REFINEMENTS AND REVERSES OF FÉJER'S INEQUALITIES FOR CONVEX FUNCTIONS ON LINEAR SPACES

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Abstract. Let f be an convex function on the convex set C in a linear space and x, $y\in C$, with $x\neq y$. If $p:[0,1]\rightarrow[0,\infty)$ is Lebesgue integrable and symmetric, namely $p(1-t)=p(t)$ for all $t\in[0,1]$, then

$$\begin{aligned}0&\leq\frac{1}{2}\left[\nabla_{+}f_{\frac{x+y}{2}}(y-x)-\nabla_{-}f_{\frac{x+y}{2}}(y-x)\right]\int_{0}^{1}\left|t-\frac{1}{2}\right|p(t)\,dt\\&\leq\int_{0}^{1}f((1-t)x+ty)\,p(t)\,dt-f\left(\frac{x+y}{2}\right)\int_{0}^{1}p(t)\,dt\\&\leq\frac{1}{2}\left[\nabla_{-}f_{y}(y-x)-\nabla_{+}f_{x}(y-x)\right]\left(\int_{0}^{1}\left|t-\frac{1}{2}\right|p(t)\,dt\right)\end{aligned}$$

and

$$\begin{aligned}0&\leq\frac{1}{2}\left[\nabla_{+}f_{\frac{x+y}{2}}(y-x)-\nabla_{-}f_{\frac{x+y}{2}}(y-x)\right]\int_{0}^{1}\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\\&\leq\frac{f(x)+f(y)}{2}\int_{0}^{1}p(t)\,dt-\int_{0}^{1}f((1-t)x+ty)\,p(t)\,dt\\&\leq\frac{1}{2}\left[\nabla_{-}f_{y}(y-x)-\nabla_{+}f_{x}(y-x)\right]\int_{0}^{1}\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt,\end{aligned}$$

where $\nabla_{\pm}f_{\cdot}(\cdot)$ are the *Gâteaux lateral derivatives.*

Some applications for norms and semi-inner products are also provided.

1. Introduction

Let X be a real linear space, $x,y\in X$, $x\neq y$ and let $[x,y]:=\{(1-\lambda)x+\lambda y,\ \lambda\in[0,1]\}$ be the *segment* generated by x and y. We consider the function $f:[x,y]\rightarrow\mathbb{R}$ and the attached function $\varphi_{(x,y)}:[0,1]\rightarrow\mathbb{R}$, $\varphi_{(x,y)}(t):=f[(1-t)x+ty]$, $t\in[0,1]$.

It is well known that f is convex on $[x,y]$ iff $\varphi(x,y)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy

(i) $\varphi'_{\pm(x,y)}(s)=\nabla_{\pm}f_{(1-s)x+sy}(y-x)$, $s\in[0,1)$,
(ii) $\varphi'_{+(x,y)}(0)=\nabla_{+}f_{x}(y-x)$,
(iii) $\varphi'_{-(x,y)}(1)=\nabla_{-}f_{y}(y-x)$,

where $\nabla_{\pm}f_{x}(y)$ are the *Gâteaux lateral derivatives,* we recall that

$$\begin{aligned}\nabla_{+}f_{x}(y)&:=\lim_{h\rightarrow0+}\frac{f(x+hy)-f(x)}{h},\\\nabla_{-}f_{x}(y)&:=\lim_{k\rightarrow0-}\frac{f(x+ky)-f(x)}{k},\quad x,\ y\in X.\end{aligned}$$

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The following inequality is the well-known Hermite-Hadamard integral inequality for convex functions defined on a segment $[x,y]\subset X$:

$$f\left(\frac{x+y}{2}\right)\leq\int_0^1 f\left[(1-t)\,x+ty\right]dt\leq\frac{f(x)+f(y)}{2}, \tag{HH}$$

which easily follows by the classical Hermite-Hadamard inequality for the convex function $\varphi(x,y):[0,1]\to\mathbb{R}$

$$\varphi_{(x,y)}\left(\frac{1}{2}\right)\leq\int_0^1\varphi_{(x,y)}(t)\,dt\leq\frac{\varphi_{(x,y)}(0)+\varphi_{(x,y)}(1)}{2}.$$

For other related results see the monograph on line [6]. For some recent results in linear spaces see [1], [2] and [8]-[11].

We have the following result [4] related to the first Hermite-Hadamard inequality in (HH):

Theorem 1. *Let X be a linear space, $x,\ y\in X,\ x\neq y$ and $f:[x,y]\subset X\to\mathbb{R}$ be a convex function on the segment $[x,y]$. Then for any $s\in(0,1)$ one has the inequality*

$$\begin{aligned}&\frac{1}{2}\left[(1-s)^2\,\nabla_+f_{(1-s)x+sy}(y-x)-s^2\nabla_-f_{(1-s)x+sy}(y-x)\right]\\&\leq\int_0^1 f\left[(1-t)\,x+ty\right]dt-f\left[(1-s)\,x+sy\right]\\&\leq\frac{1}{2}\left[(1-s)^2\,\nabla_-f_y(y-x)-s^2\nabla_+f_x(y-x)\right].\end{aligned}\tag{1.1}$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $s=0$ or $s=1$.

If $f:[x,y]\to\mathbb{R}$ is as in Theorem 1 and Gâteaux differentiable in $c:=(1-\lambda)\,x+\lambda y$, $\lambda\in(0,1)$ along the direction $y-x$, then we have the inequality:

$$\left(\frac{1}{2}-\lambda\right)\nabla f_c(y-x)\leq\int_0^1 f\left[(1-t)\,x+ty\right]dt-f(c). \tag{1.2}$$

If f is as in Theorem 1, then

$$\begin{aligned}0&\leq\frac{1}{8}\left[\nabla_+f_{\frac{x+y}{2}}(y-x)-\nabla_-f_{\frac{x+y}{2}}(y-x)\right]\\&\leq\int_0^1 f\left[(1-t)\,x+ty\right]dt-f\left(\frac{x+y}{2}\right)\\&\leq\frac{1}{8}\left[\nabla_-f_y(y-x)-\nabla_+f_x(y-x)\right].\end{aligned}\tag{1.3}$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

Also we have the following result [5] related to the second Hermite-Hadamard inequality in (HH):

Theorem 2. *Let X be a linear space, $x,\ y\in X,\ x\neq y$ and $f:[x,y]\subset X\to\mathbb{R}$ be a convex function on the segment $[x,y]$. Then for any $s\in(0,1)$ one has the* *inequality*

$$\begin{aligned}(1.4)\qquad &\frac{1}{2}\left[(1-s)^2\nabla_+f_{(1-s)x+sy}(y-x)-s^2\nabla_-f_{(1-s)x+sy}(y-x)\right]\\ &\le (1-s)f(x)+sf(y)-\int_0^1 f[(1-t)x+ty]\,dt\\ &\le \frac{1}{2}\left[(1-s)^2\nabla_-f_y(y-x)-s^2\nabla_+f_x(y-x)\right].\end{aligned}$$

The constant $\frac{1}{2}$ *is sharp in both inequalities. The second inequality also holds for* $s=0$ *or* $s=1$.

If $f:[x,y]\to\mathbb{R}$ is as in Theorem 2 and Gâteaux differentiable in $c:=(1-\lambda)x+\lambda y$, $\lambda\in(0,1)$ along the direction $y-x$, then we have the inequality:

$$(1.5)\qquad \left(\frac{1}{2}-\lambda\right)\nabla f_c(y-x)\le(1-\lambda)f(x)+\lambda f(y)-\int_0^1 f[(1-t)x+ty]\,dt.$$

If f is as in Theorem 2, then

$$\begin{aligned}(1.6)\qquad 0&\le\frac{1}{8}\left[\nabla_+f_{\frac{x+y}{2}}(y-x)-\nabla_-f_{\frac{x+y}{2}}(y-x)\right]\\ &\le\frac{f(x)+f(y)}{2}-\int_0^1 f[(1-t)x+ty]\,dt\\ &\le\frac{1}{8}\left[\nabla_-f_y(y-x)-\nabla_+f_x(y-x)\right].\end{aligned}$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

By the convexity of f we have for all $t\in[0,1]$ that

$$f\left(\frac{x+y}{2}\right)\le\frac{f[(1-t)x+ty]+f[(1-t)y+tx]}{2}\le\frac{f(x)+f(y)}{2}.$$

If we multiply this inequality by $p:[0,1]\to[0,\infty)$, a Lebesgue integrable function on $[0,1]$, and integrate on $[0,1]$ over $t\in[0,1]$, then we get

$$\begin{aligned}(1.7)\qquad &f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt\\ &\le\frac{\int_0^1 f[(1-t)x+ty]p(t)\,dt+\int_0^1 f[(1-t)y+tx]p(t)\,dt}{2}\\ &\le\frac{f(x)+f(y)}{2}\int_0^1 p(t)\,dt.\end{aligned}$$

By changing the variable $s=1-t$, then we get

$$\int_0^1 f[(1-t)y+tx]p(t)\,dt=\int_0^1 f[sy+(1-s)x]p(1-s)\,dt$$

and by (1.7) we obtain

$$\begin{aligned}(1.8)\qquad f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt&\le\int_0^1 f[(1-t)x+ty]\breve{p}(t)\,dt\\ &\le\frac{f(x)+f(y)}{2}\int_0^1 p(t)\,dt,\end{aligned}$$

where $\breve{p}(t):=\frac{1}{2}[p(t)+p(1-t)]$, $t\in[0,1]$.

If p is symmetric on $[0,1]$, namely $p(t) = p(1-t)$ for $t \in [0,1]$, then (1.8) becomes the Féjer's inequality

$$f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt \le \int_0^1 f[(1-t)x+ty]\,p(t)\,dt \le \frac{f(x)+f(y)}{2}\int_0^1 p(t)\,dt. \tag{1.9}$$

Motivated by the above results, we establish in this paper some refinements and reverses of Féjer's inequalities (1.9). Some applications for norms and semi-inner products are also provided.

2. Refinements and Reverse Féjer Inequalities

We have:

Theorem 3. *Let f be an convex function on C and $x,\ y \in C$ with $x \neq y$. If $p : [0,1] \to [0,\infty)$ is Lebesgue integrable and symmetric, namely $p(1-t) = p(t)$ for all $t \in [0,1]$, then*

$$\begin{aligned}0 &\le \frac{1}{2}\left[\nabla_+ f_{\frac{x+y}{2}}(y-x) - \nabla_- f_{\frac{x+y}{2}}(y-x)\right]\int_0^1 \left|t-\frac{1}{2}\right| p(t)\,dt \\ &\le \int_0^1 f((1-t)x+ty)\,p(t)\,dt - f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt \\ &\le \frac{1}{2}\left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x)\right]\left(\int_0^1 \left|t-\frac{1}{2}\right| p(t)\,dt\right).\end{aligned} \tag{2.1}$$

Proof. Let $x,\ y \in C$, with $x \neq y$. Since $\varphi_{(x,y)}$ is differentiable everywhere on $[0,1]$ except a countable number of points, by using the integration by parts formula for Lebesgue integral, we have

$$\begin{aligned}&\int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt \\ &= \left(\int_t^1 p(s)\,ds\right)\varphi_{(x,y)}(t)\Big]_{1/2}^1 + \int_{1/2}^1 p(t)\,\varphi_{(x,y)}(t)\,dt \\ &= -\left(\int_{1/2}^1 p(s)\,ds\right)\varphi_{(x,y)}(1/2) + \int_{1/2}^1 p(t)\,\varphi_{(x,y)}(t)\,dt \\ &= -\left(\int_{1/2}^1 p(s)\,ds\right) f\left(\frac{x+y}{2}\right) + \int_{1/2}^1 p(t)\,\varphi_{(x,y)}(t)\,dt\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt \\
&=\left(\int_0^t p(s)\,ds\right)\varphi_{(x,y)}(t)\Big]_0^{1/2}-\int_0^{1/2}p(t)\,\varphi_{(x,y)}(t)\,dt \\
&=\left(\int_0^{1/2} p(s)\,ds\right)\varphi_{(x,y)}(1/2)-\int_0^{1/2}p(t)\,\varphi_{(x,y)}(t)\,dt \\
&=\left(\int_0^{1/2} p(s)\,ds\right)f\left(\frac{x+y}{2}\right)-\int_0^{1/2}p(t)\,\varphi_{(x,y)}(t)\,dt.
\end{aligned}$$

By subtracting the second identity from the first, we get

$$\begin{aligned}
&\int_{1/2}^1\left(\int_t^1 p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt-\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt \\
&=\int_{1/2}^1 p(t)\,\varphi_{(x,y)}(t)\,dt+\int_0^{1/2}p(t)\,\varphi_{(x,y)}(t)\,dt \\
&-\left(\int_{1/2}^1 p(s)\,ds\right)f\left(\frac{x+y}{2}\right)-\left(\int_0^{1/2} p(s)\,ds\right)f\left(\frac{x+y}{2}\right).
\end{aligned}$$

By the symmetry of p we get

$$\int_{1/2}^1 p(s)\,ds=\int_0^{1/2}p(s)\,ds=\frac{1}{2}\int_0^1 p(s)\,ds$$

and then we get the following identity of interest in itself

$$\begin{aligned}
&\int_0^1 p(t)\,\varphi_{(x,y)}(t)\,dt-\int_0^1 p(s)\,ds f\left(\frac{x+y}{2}\right) \\
&=\int_{1/2}^1\left(\int_t^1 p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt-\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt.
\end{aligned}\tag{2.2}$$

By the convexity of $\varphi_{(x,y)}$ on $[0,1]$ we have

$$\nabla_+ f_x(y-x)=\varphi'_{+(x,y)}(0)\le\varphi'_{(x,y)}(t)\le\varphi'_{-(x,y)}\left(\frac{1}{2}\right)=\nabla_- f_{\frac{x+y}{2}}(y-x),$$

for almost every $t\in[0,1/2]$ and

$$\nabla_+ f_{\frac{x+y}{2}}(y-x)=\varphi'_{+(x,y)}\left(\frac{1}{2}\right)\le\varphi'_{(x,y)}(t)\le\varphi'_{-(x,y)}(1)=\nabla_- f_y(y-x),$$

for almost every $t\in[1/2,1]$.

This implies that

$$\begin{aligned}
\left(\int_0^t p(s)\,ds\right)\nabla_+ f_x(y-x)&\le\left(\int_0^t p(s)\,ds\right)\varphi'_{(x,y)}(t) \\
&\le\left(\int_0^t p(s)\,ds\right)\nabla_- f_{\frac{x+y}{2}}(y-x),
\end{aligned}$$

for almost every $t \in [0, 1/2]$ and

$$\left(\int_t^1 p(s)\,ds\right)\nabla_+ f_{\frac{x+y}{2}}(y-x) \le \left(\int_t^1 p(s)\,ds\right)\varphi'_{(x,y)}(t) \le \left(\int_t^1 p(s)\,ds\right)\nabla_- f_y(y-x),$$

for almost every $t \in [1/2, 1]$.

By integrating these two inequalities on the corresponding intervals, we obtain

$$\int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right)dt\nabla_+ f_{\frac{x+y}{2}}(y-x) \le \int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt \le \int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right)dt\nabla_- f_y(y-x)$$

and

$$-\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt\nabla_- f_{\frac{x+y}{2}}(y-x) \le -\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt \le -\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt\nabla_+ f_x(y-x).$$

If we add these inequalities, then we get

(2.3)

$$\begin{aligned}
&\int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right)dt\nabla_+ f_{\frac{x+y}{2}}(y-x) - \int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt\nabla_- f_{\frac{x+y}{2}}(y-x)\\
&\le \int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt - \int_0^{1/2}\left(\int_0^t p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt\\
&\le \int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right)dt\nabla_- f_y(y-x) - \int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt\nabla_+ f_x(y-x)
\end{aligned}$$

for all $x, y \in C$, with $x \ne y$.

Further, integrating by parts in the Lebesgue integral, we have

$$\begin{aligned}
\int_{1/2}^1 \left(\int_t^1 p(s)\,ds\right)dt &= \left(\int_t^1 p(s)\,ds\right)t\Big]_{1/2}^1 + \int_{1/2}^1 tp(t)\,dt\\
&= \int_{1/2}^1 tp(t)\,dt - \frac{1}{2}\int_{1/2}^1 p(s)\,ds\\
&= \int_{1/2}^1 \left(t-\frac{1}{2}\right)p(t)\,dt
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt &= \left(\int_0^t p(s)\,ds\right)t\Big]_0^{1/2} - \int_0^{1/2} p(t)\,tdt\\
&= \frac{1}{2}\int_0^{1/2} p(s)\,ds - \int_0^{1/2} p(t)\,tdt\\
&= \int_0^{1/2}\left(\frac{1}{2}-t\right)p(t)\,dt.
\end{aligned}$$

We have

$$\int_0^1 \left|t-\frac{1}{2}\right| p(t)\,dt = \int_{1/2}^1 \left(t-\frac{1}{2}\right) p(t)\,dt + \int_0^{1/2} \left(\frac{1}{2}-t\right) p(t)\,dt.$$

Since p is symmetric on $[0,1]$, hence by changing the variable $s=1-t$, we have

$$\begin{aligned}\int_0^{1/2} \left(\frac{1}{2}-t\right) p(t)\,dt &= \int_{1/2}^1 \left(s-\frac{1}{2}\right) p(1-s)\,ds \\ &= \int_{1/2}^1 \left(s-\frac{1}{2}\right) p(s)\,ds = \int_{1/2}^1 \left(t-\frac{1}{2}\right) p(t)\,dt,\end{aligned}$$

which shows that

$$\int_{1/2}^1 \left(t-\frac{1}{2}\right) p(t)\,dt = \int_0^{1/2} \left(\frac{1}{2}-t\right) p(t)\,dt = \frac{1}{2}\int_0^1 \left|t-\frac{1}{2}\right| p(t)\,dt.$$

By utilising (2.3) we then obtain (2.1). □

Remark 1. *If we put $p\equiv 1$ in (2.1), then we recapture the earlier result (1.3). If we take $p(t)=\left|t-\frac{1}{2}\right|$, $t\in[0,1]$ then we get*

$$\begin{aligned}(2.4)\qquad 0 &\le \frac{1}{24}\left[\nabla_+ f_{\frac{x+y}{2}}(y-x) - \nabla_- f_{\frac{x+y}{2}}(y-x)\right] \\ &\le \int_0^1 f((1-t)x+ty)\left|t-\frac{1}{2}\right| dt - \frac{1}{4} f\left(\frac{x+y}{2}\right) \\ &\le \frac{1}{24}\left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x)\right].\end{aligned}$$

We also have:

Theorem 4. *Let f be an convex function on C and x, $y\in C$, with $x\neq y$. If $p:[0,1]\to[0,\infty)$ is Lebesgue integrable and symmetric, namely $p(1-t)=p(t)$ for all $t\in[0,1]$, then*

$$\begin{aligned}(2.5)\qquad 0 &\le \frac{1}{2}\left[\nabla_+ f_{\frac{x+y}{2}}(y-x) - \nabla_- f_{\frac{x+y}{2}}(y-x)\right]\int_0^1 \left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right) p(t)\,dt \\ &\le \frac{f(x)+f(y)}{2}\int_0^1 p(t)\,dt - \int_0^1 f((1-t)x+ty)\,p(t)\,dt \\ &\le \frac{1}{2}\left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x)\right]\int_0^1 \left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right) p(t)\,dt.\end{aligned}$$

Proof. Using the integration by parts for Lebesgue's integral, we have

$$\begin{aligned}
&\int_0^1 \left(\int_0^t p(s)\,ds - \frac{1}{2}\int_0^1 p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt \\
&= \left(\int_0^t p(s)\,ds - \frac{1}{2}\int_0^1 p(s)\,ds\right)\varphi_{(x,y)}(t)\Big]_0^1 - \int_0^1 p(t)\,\varphi_{(x,y)}(t)\,dt \\
&= \left(\int_0^1 p(s)\,ds - \frac{1}{2}\int_0^1 p(s)\,ds\right)\varphi_{(x,y)}(1) + \left(\frac{1}{2}\int_0^1 p(s)\,ds\right)\varphi_{(x,y)}(0) \\
&- \int_0^1 p(t)\,\varphi_{(x,y)}(t)\,dt \\
&= \left(\int_0^1 p(t)\,dt\right)\frac{f(x)+f(y)}{2} - \int_0^1 p(t)\,\varphi_{(x,y)}(t)\,dt.
\end{aligned}$$

We also have

$$\begin{aligned}
&\int_0^1 \left(\int_0^t p(s)\,ds - \frac{1}{2}\int_0^1 p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt \\
&= \int_0^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt \\
&= \int_0^{1/2} \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt \\
&+ \int_{1/2}^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt \\
&= \int_{1/2}^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt \\
&- \int_0^{1/2} \left(\int_0^{1/2} p(s)\,ds - \int_0^t p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt.
\end{aligned}$$

Observe that

$$\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds \geq 0 \text{ for } t \in [1/2, 1]$$

and

$$\int_0^{1/2} p(s)\,ds - \int_0^t p(s)\,ds \geq 0 \text{ for } t \in [0, 1/2]\,.$$

By the convexity of $\varphi_{(x,y)}$ on the interval $[0,1]$, we deduce

$$\begin{aligned}
&\int_{1/2}^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds\right)dt\nabla_+ f_{\frac{x+y}{2}}(y-x) \\
&\leq \int_{1/2}^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt \\
&\leq \int_{1/2}^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds\right)dt\nabla_- f_y(y-x)
\end{aligned}$$

and

$$
\begin{aligned}
&-\int_{0}^{1/2}\left(\int_{0}^{1/2}p(s)\,ds-\int_{0}^{t}p(s)\,ds\right)dt\nabla_{-}f_{\frac{x+y}{2}}(y-x)\\
&\leq-\int_{0}^{1/2}\left(\int_{0}^{1/2}p(s)\,ds-\int_{0}^{t}p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt\\
&\leq-\int_{0}^{1/2}\left(\int_{0}^{1/2}p(s)\,ds-\int_{0}^{t}p(s)\,ds\right)dt\nabla_{+}f_{x}(y-x)\,.
\end{aligned}
$$

If we add these inequalities, then we get

$$
\begin{aligned}
&\int_{1/2}^{1}\left(\int_{0}^{t}p(s)\,ds-\int_{0}^{1/2}p(s)\,ds\right)dt\nabla_{+}f_{\frac{x+y}{2}}(y-x)\\
&-\int_{0}^{1/2}\left(\int_{0}^{1/2}p(s)\,ds-\int_{0}^{t}p(s)\,ds\right)dt\nabla_{-}f_{\frac{x+y}{2}}(y-x)\\
&\leq\int_{1/2}^{1}\left(\int_{0}^{t}p(s)\,ds-\int_{0}^{1/2}p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt\\
&-\int_{1/2}^{1}\left(\int_{0}^{t}p(s)\,ds-\int_{0}^{1/2}p(s)\,ds\right)\varphi'_{(x,y)}(t)\,dt\\
&\leq\int_{1/2}^{1}\left(\int_{0}^{t}p(s)\,ds-\int_{0}^{1/2}p(s)\,ds\right)dt\nabla_{-}f_{y}(y-x)\\
&-\int_{0}^{1/2}\left(\int_{0}^{1/2}p(s)\,ds-\int_{0}^{t}p(s)\,ds\right)dt\nabla_{+}f_{x}(y-x)\,.
\end{aligned}
\tag{2.6}
$$

Since

$$
\begin{aligned}
&\int_{1/2}^{1}\left(\int_{0}^{t}p(s)\,ds-\int_{0}^{1/2}p(s)\,ds\right)dt\\
&=\int_{1/2}^{1}\left(\int_{0}^{t}p(s)\,ds\right)dt-\frac{1}{2}\int_{0}^{1/2}p(s)\,ds\\
&=\left(\int_{0}^{t}p(s)\,ds\right)t\Big]_{1/2}^{1}-\int_{1/2}^{1}tp(t)\,dt-\frac{1}{2}\int_{0}^{1/2}p(s)\,ds\\
&=\int_{0}^{1}p(s)\,ds-\frac{1}{2}\int_{0}^{1/2}p(s)\,ds-\int_{1/2}^{1}tp(t)\,dt-\frac{1}{2}\int_{0}^{1/2}p(s)\,ds\\
&=\int_{0}^{1}p(s)\,ds-\int_{0}^{1/2}p(s)\,ds-\int_{1/2}^{1}tp(t)\,dt\\
&=\int_{1/2}^{1}p(s)\,ds-\int_{1/2}^{1}tp(t)\,dt=\int_{1/2}^{1}(1-t)\,p(t)\,dt
\end{aligned}
$$

and

$$\begin{aligned}
&\int_0^{1/2}\left(\int_0^{1/2} p(s)\,ds-\int_0^t p(s)\,ds\right)dt\\
&=\frac{1}{2}\int_0^{1/2} p(s)\,ds-\int_0^{1/2}\left(\int_0^t p(s)\,ds\right)dt\\
&=\frac{1}{2}\int_0^{1/2} p(s)\,ds-\left(\left(\int_0^t p(s)\,ds\right)t\Big]_0^{1/2}-\int_0^{1/2} tp(t)\,dt\right)\\
&=\frac{1}{2}\int_0^{1/2} p(s)\,ds-\frac{1}{2}\int_0^{1/2} p(s)\,ds+\int_0^{1/2} tp(t)\,dt=\int_0^{1/2} tp(t)\,dt.
\end{aligned}$$

If we change the variable $s=1-t$, then

$$\begin{aligned}
\int_0^{1/2} tp(t)\,dt&=-\int_1^{1/2}(1-s)\,p(1-s)\,ds=\int_{1/2}^1(1-s)\,p(1-s)\,ds\\
&=\int_{1/2}^1(1-s)\,p(s)\,ds.
\end{aligned}\tag{2.7}$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\\
&=\frac{1}{2}\int_0^{1/2}\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt+\frac{1}{2}\int_{1/2}^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\\
&=\frac{1}{2}\int_0^{1/2}\left(\frac{1}{2}-\frac{1}{2}+t\right)p(t)\,dt+\frac{1}{2}\int_{1/2}^1\left(\frac{1}{2}-t+\frac{1}{2}\right)p(t)\,dt\\
&=\frac{1}{2}\int_0^{1/2} tp(t)\,dt+\frac{1}{2}\int_{1/2}^1(1-t)\,p(t)\,dt=\int_0^{1/2} tp(t)\,dt \text{ (by 2.7)}
\end{aligned}$$

and by (2.6) we get the desired result (2.5). □

Remark 2. *If we put $p\equiv 1$ in (2.5), then we recapture the earlier result (1.6). If we take $p(t)=\left|t-\frac{1}{2}\right|$, $t\in[0,1]$ in (2.5), then we get*

$$\begin{aligned}
0&\leq\frac{1}{48}\left[\nabla_+ f_{\frac{x+y}{2}}(y-x)-\nabla_- f_{\frac{x+y}{2}}(y-x)\right]\\
&\leq\frac{f(x)+f(y)}{8}-\int_0^1 f((1-t)\,x+ty)\left|t-\frac{1}{2}\right|dt\\
&\leq\frac{1}{48}\left[\nabla_- f_y(y-x)-\nabla_+ f_x(y-x)\right].
\end{aligned}\tag{2.8}$$

3. Examples for Norms

Now, assume that $(X,\|\cdot\|)$ is a normed linear space. The function $f_0(s)=\frac{1}{2}\|x\|^2$, $x\in X$ is convex and thus the following limits exist

(iv) $\langle x,y\rangle_s:=\nabla_+ f_{0,y}(x)=\lim\limits_{t\to 0+}\frac{\|y+tx\|^2-\|y\|^2}{2t}$;

(v) $\langle x,y\rangle_i:=\nabla_- f_{0,y}(x)=\lim\limits_{s\to 0-}\frac{\|y+sx\|^2-\|y\|^2}{2s}$;

for any x, $y \in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [2] or [7]), assuming that p, $q \in \{s, i\}$ and $p \neq q$:

(a) $\langle x, x\rangle_p = \|x\|^2$ for all $x \in X$;
(aa) $\langle \alpha x, \beta y\rangle_p = \alpha\beta \langle x, y\rangle_p$ if α, $\beta \geq 0$ and x, $y \in X$;
(aaa) $\left|\langle x, y\rangle_p\right| \leq \|x\| \|y\|$ for all x, $y \in X$;
(av) $\langle \alpha x + y, x\rangle_p = \alpha \langle x, x\rangle_p + \langle y, x\rangle_p$ if x, $y \in X$ and $\alpha \in \mathbb{R}$;
(v) $\langle -x, y\rangle_p = -\langle x, y\rangle_q$ for all x, $y \in X$;
(va) $\langle x + y, z\rangle_p \leq \|x\| \|z\| + \langle y, z\rangle_p$ for all x, y, $z \in X$;
(vaa) The mapping $\langle \cdot, \cdot\rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p = s$ (or $p = i$);
(vaaa) The normed linear space $(X, \|\cdot\|)$ is smooth at the point $x_0 \in X \setminus \{0\}$ if and only if $\langle y, x_0\rangle_s = \langle y, x_0\rangle_i$ for all $y \in X$; in general $\langle y, x\rangle_i \leq \langle y, x\rangle_s$ for all x, $y \in X$;
(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle \cdot, \cdot\rangle$, then $\langle y, x\rangle_i = \langle y, x\rangle = \langle y, x\rangle_s$ for all x, $y \in X$.

The function $f_r(x) = \|x\|^r$ ($x \in X$ and $1 \leq r < \infty$) is also convex. Therefore, the following limits, which are related to the superior (inferior) semi-inner products,

$$\begin{aligned}\nabla_{\pm} f_{r,y}(x) &:= \lim_{t\to 0\pm} \frac{\|y + tx\|^r - \|y\|^r}{t} \\ &= r\|y\|^{r-1} \lim_{t\to 0\pm} \frac{\|y + tx\| - \|y\|}{t} = r\|y\|^{r-2} \langle x, y\rangle_{s(i)}\end{aligned}$$

exist for all x, $y \in X$ whenever $r \geq 2$; otherwise, they exist for any $x \in X$ and nonzero $y \in X$. In particular, if $r = 1$, then the following limits

$$\nabla_{\pm} f_{1,y}(x) := \lim_{t\to 0\pm} \frac{\|y + tx\| - \|y\|}{t} = \frac{\langle x, y\rangle_{s(i)}}{\|y\|}$$

exist for x, $y \in X$ and $y \neq 0$.

If we write the inequalities (2.1) for the function $f_r(x) = \|x\|^r$ ($x \in X$ and $1 \leq r < \infty$), then we get

$$\begin{aligned}0 &\leq \frac{1}{2} r \left\|\frac{x + y}{2}\right\|^{r-2} \left[\left\langle y - x, \frac{x + y}{2}\right\rangle_s - \left\langle y - x, \frac{x + y}{2}\right\rangle_i\right] \\ &\quad\times \int_0^1 \left|t - \frac{1}{2}\right| p(t)\, dt \\ &\leq \int_0^1 \|(1 - t)x + ty\|^r p(t)\, dt - \left\|\frac{x + y}{2}\right\|^r \int_0^1 p(t)\, dt \\ &\leq \frac{1}{2} r \left[\|y\|^{r-2} \langle y - x, y\rangle_s - \|x\|^{r-2} \langle y - x, x\rangle_i\right] \\ &\quad\times \int_0^1 \left|t - \frac{1}{2}\right| p(t)\, dt,\end{aligned} \tag{3.1}$$

for any $p : [0, 1] \to [0, \infty)$ a Lebesgue integrable and symmetric function.

If $r\geq 2$, then the inequality (3.1) holds for any $x,\,y\in X$. If $r\in[1,2)$, then the inequality (3.1) holds for any $x,\,y\in X$ with $x,\,y,\,x+y\neq 0$.

If we take $r=2$, then we get the simpler inequality

$$0\leq\left[\left\langle y-x,\frac{x+y}{2}\right\rangle_s-\left\langle y-x,\frac{x+y}{2}\right\rangle_i\right]\int_0^1\left|t-\frac{1}{2}\right|p(t)\,dt \tag{3.2}$$
$$\leq\int_0^1\|(1-t)\,x+ty\|^2\,p(t)\,dt-\left\|\frac{x+y}{2}\right\|^2\int_0^1 p(t)\,dt$$
$$\leq\left[\langle y-x,y\rangle_s-\langle y-x,x\rangle_i\right]\int_0^1\left|t-\frac{1}{2}\right|p(t)\,dt,$$

for any $x,\,y\in X$.

If we write the inequalities (2.5) for the function $f_r(x)=\|x\|^r$ ($x\in X$ and $1\leq r<\infty$), then for any $p:[0,1]\to[0,\infty)$ a Lebesgue integrable and symmetric function we get

$$0\leq\frac{1}{2}r\left\|\frac{x+y}{2}\right\|^{r-2}\left[\left\langle y-x,\frac{x+y}{2}\right\rangle_s-\left\langle y-x,\frac{x+y}{2}\right\rangle_i\right] \tag{3.3}$$
$$\times\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt$$
$$\leq\frac{\|x\|^r+\|y\|^r}{2}\int_0^1 p(t)\,dt-\int_0^1\|(1-t)\,x+ty\|^r\,p(t)\,dt$$
$$\leq\frac{1}{2}r\left[\|y\|^{r-2}\langle y-x,y\rangle_s-\|x\|^{r-2}\langle y-x,x\rangle_i\right]$$
$$\times\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt.$$

If $r\geq 2$, then the inequality (3.3) holds for any $x,\,y\in X$. If $r\in[1,2)$, then the inequality (3.3) holds for any $x,\,y\in X$ with $x,\,y,\,x+y\neq 0$.

If $(H;\langle\cdot,\cdot\rangle)$ is a real inner product space and $p:[0,1]\to[0,\infty)$ a Lebesgue integrable and symmetric function on $[0,1]$, then for (3.1) we have

$$0\leq\int_0^1\|(1-t)\,x+ty\|^r\,p(t)\,dt-\left\|\frac{x+y}{2}\right\|^r\int_0^1 p(t)\,dt \tag{3.4}$$
$$\leq\frac{1}{2}r\left\langle y-x,\|y\|^{r-2}\,y-\|x\|^{r-2}\,x\right\rangle\int_0^1\left|t-\frac{1}{2}\right|p(t)\,dt,$$

while from (3.3) we get

$$0\leq\frac{\|x\|^r+\|y\|^r}{2}\int_0^1 p(t)\,dt-\int_0^1\|(1-t)\,x+ty\|^r\,p(t)\,dt \tag{3.5}$$
$$\leq\frac{1}{2}r\left\langle y-x,\|y\|^{r-2}\,y-\|x\|^{r-2}\,x\right\rangle\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt.$$

In particular, for $r=2$, we derive the simpler inequalities

$$0\leq\int_0^1\|(1-t)\,x+ty\|^2\,p(t)\,dt-\left\|\frac{x+y}{2}\right\|^2\int_0^1 p(t)\,dt \tag{3.6}$$
$$\leq\|y-x\|^2\int_0^1\left|t-\frac{1}{2}\right|p(t)\,dt,$$

while from (3.3) we get

$$0 \le \frac{\|x\|^2 + \|y\|^2}{2} \int_0^1 p(t)\,dt - \int_0^1 \|(1-t)x + ty\|^2 p(t)\,dt \le \|y-x\|^2 \int_0^1 \left(\frac{1}{2} - \left|t - \frac{1}{2}\right|\right) p(t)\,dt \tag{3.7}$$

for all x, $y \in H$.

4. Examples for Functions of Several Variables

Now, let $\Omega \subset \mathbb{R}^n$ be an open convex set in $\mathbb{R}^n$. If $F : \Omega \to \mathbb{R}$ is a differentiable convex function on Ω, then, obviously, for any $\bar{c} \in \Omega$ we have

$$\nabla F_{\bar{c}}(\bar{y}) = \sum_{i=1}^n \frac{\partial F(\bar{c})}{\partial x_i} \cdot y_i, \;\; \bar{y} = (y_1, ..., y_n) \in \mathbb{R}^n,$$

where $\frac{\partial F}{\partial x_i}$ are the partial derivatives of F with respect to the variable x_i $(i = 1, ..., n)$.

Using the inequalities (2.1), we get for all $\bar{a}$, $\bar{b} \in \Omega$ and $p : [0,1] \to [0,\infty)$ a Lebesgue integrable and symmetric function on $[0,1]$ that

$$0 \le \int_0^1 F\left((1-t)\bar{a} + t\bar{b}\right) p(t)\,dt - F\left(\frac{\bar{a}+\bar{b}}{2}\right) \int_0^1 p(t)\,dt \le \frac{1}{2}\left(\int_0^1 \left|t - \frac{1}{2}\right| p(t)\,dt\right) \sum_{i=1}^n \left(\frac{\partial F(\bar{b})}{\partial x_i} - \frac{\partial F(\bar{a})}{\partial x_i}\right)(b_i - a_i) \tag{4.1}$$

and by (2.5) we obtain

$$0 \le \frac{F(\bar{a}) + F(\bar{b})}{2} \int_0^1 p(t)\,dt - \int_0^1 F\left((1-t)\bar{a} + t\bar{b}\right) p(t)\,dt \le \frac{1}{2} \int_0^1 \left(\frac{1}{2} - \left|t - \frac{1}{2}\right|\right) p(t)\,dt \sum_{i=1}^n \left(\frac{\partial F(\bar{b})}{\partial x_i} - \frac{\partial F(\bar{a})}{\partial x_i}\right)(b_i - a_i). \tag{4.2}$$

For $p \equiv 1$ we recapture the results obtained in [4] and [5] while for $p(t) = \left|t - \frac{1}{2}\right|$, $t \in [0,1]$ we get

$$0 \le \int_0^1 F\left((1-t)\bar{a} + t\bar{b}\right) \left|t - \frac{1}{2}\right| dt - \frac{1}{4} F\left(\frac{\bar{a}+\bar{b}}{2}\right) \le \frac{1}{24} \sum_{i=1}^n \left(\frac{\partial F(\bar{b})}{\partial x_i} - \frac{\partial F(\bar{a})}{\partial x_i}\right)(b_i - a_i) \tag{4.3}$$

and

$$0 \le \frac{F(\bar{a}) + F(\bar{b})}{8} - \int_0^1 F\left((1-t)\bar{a} + t\bar{b}\right) \left|t - \frac{1}{2}\right| dt \le \frac{1}{48} \sum_{i=1}^n \left(\frac{\partial F(\bar{b})}{\partial x_i} - \frac{\partial F(\bar{a})}{\partial x_i}\right)(b_i - a_i) \tag{4.4}$$

for all $\bar{a}$, $\bar{b} \in \Omega$.

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