f^{\prime}(t)\, dt \\
& \leq f_{-}^{\prime}(b) \int_{\frac{a+b}{2}}^{b}\left(g(t)-\frac{g(a)+g(b)}{2}\right) dt \\
& -f_{+}^{\prime}(a) \int_{a}^{\frac{a+b}{2}}\left(\frac{g(a)+g(b)}{2}-g(t)\right) dt
\end{aligned}
$$

and by (2.5) we get (2.3).

Using integration by parts, we have

$$
\begin{aligned}
& \int_{\frac{a+b}{2}}^{b}\left(g(t)-\frac{g(a)+g(b)}{2}\right) dt-\left(\frac{b-a}{2}\right) \frac{g(b)-g(a)}{2} \\
& =\int_{\frac{a+b}{2}}^{b}\left(g(t)-\frac{g(a)+g(b)}{2}-\frac{g(b)-g(a)}{2}\right) dt \\
& =\int_{\frac{a+b}{2}}^{b}(g(t)-g(b))\, dt=\int_{\frac{a+b}{2}}^{b}(g(t)-g(b))\, d\left(t-\frac{a+b}{2}\right) \\
& =\left.(g(t)-g(b))\left(t-\frac{a+b}{2}\right)\right|_{\frac{a+b}{2}}^{b}-\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right) dg(t) \\
& =-\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right) dg(t),
\end{aligned}
$$

which gives that

$$
\begin{aligned}
& \int_{\frac{a+b}{2}}^{b}\left(g(t)-\frac{g(a)+g(b)}{2}\right) dt \\
& =\left(\frac{b-a}{2}\right) \frac{g(b)-g(a)}{2}-\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right) dg(t) .
\end{aligned}
$$

Also, we have

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} \left(\frac{g(a)+g(b)}{2} - g(t) \right) dt - \left(\frac{b-a}{2} \right) \frac{g(b)-g(a)}{2} \\
&= \int_a^{\frac{a+b}{2}} \left(\frac{g(a)+g(b)}{2} - \frac{g(b)-g(a)}{2} - g(t) \right) dt \\
&= \int_a^{\frac{a+b}{2}} (g(a)-g(t))\, dt = \int_a^{\frac{a+b}{2}} (g(a)-g(t))\, d\left(t - \frac{a+b}{2} \right) \\
&= (g(a)-g(t)) \left(t - \frac{a+b}{2} \right) \Big|_a^{\frac{a+b}{2}} + \int_a^{\frac{a+b}{2}} \left(t - \frac{a+b}{2} \right) dg(t) \\
&= \int_a^{\frac{a+b}{2}} \left(t - \frac{a+b}{2} \right) dg(t),
\end{aligned}$$

which gives

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} \left(\frac{g(a)+g(b)}{2} - g(t) \right) dt \\
&= \left(\frac{b-a}{2} \right) \frac{g(b)-g(a)}{2} + \int_a^{\frac{a+b}{2}} \left(t - \frac{a+b}{2} \right) dg(t).
\end{aligned}$$

Then by (2.3) we get

$$\begin{aligned}
&f'_+\left(\frac{a+b}{2} \right) \left[\left(\frac{b-a}{2} \right) \frac{g(b)-g(a)}{2} - \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2} \right) dg(t) \right] \\
&- f'_-\left(\frac{a+b}{2} \right) \left[\left(\frac{b-a}{2} \right) \frac{g(b)-g(a)}{2} + \int_a^{\frac{a+b}{2}} \left(t - \frac{a+b}{2} \right) dg(t) \right] \\
&\le \frac{f(b)+f(a)}{2} [g(b)-g(a)] - \int_a^b f(t)\, dg(t) \\
&\le f'_-(b) \left[\left(\frac{b-a}{2} \right) \frac{g(b)-g(a)}{2} - \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2} \right) dg(t) \right] \\
&- f'_+(a) \left[\left(\frac{b-a}{2} \right) \frac{g(b)-g(a)}{2} + \int_a^{\frac{a+b}{2}} \left(t - \frac{a+b}{2} \right) dg(t) \right]
\end{aligned}$$

that is equivalent to (2.4). □

Remark 1. *If f is convex and differentiable at $\frac{a+b}{2}$, then the first inequality in (2.4) becomes*

$$\begin{aligned}
(2.6) \qquad &f'\left(\frac{a+b}{2} \right) \int_a^b \left(\frac{a+b}{2} - t \right) dg(t) \\
&\le \frac{f(b)+f(a)}{2} [g(b)-g(a)] - \int_a^b f(t)\, dg(t)
\end{aligned}$$

provided that g satisfy the conditions (2.1) and (2.2).

If $g(t)=\int_a^t p(s)\,dv(s)$ then the condition (2.1) is equivalent to

$$\int_a^t p(s)\,dv(s) \geq \frac{1}{2}\int_a^b p(s)\,dv(s)$$

or to

$$\int_a^t p(s)\,dv(s) \geq \int_t^b p(s)\,dv(s),\ \text{ for } t\in\left[\frac{a+b}{2},b\right], \tag{2.7}$$

while (2.2) is equivalent to

$$\int_t^b p(s)\,dv(s) \geq \int_a^t p(s)\,dv(s)\ \text{ for } t\in\left[a,\frac{a+b}{2}\right]. \tag{2.8}$$

Corollary 1. *Let $p:[a,b]\to\mathbb{R}$ be continuous and v of bounded variation on $[a,b]$ and such that the conditions (2.7) and (2.8) are valid, then for any f a continuous convex function on $[a,b]$, we have*

$$\begin{aligned}
&f'_+\left(\frac{a+b}{2}\right)\left[\frac{b-a}{4}\int_a^b p(s)\,dv(s)-\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)p(t)\,dv(t)\right]\\
&-f'_-\left(\frac{a+b}{2}\right)\left[\frac{b-a}{4}\int_a^b p(s)\,dv(s)-\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)p(t)\,dv(t)\right]\\
&\leq \frac{f(b)+f(a)}{2}\int_a^b p(s)\,dv(s)-\int_a^b f(t)\,p(t)\,dv(t)\\
&\leq f'_-(b)\left[\frac{b-a}{4}\int_a^b p(s)\,dv(s)-\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)p(t)\,dv(t)\right]\\
&-f'_+(a)\left[\frac{b-a}{4}\int_a^b p(s)\,dv(s)-\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)p(t)\,dv(t)\right].
\end{aligned} \tag{2.9}$$

If f is differentiable in $\frac{a+b}{2}$, then the first inequality in (2.9) becomes

$$\begin{aligned}
&f'\left(\frac{a+b}{2}\right)\int_a^b\left(\frac{a+b}{2}-t\right)p(t)\,dv(t)\\
&\leq \frac{f(b)+f(a)}{2}\int_a^b p(s)\,dv(s)-\int_a^b f(t)\,p(t)\,dv(t)
\end{aligned} \tag{2.10}$$

Remark 2. *If $v(t) = t$, $t \in [a,b]$, p is symmetric on $[a,b]$ in the sense that $p(a+b-t) = p(t)$ for all $t \in [a,b]$ and p is nonnegative on $[a,b]$, then the conditions (2.7) and (2.8) hold for $v(t) = t$ and by (2.9) we get*

$$\begin{aligned}(2.11)\qquad & f'_+\left(\frac{a+b}{2}\right)\left[\frac{b-a}{4}\int_a^b p(s)\,ds - \int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right)p(t)\,dt\right] \\ & - f'_-\left(\frac{a+b}{2}\right)\left[\frac{b-a}{4}\int_a^b p(s)\,ds - \int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)p(t)\,dt\right] \\ & \le \frac{f(b)+f(a)}{2}\int_a^b p(s)\,ds - \int_a^b f(t)\,p(t)\,dt \\ & \le f'_-(b)\left[\frac{b-a}{4}\int_a^b p(s)\,ds - \int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right)p(t)\,dt\right] \\ & - f'_+(a)\left[\frac{b-a}{4}\int_a^b p(s)\,ds - \int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)p(t)\,dt\right].\end{aligned}$$

If f is differentiable in $\frac{a+b}{2}$, then

$$\begin{aligned}(2.12)\qquad & f'\left(\frac{a+b}{2}\right)\int_a^b \left(\frac{a+b}{2}-t\right)p(t)\,dt \\ & \le \frac{f(b)+f(a)}{2}\int_a^b p(s)\,ds - \int_a^b f(t)\,p(t)\,dt.\end{aligned}$$

We provide now some inequalities for the symmetric transform of a convex function $f : [a,b] \to \mathbb{R}$ defined by

$$\breve{f}(t) := \frac{1}{2}\left[f(t) + f(a+b-t)\right].$$

Theorem 2. *Let f be a continuous convex function on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ a function of bounded variation with the properties (2.1) and (2.2), then*

$$\begin{aligned}(2.13)\qquad 0 & \le \frac{1}{2}\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right] \\ & \times \int_a^b \left[\frac{1}{2}(b-a) - \left|\frac{a+b}{2}-t\right|\right] dg(t) \\ & \le \frac{f(b)+f(a)}{2}\left[g(b)-g(a)\right] - \int_a^b \breve{f}(t)\,dg(t) \\ & \le \frac{1}{2}\left[f'_-(b) - f'_+(a)\right] \\ & \times \int_a^b \left[\frac{1}{2}(b-a) - \left|\frac{a+b}{2}-t\right|\right] dg(t).\end{aligned}$$

Proof. The function $h:[a,b]\to\mathbb{R}$ defined by $h(t)=f(a+b-t)$ is convex and

$$\begin{aligned}
h_{+}(a) &= \lim_{s\to 0+}\frac{h(a+s)-h(a)}{s}=\lim_{s\to 0+}\frac{f(b-s)-f(b)}{s}\\
&= -\lim_{s\to 0+}\frac{f(b-s)-f(b)}{-s}=-\lim_{u\to 0-}\frac{f(b+u)-f(b)}{u}\\
&= -f'_{-}(b).
\end{aligned}$$

Similarly

$$h_{-}\left(\frac{a+b}{2}\right)=-f_{+}\left(\frac{a+b}{2}\right),\ h_{+}\left(\frac{a+b}{2}\right)=-f_{-}\left(\frac{a+b}{2}\right),$$

and

$$h_{-}(b)=-f'_{+}(a).$$

By writing the inequality (2.3) for the function h we have

$$\begin{aligned}
& h'_{+}\left(\frac{a+b}{2}\right)\left[\left(\frac{b-a}{2}\right)\frac{g(b)-g(a)}{2}-\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)dg(t)\right]\\
&- h'_{-}\left(\frac{a+b}{2}\right)\left[\left(\frac{b-a}{2}\right)\frac{g(b)-g(a)}{2}-\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)dg(t)\right]\\
&\le \frac{h(b)+h(a)}{2}\left[g(b)-g(a)\right]-\int_{a}^{b}h(t)\,dg(t)\\
&\le h'_{-}(b)\left[\left(\frac{b-a}{2}\right)\frac{g(b)-g(a)}{2}-\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)dg(t)\right]\\
&- h'_{+}(a)\left[\left(\frac{b-a}{2}\right)\frac{g(b)-g(a)}{2}-\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)dg(t)\right].
\end{aligned}$$

namely

$$\begin{aligned}
& -f_{-}\left(\frac{a+b}{2}\right)\left[\left(\frac{b-a}{2}\right)\frac{g(b)-g(a)}{2}-\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)dg(t)\right]\\
&+ f_{+}\left(\frac{a+b}{2}\right)\left[\left(\frac{b-a}{2}\right)\frac{g(b)-g(a)}{2}-\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)dg(t)\right]\\
&\le \frac{f(b)+f(a)}{2}\left[g(b)-g(a)\right]-\int_{a}^{b}f(a+b-t)\,dg(t)\\
&\le -f'_{+}(a)\left[\left(\frac{b-a}{2}\right)\frac{g(b)-g(a)}{2}-\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)dg(t)\right]\\
&+ f'_{-}(b)\left[\left(\frac{b-a}{2}\right)\frac{g(b)-g(a)}{2}-\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)dg(t)\right].
\end{aligned}\tag{2.14}$$

If we add this inequality and (2.4) and divide by 2, then we get

$$\begin{aligned}
&\frac{1}{2}f'_{+}\left(\frac{a+b}{2}\right)\left[(b-a)\frac{g(b)-g(a)}{2}-\int_a^b\left|t-\frac{a+b}{2}\right|dg(t)\right]\\
&-\frac{1}{2}f'_{-}\left(\frac{a+b}{2}\right)\left[(b-a)\frac{g(b)-g(a)}{2}-\int_a^b\left|\frac{a+b}{2}-t\right|dg(t)\right]\\
&\leq\frac{f(b)+f(a)}{2}[g(b)-g(a)]-\int_a^b\breve{f}(t)\,dg(t)\\
&\leq\frac{1}{2}f'_{-}(b)\left[(b-a)\frac{g(b)-g(a)}{2}-\int_a^b\left|t-\frac{a+b}{2}\right|dg(t)\right]\\
&-\frac{1}{2}f'_{+}(a)\left[(b-a)\frac{g(b)-g(a)}{2}-\int_a^b\left|\frac{a+b}{2}-t\right|dg(t)\right]
\end{aligned}$$

that is equivalent to the second and third inequalities in (2.13).

Observe that

$$\begin{aligned}
0&\leq\int_{\frac{a+b}{2}}^b\left(g(t)-\frac{g(a)+g(b)}{2}\right)dt\\
&=\left(\frac{b-a}{2}\right)\frac{g(b)-g(a)}{2}-\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)dg(t)
\end{aligned}$$

and

$$\begin{aligned}
0&\leq\int_a^{\frac{a+b}{2}}\left(\frac{g(a)+g(b)}{2}-g(t)\right)dt\\
&=\left(\frac{b-a}{2}\right)\frac{g(b)-g(a)}{2}+\int_a^{\frac{a+b}{2}}\left(t-\frac{a+b}{2}\right)dg(t)\\
&=\left(\frac{b-a}{2}\right)\frac{g(b)-g(a)}{2}-\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)dg(t).
\end{aligned}$$

If we add these inequalities, we get

$$\begin{aligned}
0&\leq(b-a)\frac{g(b)-g(a)}{2}-\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)dg(t)-\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)dg(t)\\
&=(b-a)\frac{g(b)-g(a)}{2}-\int_a^b\left|t-\frac{a+b}{2}\right|dg(t)\\
&=\int_a^b\left[\frac{1}{2}(b-a)-\left|\frac{a+b}{2}-t\right|\right]dg(t),
\end{aligned}$$

which shows that the first inequality in (2.13) is also true. □

Corollary 2. *Let $p:[a,b]\to\mathbb{R}$ be continuous and v of bounded variation on $[a,b]$ and such that the conditions (2.7) and (2.8) are valid, then for any f a continuous* *convex function on* $[a,b]$, *we have*

$$0 \leq \frac{1}{2}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right] \times \int_a^b \left[\frac{1}{2}(b-a)-\left|\frac{a+b}{2}-t\right|\right] p(t)\,dv(t) \leq \frac{f(b)+f(a)}{2}\int_a^b p(s)\,dv(s)-\int_a^b \breve{f}(t)\,p(t)\,dv(t) \leq \frac{1}{2}\left[f'_{-}(b)-f'_{+}(a)\right] \times \int_a^b \left[\frac{1}{2}(b-a)-\left|\frac{a+b}{2}-t\right|\right] p(t)\,dv(t). \tag{2.15}$$

Remark 3. *If* $v(t)=t$, $t\in[a,b]$, p *is symmetric on* $[a,b]$ *in the sense that* $p(a+b-t)=p(t)$ *for all* $t\in[a,b]$ *and* p *is nonnegative on* $[a,b]$, *then the conditions (2.7) and (2.8) hold for* $v(t)=t$ *and by (2.15) we get*

$$0 \leq \frac{1}{2}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right] \times \int_a^b \left[\frac{1}{2}(b-a)-\left|\frac{a+b}{2}-t\right|\right] p(t)\,dt \leq \frac{f(b)+f(a)}{2}\int_a^b p(s)\,ds-\int_a^b \breve{f}(t)\,p(t)\,dt \leq \frac{1}{2}\left[f'_{-}(b)-f'_{+}(a)\right] \times \int_a^b \left[\frac{1}{2}(b-a)-\left|\frac{a+b}{2}-t\right|\right] p(t)\,dt. \tag{2.16}$$

Observe that, by the change of variable $s=a+b-t$, *we have*

$$\int_a^b f(a+b-t)\,p(t)\,dt=\int_a^b f(s)\,p(a+b-s)\,ds=\int_a^b f(s)\,p(s)\,ds,$$

which implies that

$$\int_a^b \breve{f}(t)\,p(t)\,dt=\int_a^b f(t)\,p(t)\,dt$$

and by (2.16) we get

$$0 \leq \frac{1}{2}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right] \times \int_a^b \left[\frac{1}{2}(b-a)-\left|\frac{a+b}{2}-t\right|\right] p(t)\,dt \leq \frac{f(b)+f(a)}{2}\int_a^b p(s)\,ds-\int_a^b f(t)\,p(t)\,dt \leq \frac{1}{2}\left[f'_{-}(b)-f'_{+}(a)\right] \times \int_a^b \left[\frac{1}{2}(b-a)-\left|\frac{a+b}{2}-t\right|\right] p(t)\,dt. \tag{2.17}$$

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