REVERSES OF OPERATOR FÉJER'S INEQUALITIES

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Abstract. Let f be an operator convex function on I and $A,\, B\in \mathcal{SA}_{I}(H)$, the convex set of selfadjoint operators with spectra in I. If $A\neq B$ and f, as an operator function, is Gâteaux differentiable on

$$[A,B]:=\{(1-t)A+tB \mid t\in[0,1]\},$$

while $p:[0,1]\to[0,\infty)$ is Lebesgue integrable and symmetric, namely $p(1-t) = p(t)$ for all $t\in[0,1]$, then

$$0\leq \int_0^1 p(t) f((1-t)A+tB)\,dt-\left(\int_0^1 p(t)\,dt\right) f\left(\frac{A+B}{2}\right)$$
$$\leq \frac{1}{2}\left(\int_0^1 \left|t-\frac{1}{2}\right| p(t)\,dt\right)[\nabla f_B(B-A)-\nabla f_A(B-A)]$$

and

$$0\leq \left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2}-\int_0^1 p(t) f((1-t)A+tB)\,dt$$
$$\leq \frac{1}{2}\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\,[\nabla f_B(B-A)-\nabla f_A(B-A)].$$

Two particular examples of interest are also given.

1. Introduction

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{1.1}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A\leq B$ with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset I$ imply $f(A)\leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [7] and the references therein.

As examples of such functions, we note that $f(t)=t^r$ is operator monotone on $[0,\infty)$ if and only if $0\leq r\leq 1$. The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\leq r\leq 2$ or $-1\leq r\leq 0$ and is operator concave on $(0,\infty)$ if $0\leq r\leq 1$. The logarithmic function $f(t)=\ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t)=-t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t)=e^t$ is neither operator convex nor operator monotone.

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In [5] we obtained among others the following Hermite-Hadamard type inequalities for operator convex functions $f:I\rightarrow\mathbb{R}$

$$f\left(\frac{A+B}{2}\right)\leq\int_{0}^{1}f\left((1-s)A+sB\right)ds\leq\frac{f(A)+f(B)}{2}, \tag{1.2}$$

where A, B are selfadjoint operators with spectra included in I.

From the operator convexity of the function f we have

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leq\frac{1}{2}\left[f\left((1-s)A+sB\right)+f\left(sA+(1-s)B\right)\right] \\ &\leq\frac{f(A)+f(B)}{2} \end{aligned} \tag{1.3}$$

for all $s\in[0,1]$ and A, B selfadjoint operators with spectra included in I.

If $p:[0,1]\rightarrow[0,\infty)$ is Lebesgue integrable and symmetric in the sense that $p(1-s)=p(s)$ for all $s\in[0,1]$, then by multiplying (1.3) with $p(s)$, integrating on $[0,1]$ and taking into account that

$$\int_{0}^{1}p(s)f\left((1-s)A+sB\right)ds=\int_{0}^{1}p(s)f\left(sA+(1-s)B\right)ds,$$

we get the weighted version of (1.2) for A, B selfadjoint operators with spectra included in I

$$\begin{aligned} \left(\int_{0}^{1}p(s)ds\right)f\left(\frac{A+B}{2}\right) &\leq\int_{0}^{1}p(s)f\left(sA+(1-s)B\right)ds \\ &\leq\left(\int_{0}^{1}p(s)ds\right)\frac{f(A)+f(B)}{2}, \end{aligned} \tag{1.4}$$

which are the operator version of the well known *Féjer's inequalities* for scalar convex functions.

For recent inequalities for operator convex functions see [1]-[6] and [8]-[17].

Motivated by the above results, we establish in this paper a reverse for each inequality in (1.4) in the case when the operator convex function is Gâteaux differentiable as a function of selfadjoint operators. Two particular examples of interest for $f(x)=-\ln x$ and $f(x)=x^{-1}$ are also given.

2. Some Preliminary Facts

Let f be an operator convex function on I. For A, $B\in\mathcal{SA}_{I}(H)$, the class of all selfadjoint operators with spectra in I, we consider the auxiliary function $\varphi_{(A,B)}:[0,1]\rightarrow\mathcal{SA}_{I}(H)$ defined by

$$\varphi_{(A,B)}(t):=f\left((1-t)A+tB\right). \tag{2.1}$$

For $x\in H$ we can also consider the auxiliary function $\varphi_{(A,B);x}:[0,1]\rightarrow\mathbb{R}$ defined by

$$\varphi_{(A,B);x}(t):=\left\langle\varphi_{(A,B)}(t)x,x\right\rangle=\left\langle f\left((1-t)A+tB\right)x,x\right\rangle. \tag{2.2}$$

We have the following basic fact:

Lemma 1. *Let f be an operator convex function on I. For any A, $B\in\mathcal{SA}_{I}(H)$, $\varphi_{(A,B)}$ is well defined and convex in the operator order. For any $(A,B)\in\mathcal{SA}_{I}(H)$ and $x\in H$ the function $\varphi_{(A,B);x}$ is convex in the usual sense on $[0,1]$.*

Proof. If $(A,B) \in \mathcal{SA}_I(H)$ and $t \in [0,1]$ the convex combination $(1-t)A + tB$ is a selfadjoint operator with the spectrum in I showing that $\mathcal{SA}_I(H)$ in the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators on H. By the continuous functional calculus of selfadjoint operator we also conclude that $f((1-t)A+tB)$ is a selfadjoint operator with spectrum in I.

Let $A,\ B \in \mathcal{SA}_I(H)$ and $t_1,\ t_2 \in [0,1]$. If $\alpha,\ \beta > 0$ with $\alpha+\beta = 1$, then

$$\begin{aligned}\varphi_{(A,B)}(\alpha t_1 + \beta t_2) &:= f((1-\alpha t_1 - \beta t_2)A + (\alpha t_1 + \beta t_2)B) \\ &= f((\alpha+\beta-\alpha t_1 - \beta t_2)A + (\alpha t_1 + \beta t_2)B) \\ &= f(\alpha[(1-t_1)A + t_1 B] + \beta[(1-t_2)A + t_2 B]) \\ &\leq \alpha f((1-t_1)A + t_1 B) + \beta f((1-t_2)A + t_2 B) \\ &= \alpha\varphi_{(A,B)}(t_1) + \beta\varphi_{(A,B)}(t_2),\end{aligned}$$

which proves the convexity $\varphi_{(A,B)}$ in the operator order.

Ley $A,\ B \in \mathcal{SA}_I(H)$ and $x \in H$. If $t_1,\ t_2 \in [0,1]$ and $\alpha,\ \beta > 0$ with $\alpha + \beta = 1$, then

$$\begin{aligned}\varphi_{(A,B);x}(\alpha t_1 + \beta t_2) &= \left\langle \varphi_{(A,B)}(\alpha t_1 + \beta t_2)x, x\right\rangle \\ &\leq \left\langle \left[\alpha\varphi_{(A,B)}(t_1) + \beta\varphi_{(A,B)}(t_2)\right]x, x\right\rangle \\ &= \alpha\left\langle \varphi_{(A,B)}(t_1)x,x\right\rangle + \beta\left\langle\varphi_{(A,B)}(t_2)x,x\right\rangle \\ &= \alpha\varphi_{(A,B);x}(t_1) + \beta\varphi_{(A,B);x}(t_2),\end{aligned}$$

which proves the convexity of $\varphi_{(A,B);x}$ on $[0,1]$. □

A continuous function $g : \mathcal{SA}_I(H) \to \mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A \in \mathcal{SA}_I(H)$ along the direction $B \in \mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla g_A(B) := \lim_{s\to 0}\frac{g(A+sB) - g(A)}{s} \in \mathcal{B}(H). \tag{2.3}$$

If the limit (2.3) exists for all $B \in \mathcal{B}(H)$, then we say that f is *Gâteaux differentiable* in A and we can write $g \in \mathcal{G}(A)$. If this is true for any A in an open set $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $g \in \mathcal{G}(\mathcal{S})$.

If g is a continuous function on I, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators $A,\ B \in \mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators

$$[A,B] := \{(1-t)A + tB \mid t \in [0,1]\}.$$

We observe that $A,\ B \in [A,B]$ and $[A,B] \subset \mathcal{SA}_I(H)$.

Lemma 2. *Let f be an operator convex function on I and $A,\ B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then the auxiliary function $\varphi_{(A,B)}$ is differentiable on $(0,1)$ and*

$$\varphi'_{(A,B)}(t) = \nabla f_{(1-t)A+tB}(B-A). \tag{2.4}$$

Also we have for the lateral derivative that

$$\varphi'_{(A,B)}(0+) = \nabla f_A(B-A) \tag{2.5}$$

and

$$\varphi'_{(A,B)}(1-) = \nabla f_B(B-A). \tag{2.6}$$

Proof. Let $t \in (0,1)$ and $h \neq 0$ small enough such that $t+h \in (0,1)$. Then

$$\begin{aligned} & \frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h} \\ & = \frac{f((1-t-h)A + (t+h)B) - f((1-t)A + tB)}{h} \\ & = \frac{f((1-t)A + tB + h(B-A)) - f((1-t)A + tB)}{h}. \end{aligned} \tag{2.7}$$

Since $f \in \mathcal{G}([A,B])$, hence by taking the limit over $h \to 0$ in (2.7) we get

$$\begin{aligned} \varphi'_{(A,B)}(t) &= \lim_{h\to 0} \frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h} \\ &= \lim_{h\to 0} \frac{f((1-t)A + tB + h(B-A)) - f((1-t)A + tB)}{h} \\ &= \nabla g_{(1-t)A+tB}(B-A), \end{aligned}$$

which proves (2.8).

Also, we have

$$\begin{aligned} \varphi'_{(A,B)}(0+) &= \lim_{h\to 0+} \frac{\varphi_{(A,B)}(h) - \varphi_{(A,B)}(0)}{h} \\ &= \lim_{h\to 0+} \frac{f((1-h)A + hB) - f(A)}{h} \\ &= \lim_{h\to 0+} \frac{f(A + h(B-A)) - f(A)}{h} \\ &= \nabla f_A(B-A) \end{aligned}$$

since f is assumed to be Gâteaux differentiable in A. This proves (2.5).

The equality (2.6) follows in a similar way. □

Lemma 3. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then for $0 < t_1 < t_2 < 1$ we have*

$$\nabla g_{(1-t_1)A+t_1B}(B-A) \leq \nabla g_{(1-t_2)A+t_2B}(B-A) \tag{2.8}$$

in the operator order.

We also have

$$\nabla f_A(B-A) \leq \nabla g_{(1-t_1)A+t_1B}(B-A) \tag{2.9}$$

and

$$\nabla g_{(1-t_2)A+t_2B}(B-A) \leq \nabla f_B(B-A). \tag{2.10}$$

Proof. Let $x \in H$. The auxiliary function $\varphi_{(A,B);x}$ is convex in the usual sense on $[0,1]$ and differentiable on $(0,1)$ and for $t \in (0,1)$

$$\begin{aligned}\varphi'_{(A,B),x}(t) &= \lim_{h\to 0}\frac{\varphi_{(A,B),x}(t+h)-\varphi_{(A,B),x}(t)}{h}\\ &= \lim_{h\to 0}\left\langle \frac{\varphi_{(A,B)}(t+h)-\varphi_{(A,B)}(t)}{h}x,x\right\rangle\\ &= \left\langle \lim_{h\to 0}\frac{\varphi_{(A,B)}(t+h)-\varphi_{(A,B)}(t)}{h}x,x\right\rangle\\ &= \left\langle \nabla g_{(1-t)A+tB}(B-A)x,x\right\rangle.\end{aligned}$$

Since for $0 < t_1 < t_2 < 1$ we have by the gadient inequality for scalar convex functions that

$$\varphi'_{(A,B),x}(t_1) \le \varphi'_{(A,B),x}(t_2)$$

then we get

$$\left\langle \nabla g_{(1-t_1)A+t_1B}(B-A)x,x\right\rangle \le \left\langle \nabla g_{(1-t_2)A+t_2B}(B-A)x,x\right\rangle \tag{2.11}$$

for all $x \in H$, which is equivalent to the inequality (2.8) in the operator order.

Let $0 < t_1 < 1$. By the gadient inequality for scalar convex functions we also have

$$\varphi'_{(A,B),x}(0+) \le \varphi'_{(A,B),x}(t_1),$$

which, as above implies that

$$\langle \nabla f_A(B-A)x,x\rangle \le \left\langle \nabla g_{(1-t_1)A+t_1B}(B-A)x,x\right\rangle$$

for all $x \le H$, that is equivalent to the operator inequality (2.9).

The inequality (2.10) follows in a similar way. □

Corollary 1. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \ne B$. If $f \in \mathcal{G}([A,B])$, then for all $t \in (0,1)$ we have*

$$\nabla f_A(B-A) \le \nabla f_{(1-t)A+tB}(B-A) \le \nabla f_B(B-A). \tag{2.12}$$

3. Reverse Operator Féjer Inequalities

We have:

Theorem 1. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \ne B$. If $f \in \mathcal{G}([A,B])$ and $p:[0,1]\to[0,\infty)$ is Lebesgue integrable and symmetric, namely $p(1-t) = p(t)$ for all $t \in [0,1]$, then*

$$\begin{aligned}0 &\le \int_0^1 p(t) f((1-t)A+tB)\,dt - \left(\int_0^1 p(t)\,dt\right) f\left(\frac{A+B}{2}\right)\\ &\le \frac{1}{2}\left(\int_0^1 \left|t-\frac{1}{2}\right| p(t)\,dt\right)[\nabla f_B(B-A) - \nabla f_A(B-A)].\end{aligned} \tag{3.1}$$

In particular, for $p \equiv 1$ we get

$$\begin{aligned}0 &\le \int_0^1 f((1-t)A+tB)\,dt - f\left(\frac{A+B}{2}\right)\\ &\le \frac{1}{8}[\nabla f_B(B-A) - \nabla f_A(B-A)].\end{aligned} \tag{3.2}$$

Proof. Let $A,\ B\in\mathcal{SA}_I(H)$, with $A\neq B$. Using the integration by parts formula for Bochner's integral, we have

$$\begin{aligned}&\int_{1/2}^{1}\left(\int_t^1 p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt\\&=\left(\int_t^1 p(s)\,ds\right)\varphi_{(A,B)}(t)\Big]_{1/2}^{1}+\int_{1/2}^{1}p(t)\,\varphi_{(A,B)}(t)\,dt\\&=-\left(\int_{1/2}^{1}p(s)\,ds\right)\varphi_{(A,B)}(1/2)+\int_{1/2}^{1}p(t)\,\varphi_{(A,B)}(t)\,dt\\&=-\left(\int_{1/2}^{1}p(s)\,ds\right)f\left(\frac{A+B}{2}\right)+\int_{1/2}^{1}p(t)\,\varphi_{(A,B)}(t)\,dt\end{aligned}$$

and

$$\begin{aligned}&\int_{0}^{1/2}\left(\int_0^t p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt\\&=\left(\int_0^t p(s)\,ds\right)\varphi_{(A,B)}(t)\Big]_{0}^{1/2}-\int_{0}^{1/2}p(t)\,\varphi_{(A,B)}(t)\,dt\\&=\left(\int_{0}^{1/2}p(s)\,ds\right)\varphi_{(A,B)}(1/2)-\int_{0}^{1/2}p(t)\,\varphi_{(A,B)}(t)\,dt\\&=\left(\int_{0}^{1/2}p(s)\,ds\right)f\left(\frac{A+B}{2}\right)-\int_{0}^{1/2}p(t)\,\varphi_{(A,B)}(t)\,dt.\end{aligned}$$

By subtracting the second identity from the first, we get

$$\begin{aligned}&\int_{1/2}^{1}\left(\int_t^1 p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt-\int_{0}^{1/2}\left(\int_0^t p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt\\&=\int_{1/2}^{1}p(t)\,\varphi_{(A,B)}(t)\,dt+\int_{0}^{1/2}p(t)\,\varphi_{(A,B)}(t)\,dt\\&-\left(\int_{1/2}^{1}p(s)\,ds\right)f\left(\frac{A+B}{2}\right)-\left(\int_{0}^{1/2}p(s)\,ds\right)f\left(\frac{A+B}{2}\right).\end{aligned}$$

By the symmetry of p we obtain

$$\int_{1/2}^{1}p(s)\,ds=\int_{0}^{1/2}p(s)\,ds=\frac{1}{2}\int_0^1 p(s)\,ds$$

and then we get the following identity of interest in itself

$$\begin{aligned}&\int_0^1 p(t)\,\varphi_{(A,B)}(t)\,dt-\int_0^1 p(s)\,ds f\left(\frac{A+B}{2}\right)\\&=\int_{1/2}^{1}\left(\int_t^1 p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt-\int_{0}^{1/2}\left(\int_0^t p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt.\end{aligned}\tag{3.3}$$

By Lemma 3 we have

$$\nabla f_A(B-A)\leq\varphi'_{(A,B)}(t)\leq\varphi'_{(A,B)}\left(\frac{1}{2}\right),\ \ t\in[0,1/2]$$

and

$$\varphi'_{(A,B)}\left(\frac{1}{2}\right) \le \varphi'_{(A,B)}(t) \le \nabla f_B(B-A),\ t\in[1/2,1]$$

in the operator order.

These imply that

$$\left(\int_0^t p(s)\,ds\right)\nabla f_A(B-A) \le \left(\int_0^t p(s)\,ds\right)\varphi'_{(A,B)}(t) \le \left(\int_0^t p(s)\,ds\right)\varphi'_{(A,B)}\left(\frac{1}{2}\right),\ t\in[0,1/2]$$

and

$$\left(\int_t^1 p(s)\,ds\right)\varphi'_{(A,B)}\left(\frac{1}{2}\right) \le \left(\int_t^1 p(s)\,ds\right)\varphi'_{(A,B)}(t) \le \left(\int_t^1 p(s)\,ds\right)\nabla f_B(B-A),\ t\in[1/2,1],$$

and by integration

$$\int_{1/2}^1\left(\int_t^1 p(s)\,ds\right)dt\varphi'_{(A,B)}\left(\frac{1}{2}\right) \le \int_{1/2}^1\left(\int_t^1 p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt \le \int_{1/2}^1\left(\int_t^1 p(s)\,ds\right)dt\nabla f_B(B-A)$$

and

$$-\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt\varphi'_{(A,B)}\left(\frac{1}{2}\right) \le -\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt \le -\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt\nabla f_A(B-A).$$

If we add these inequalities, then we get

$$\left[\int_{1/2}^1\left(\int_t^1 p(s)\,ds\right)dt - \int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt\right]\varphi'_{(A,B)}\left(\frac{1}{2}\right) \tag{3.4}$$
$$\le \int_{1/2}^1\left(\int_t^1 p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt - \int_0^{1/2}\left(\int_0^t p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt$$
$$\le \int_{1/2}^1\left(\int_t^1 p(s)\,ds\right)dt\nabla f_B(B-A) - \int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt\nabla f_A(B-A)$$

in the operator order.

Integrating by parts in the Lebesgue integral, we have

$$\int_{1/2}^1\left(\int_t^1 p(s)\,ds\right)dt = \left(\int_t^1 p(s)\,ds\right)t\Big]_{1/2}^1 + \int_{1/2}^1 tp(t)\,dt = \int_{1/2}^1 tp(t)\,dt - \frac{1}{2}\int_{1/2}^1 p(s)\,ds = \int_{1/2}^1\left(t-\frac{1}{2}\right)p(t)\,dt$$

and

$$\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt = \left(\int_0^t p(s)\,ds\right)t\Big]_0^{1/2} - \int_0^{1/2} p(t)\,tdt$$
$$= \frac{1}{2}\int_0^{1/2} p(s)\,ds - \int_0^{1/2} p(t)\,tdt$$
$$= \int_0^{1/2}\left(\frac{1}{2}-t\right)p(t)\,dt.$$

Then

$$\int_{1/2}^1\left(\int_t^1 p(s)\,ds\right)dt - \int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt$$
$$= \int_{1/2}^1 tp(t)\,dt - \frac{1}{2}\int_{1/2}^1 p(s)\,ds - \frac{1}{2}\int_0^{1/2} p(s)\,ds + \int_0^{1/2} p(t)\,tdt$$
$$= \int_0^1 tp(t)\,dt - \int_{1/2}^1 p(s)\,ds = \int_0^1 tp(t)\,dt - \frac{1}{2}\int_0^1 p(t)\,dt$$
$$= \int_0^1\left(t-\frac{1}{2}\right)p(t)\,dt = 0,$$

since the function $q(t) = \left(t-\frac{1}{2}\right)p(t)$ is asymmetric on $[0,1]$.

Also by changing the variable $s = 1-t$, we have

$$\int_0^{1/2}\left(\frac{1}{2}-t\right)p(t)\,dt = \int_{1/2}^1\left(s-\frac{1}{2}\right)p(1-s)\,ds = \int_{1/2}^1\left(s-\frac{1}{2}\right)p(s)\,ds$$
$$= \frac{1}{2}\int_0^1\left|s-\frac{1}{2}\right|p(s)\,ds.$$

□

Remark 1. *If we take $p(t) = \left|t-\frac{1}{2}\right|$, $t\in[0,1]$ in (3.1), then we get*

$$0 \le \int_0^1\left|t-\frac{1}{2}\right|f((1-t)A+tB)\,dt - \frac{1}{4}f\left(\frac{A+B}{2}\right) \tag{3.5}$$
$$\le \frac{1}{24}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right].$$

We also have:

Theorem 2. *Let f be an operator convex function on I and A, $B\in\mathcal{SA}_I(H)$, with $A\neq B$. If $f\in\mathcal{G}([A,B])$ and $p:[0,1]\to[0,\infty)$ is Lebesgue integrable and symmetric, namely $p(1-t)=p(t)$ for all $t\in[0,1]$, then*

$$0 \le \left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2} - \int_0^1 p(t)\,f((1-t)A+tB)\,dt \tag{3.6}$$
$$\le \frac{1}{2}\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right].$$

In particular, for $p \equiv 1$ we get

$$0 \leq \frac{f(A)+f(B)}{2} - \int_0^1 f((1-t)A+tB)\,dt \leq \frac{1}{8}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right]. \tag{3.7}$$

Proof. Using the integration by parts for Bochner's integral, we have

$$\begin{aligned}
&\int_0^1 \left(\int_0^t p(s)\,ds - \frac{1}{2}\int_0^1 p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt \\
&= \left(\int_0^t p(s)\,ds - \frac{1}{2}\int_0^1 p(s)\,ds\right)\varphi_{(A,B)}(t)\Big]_0^1 - \int_0^1 p(t)\,\varphi_{(A,B)}(t)\,dt \\
&= \left(\int_0^1 p(s)\,ds - \frac{1}{2}\int_0^1 p(s)\,ds\right)\varphi_{(A,B)}(1) + \left(\frac{1}{2}\int_0^1 p(s)\,ds\right)\varphi_{(A,B)}(0) \\
&- \int_0^1 p(t)\,\varphi_{(A,B)}(t)\,dt \\
&= \left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2} - \int_0^1 p(t)\,\varphi_{(A,B)}(t)\,dt.
\end{aligned}$$

We also have

$$\begin{aligned}
&\int_0^1 \left(\int_0^t p(s)\,ds - \frac{1}{2}\int_0^1 p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt \\
&= \int_0^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt \\
&= \int_0^{1/2} \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt \\
&+ \int_{1/2}^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt \\
&= \int_{1/2}^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt \\
&- \int_0^{1/2} \left(\int_0^{1/2} p(s)\,ds - \int_0^t p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt.
\end{aligned}$$

Observe that

$$\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds \geq 0 \text{ for } t \in [1/2, 1]$$

and

$$\int_0^{1/2} p(s)\,ds - \int_0^t p(s)\,ds \geq 0 \text{ for } t \in [0, 1/2].$$

By Lemma 3 then have in the operatorial order the following inequalities

$$\begin{aligned}
&\int_{1/2}^{1}\left(\int_0^t p(s)\,ds-\int_0^{1/2}p(s)\,ds\right)dt\varphi'_{(A,B)}\left(\frac{1}{2}\right)\\
&\leq\int_{1/2}^{1}\left(\int_0^t p(s)\,ds-\int_0^{1/2}p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt\\
&\leq\int_{1/2}^{1}\left(\int_0^t p(s)\,ds-\int_0^{1/2}p(s)\,ds\right)dt\nabla f_B(B-A)
\end{aligned}$$

and

$$\begin{aligned}
&-\int_0^{1/2}\left(\int_0^{1/2}p(s)\,ds-\int_0^t p(s)\,ds\right)dt\varphi'_{(A,B)}\left(\frac{1}{2}\right)\\
&\leq-\int_0^{1/2}\left(\int_0^{1/2}p(s)\,ds-\int_0^t p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt\\
&\leq-\int_0^{1/2}\left(\int_0^{1/2}p(s)\,ds-\int_0^t p(s)\,ds\right)dt\nabla f_A(B-A).
\end{aligned}$$

If we add these inequalities, then we get

$$\begin{aligned}
&\left[\int_{1/2}^{1}\left(\int_0^t p(s)\,ds-\int_0^{1/2}p(s)\,ds\right)dt\right.\\
&\left.-\int_0^{1/2}\left(\int_0^{1/2}p(s)\,ds-\int_0^t p(s)\,ds\right)dt\right]\times\varphi'_{(A,B)}\left(\frac{1}{2}\right)\\
&\leq\int_{1/2}^{1}\left(\int_0^t p(s)\,ds-\int_0^{1/2}p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt\\
&-\int_{1/2}^{1}\left(\int_0^t p(s)\,ds-\int_0^{1/2}p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt\\
&\leq\int_{1/2}^{1}\left(\int_0^t p(s)\,ds-\int_0^{1/2}p(s)\,ds\right)dt\nabla f_B(B-A)\\
&-\int_0^{1/2}\left(\int_0^{1/2}p(s)\,ds-\int_0^t p(s)\,ds\right)dt\nabla f_A(B-A).
\end{aligned}\tag{3.8}$$

Observe that

$$
\begin{aligned}
&\int_{1/2}^{1}\left(\int_0^t p(s)\,ds-\int_0^{1/2} p(s)\,ds\right)dt\\
&-\int_0^{1/2}\left(\int_0^{1/2} p(s)\,ds-\int_0^{t} p(s)\,ds\right)dt\\
&=\int_{1/2}^{1}\left(\int_0^t p(s)\,ds\right)dt-\frac{1}{2}\int_0^{1/2} p(s)\,ds\\
&-\frac{1}{2}\int_0^{1/2} p(s)\,ds+\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt\\
&=\int_0^1\left(\int_0^t p(s)\,ds\right)dt-\int_0^{1/2} p(s)\,ds\\
&=\int_0^1\left(\int_0^t p(s)\,ds\right)dt-\frac{1}{2}\int_0^1 p(t)\,dt
\end{aligned}
$$

$$
\begin{aligned}
&=\left(\int_0^t p(s)\,ds\right)t\Big]_0^1-\int_0^1 p(t)\,tdt-\frac{1}{2}\int_0^1 p(t)\,dt\\
&=\int_0^1 p(t)\,dt-\int_0^1 p(t)\,tdt-\frac{1}{2}\int_0^1 p(t)\,dt=\int_0^1\left(\frac{1}{2}-t\right)p(t)\,dt=0
\end{aligned}
$$

since the function $q(t)=\left(t-\frac{1}{2}\right)p(t)$ is asymmetric on $[0,1]$.

Therefore, by the first inequality in (3.8) we get the first inequality in (3.6).

We also have

$$
\begin{aligned}
&\int_{1/2}^{1}\left(\int_0^t p(s)\,ds-\int_0^{1/2} p(s)\,ds\right)dt\\
&=\int_{1/2}^{1}\left(\int_0^t p(s)\,ds\right)dt-\frac{1}{2}\int_0^{1/2} p(s)\,ds\\
&=\left(\int_0^t p(s)\,ds\right)t\Big]_{1/2}^1-\int_{1/2}^1 tp(t)\,dt-\frac{1}{2}\int_0^{1/2} p(s)\,ds\\
&=\int_0^1 p(s)\,ds-\frac{1}{2}\int_0^{1/2} p(s)\,ds-\int_{1/2}^1 tp(t)\,dt-\frac{1}{2}\int_0^{1/2} p(s)\,ds\\
&=\int_0^1 p(s)\,ds-\int_0^{1/2} p(s)\,ds-\int_{1/2}^1 tp(t)\,dt\\
&=\int_{1/2}^1 p(s)\,ds-\int_{1/2}^1 tp(t)\,dt=\int_{1/2}^1(1-t)\,p(t)\,dt
\end{aligned}
$$

and

$$\int_0^{1/2}\left(\int_0^{1/2} p(s)\,ds-\int_0^t p(s)\,ds\right)dt$$
$$=\frac{1}{2}\int_0^{1/2} p(s)\,ds-\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt$$
$$=\frac{1}{2}\int_0^{1/2} p(s)\,ds-\left(\left(\int_0^t p(s)\,ds\right)t\bigg]_0^{1/2}-\int_0^{1/2} tp(t)\,dt\right)$$
$$=\frac{1}{2}\int_0^{1/2} p(s)\,ds-\frac{1}{2}\int_0^{1/2} p(s)\,ds+\int_0^{1/2} tp(t)\,dt=\int_0^{1/2} tp(t)\,dt.$$

If we change the variable $s=1-t$, then

$$\int_0^{1/2} tp(t)\,dt=-\int_1^{1/2}(1-s)\,p(1-s)\,ds=\int_{1/2}^1(1-s)\,p(1-s)\,ds \tag{3.9}$$
$$=\int_{1/2}^1(1-s)\,p(s)\,ds.$$

Therefore

$$\frac{1}{2}\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt$$
$$=\frac{1}{2}\int_0^{1/2}\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt+\frac{1}{2}\int_{1/2}^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt$$
$$=\frac{1}{2}\int_0^{1/2}\left(\frac{1}{2}-\frac{1}{2}+t\right)p(t)\,dt+\frac{1}{2}\int_{1/2}^1\left(\frac{1}{2}-t+\frac{1}{2}\right)p(t)\,dt$$
$$=\frac{1}{2}\int_0^{1/2} tp(t)\,dt+\frac{1}{2}\int_{1/2}^1(1-t)\,p(t)\,dt=\int_0^{1/2} tp(t)\,dt \text{ (by 3.9)}$$

and by the second inequality in (3.8) we get the second part of (3.6). □

Remark 2. *If we take* $p(t)=\left|t-\frac{1}{2}\right|$, $t\in[0,1]$ *in (3.6), then we get*

$$0\le\frac{f(A)+f(B)}{8}-\int_0^1\left|t-\frac{1}{2}\right|f((1-t)A+tB)\,dt \tag{3.10}$$
$$\le\frac{1}{48}\left[\nabla f_B(B-A)-\nabla f_A(B-A)\right].$$

Remark 3. *It is well known that, if f is a C^1-function defined on an open interval, then the operator function $f(X)$ is Fréchet differentiable and the derivative $Df(A)(B)$ equals the Gâteaux derivative $\nabla f_A(B)$. So for operator convex functions f that are of class C^1 on I and $p:[0,1]\to[0,\infty)$ a Lebesgue integrable and symmetric weight on $[0,1]$ we have the inequalities*

$$0\le\int_0^1 p(t)\,f((1-t)A+tB)\,dt-\left(\int_0^1 p(t)\,dt\right)f\left(\frac{A+B}{2}\right) \tag{3.11}$$
$$\le\frac{1}{2}\left(\int_0^1\left|t-\frac{1}{2}\right|p(t)\,dt\right)\left[Df(B)(B-A)-Df(A)(B-A)\right]$$

and

$$0\le\left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2}-\int_0^1 p(t)\,f((1-t)A+tB)\,dt \tag{3.12}$$
$$\le\frac{1}{2}\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\,[Df(B)(B-A)-Df(A)(B-A)]$$

for all $A,\ B\in\mathcal{SA}_I(H)$.

4. Some Examples

We note that the function $f(x)=-\ln x$ is operator convex on $(0,\infty)$. The ln function is operator Gâteaux differentiable with the following explicit formula for the derivative (cf. Pedersen [12, p. 155]):

$$\nabla\ln_T(S)=\int_0^\infty(s1_H+T)^{-1}S(s1_H+T)^{-1}ds \tag{4.1}$$

for $T,\ S>0$

If we write the inequalities (3.1) and (3.6) for $-\ln$ and $p:[0,1]\to[0,\infty)$ is Lebesgue integrable and symmetric we get

$$0\le\left(\int_0^1 p(t)\,dt\right)\ln\left(\frac{A+B}{2}\right)-\int_0^1 p(t)\ln((1-t)A+tB)\,dt \tag{4.2}$$
$$\le\frac{1}{2}\left(\int_0^1\left|t-\frac{1}{2}\right|p(t)\,dt\right)\left[\int_0^\infty(s1_H+A)^{-1}(B-A)(s1_H+A)^{-1}ds\right.$$
$$\left.-\int_0^\infty(s1_H+B)^{-1}(B-A)(s1_H+B)^{-1}ds\right]$$

and

$$0\le\int_0^1 p(t)\ln((1-t)A+tB)\,dt-\left(\int_0^1 p(t)\,dt\right)\frac{\ln A+\ln B}{2} \tag{4.3}$$
$$\le\frac{1}{2}\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\left[\int_0^\infty(s1_H+A)^{-1}(B-A)(s1_H+A)^{-1}ds\right.$$
$$\left.-\int_0^\infty(s1_H+B)^{-1}(B-A)(s1_H+B)^{-1}ds\right]$$

for all $A,\ B>0$.

The function $f(x)=x^{-1}$ is also operator convex on $(0,\infty)$, operator Gâteaux differentiable and

$$\nabla f_T(S)=-T^{-1}ST^{-1}$$

for $T,\ S>0$.

If we write the inequalities (3.1) and (3.6) for this function, then we get

$$0\le\int_0^1 p(t)((1-t)A+tB)^{-1}dt-\left(\int_0^1 p(t)\,dt\right)\left(\frac{A+B}{2}\right)^{-1} \tag{4.4}$$
$$\le\frac{1}{2}\left(\int_0^1\left|t-\frac{1}{2}\right|p(t)\,dt\right)\left[A^{-1}(B-A)A^{-1}-B^{-1}(B-A)B^{-1}\right]$$

and

$$0 \le \left(\int_0^1 p(t)\,dt \right) \frac{A^{-1} + B^{-1}}{2} - \int_0^1 p(t)\left((1-t)A + tB \right)^{-1} dt \le \frac{1}{2} \int_0^1 \left(\frac{1}{2} - \left| t - \frac{1}{2} \right| \right) p(t)\,dt \left[A^{-1}(B-A)A^{-1} - B^{-1}(B-A)B^{-1} \right] \tag{4.5}$$

for all $A, B > 0$.

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