COMPARING WEIGHTED AND INTEGRAL MEANS FOR CONVEX FUNCTIONS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let f be a convex function on I and $a,\ b\in I$ with $a<b$. If $p:[a,b]\to[0,\infty)$ is Lebesgue integrable and symmetric, namely $p(b+a-t)=p(t)$ for all $t\in[a,b]$, then we show in this paper among others that

$$\left|\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\,f(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\frac{1}{2}\left[\frac{f'_-(b)-f'_+(a)}{b-a}\right]\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)(x-a)^2\,dx.$$

Some examples are given as well.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)dt\leq\frac{f(a)+f(b)}{2},\quad a,\ b\in\mathbb{R},\ a<b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [8]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [1]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [8]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the *Hermite-Hadamard inequality*. For a monograph devoted to this result see [6]. Reverses of the Hermite-Hadamard inequality are provided in [2] and [3]. The recent survey paper [4] provides other related results.

In 1906, Fejér [7], while studying trigonometric polynomials, obtained inequalities which generalize that of Hermite & Hadamard:

Theorem 1. *Consider the integral $\int_a^b f(t)\,p(t)\,dt$, where f is a convex function in the interval (a,b) and p is a positive function in the same interval such that*

$$p(a+t)=p(b-t),\ \ 0\leq t\leq\frac{1}{2}(b-a),$$

i.e., $y=p(t)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b),0\right)$ and is normal to the t-axis. Under those conditions the

1991 *Mathematics Subject Classification.* 26D15, 26D10.

Key words and phrases. Convex functions, Integral inequalities, Hermite-Hadamard inequality, Féjer's inequalities, Integral mean, Weighted integral mean.

RGMIA Res. Rep. Coll. 22 (2019), Art. 93, 10 pp. Received 30/09/19

following inequalities are valid:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{\int_a^b p(t)\,dt}\int_a^b f(t)\,p(t)\,dt \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

If f is concave on (a,b), then the inequalities reverse in (1.2)

In the recent paper [5] we obtained the following refinement and reverse of Féjer's first inequality:

Theorem 2. *Let f be a convex function on I and a, $b \in I$, with $a < b$. If $p : [a,b] \to [0,\infty)$ is Lebesgue integrable and symmetric, namely $p(b+a-t) = p(t)$ for all $t \in [a,b]$, then*

$$\begin{aligned} 0 &\leq \frac{1}{2}\frac{1}{\int_a^b p(t)\,dt}\int_a^b \left|t-\frac{a+b}{2}\right| p(t)\,dt \left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right] \\ &\leq \frac{1}{\int_a^b p(t)\,dt}\int_a^b p(t)\,f(t)\,dt - f\left(\frac{a+b}{2}\right) \\ &\leq \frac{1}{2}\frac{1}{\int_a^b p(t)\,dt}\int_a^b \left|t-\frac{a+b}{2}\right| p(t)\,dt\left[f'_-(b) - f'_+(a)\right]. \end{aligned} \tag{1.3}$$

In the same paper [5] we also obtained the corresponding result for the second Féjer's inequality:

Theorem 3. *Let f be a convex function on I and a, $b \in I$, with $a < b$. If $p : [a,b] \to [0,\infty)$ is Lebesgue integrable and symmetric, namely $p(b+a-t) = p(t)$ for all $t \in [a,b]$, then*

$$\begin{aligned} 0 &\leq \frac{1}{2}\frac{1}{\int_a^b p(t)\,dt}\int_a^b \left[\frac{1}{2}(b-a) - \left|t-\frac{a+b}{2}\right|\right] p(t)\,dt \\ &\times \left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right] \\ &\leq \frac{f(a)+f(b)}{2} - \frac{1}{\int_a^b p(t)\,dt}\int_a^b p(t)\,f(t)\,dt \\ &\leq \frac{1}{2}\frac{1}{\int_a^b p(t)\,dt}\int_a^b \left[\frac{1}{2}(b-a) - \left|t-\frac{a+b}{2}\right|\right] p(t)\,dt \\ &\times \left[f'_-(b) - f'_+(a)\right]. \end{aligned} \tag{1.4}$$

Motivated by the above results, in this paper we compare the weighted integral mean

$$\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\,f(x)\,dx$$

with the integral mean

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

in the case of convex functions $f : [a,b] \to \mathbb{R}$ and integrable and nonnegative wight p. The case of symmetric weights p on $[a,b]$ is also analyzed. Some examples are given as well.

2. The Main Results

We have the following equality:

Lemma 1. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ and $g:[a,b]\to\mathbb{C}$ a Lebesgue integrable function, then*

$$\begin{aligned}(2.1)\qquad &(b-a)\int_a^b g(x)f(x)\,dx-\int_a^b f(x)\,dx\int_a^b g(x)\,dx\\ &=\int_a^b g(x)\left(\int_a^x (t-a)f'(t)\,dt\right)dx+\int_a^b g(x)\left(\int_x^b (t-b)f'(t)\,dt\right)dx.\end{aligned}$$

Proof. We start to the Montgomery identity for an absolutely continuous function $f:[a,b]\to\mathbb{R}$

$$f(x)(b-a)-\int_a^b f(t)\,dt=\int_a^x (t-a)f'(t)\,dt+\int_x^b (t-b)f'(t)\,dt$$

that holds for all $x\in[a,b]$.

If we multiply this identity by $g(x)$ and integrate over x in $[a,b]$, then we get

$$\begin{aligned}(2.2)\qquad &(b-a)\int_a^b g(x)f(x)\,dx-\int_a^b f(t)\,dt\int_a^b g(x)\,dx\\ &=\int_a^b g(x)\left(\int_a^x (t-a)f'(t)\,dt\right)dx+\int_a^b g(x)\left(\int_x^b (t-b)f'(t)\,dt\right)dx,\end{aligned}$$

which proves the desired identity (2.1). □

Theorem 4. *If $f:[a,b]\to\mathbb{R}$ is convex and $p:[a,b]\to[0,\infty)$ is Lebesgue integrable, then*

$$\begin{aligned}(2.3)\qquad &\frac{1}{2}\frac{1}{(b-a)\int_a^b p(x)\,dx}\int_a^b p(x)\left\{f'_+(a)(x-a)^2-f'_-(b)(b-x)^2\right\}dx\\ &\leq\frac{1}{2}\frac{1}{(b-a)\int_a^b p(x)\,dx}\\ &\times\int_a^b p(x)\left\{(x-a)^2\Delta(f;a,x)-(b-x)^2\Delta(f;x,b)\right\}dx\\ &\leq\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)f(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx,\end{aligned}$$

where $\Delta(f;\alpha,\beta)$ is the divided difference, namely

$$\Delta(f;\alpha,\beta)=\frac{f(\alpha)-f(\beta)}{\alpha-\beta}.$$

Proof. Since f is convex, then f' exists everywhere on $[a,b]$ except a countable number of points and is nondecreasing, then by Čebyšev's inequality for synchronous functions, we have

$$\begin{aligned}\int_a^x (t-a)f'(t)\,dt&\geq\frac{1}{x-a}\int_a^x (t-a)\,dt\int_a^x f'(t)\,dt\\ &=\frac{1}{2}(x-a)[f(x)-f(a)]\end{aligned}$$

and

$$\int_x^b (t-b) f'(t)\,dt \geq \frac{1}{b-x}\int_x^b (t-b)\,dt\,[f(b)-f(x)]$$
$$= -\frac{1}{2}(b-x)[f(b)-f(x)]$$

for $x \in (a,b)$.

These imply that

$$\int_a^b p(x)\left(\int_a^x (t-a) f'(t)\,dt\right)dx \geq \frac{1}{2}\int_a^b p(x)(x-a)[f(x)-f(a)]\,dx$$

and

$$\int_a^b p(x)\left(\int_x^b (t-b) f'(t)\,dt\right)dx \geq -\frac{1}{2}\int_a^b p(x)(b-x)[f(b)-f(x)]\,dx.$$

If we add these inequalities, then we get

$$\begin{aligned}
&\int_a^b p(x)\left(\int_a^x (t-a) f'(t)\,dt\right)dx + \int_a^b p(x)\left(\int_x^b (t-b) f'(t)\,dt\right)dx \\
&\geq \frac{1}{2}\int_a^b p(x)(x-a)[f(x)-f(a)]\,dx - \frac{1}{2}\int_a^b p(x)(b-x)[f(b)-f(x)]\,dx \\
&= \frac{1}{2}\int_a^b p(x)\{(x-a)[f(x)-f(a)]-(b-x)[f(b)-f(x)]\}\,dx \\
&= \frac{1}{2}\int_a^b p(x)\left\{(x-a)^2\frac{f(x)-f(a)}{x-a}-(b-x)^2\frac{f(b)-f(x)}{b-x}\right\}dx \\
&= \frac{1}{2}\int_a^b p(x)\left\{(x-a)^2\Delta(f;a,x)-(b-x)^2\Delta(f;x,b)\right\}dx.
\end{aligned}$$

By using the first identity in (2.1) for $g=p$, we get the second inequality in (2.3).

By the convexity of f we have

$$\Delta(f;a,x) \geq f'_+(a) \text{ and } f'_-(b) \geq \Delta(f;x,b),$$

which proves the first inequality in (2.3). □

In the following we provide another direct proof of the inequality between the first and last term in (2.3) and a reverse inequality as well.

Theorem 5. *If $f:[a,b]\to\mathbb{R}$ is convex and $p:[a,b]\to[0,\infty)$ is Lebesgue integrable, then*

$$\begin{aligned}
&\frac{1}{2}\frac{1}{(b-a)\int_a^b p(x)\,dx}\int_a^b p(x)\left\{f'_+(a)(x-a)^2-f'_-(b)(b-x)^2\right\}dx \\
&\leq \frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x) f(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \\
&\leq \frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x) f'(x)\left(x-\frac{a+b}{2}\right)dx.
\end{aligned} \tag{2.4}$$

We also have

$$\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\,f(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx \tag{2.5}$$
$$\leq \frac{1}{2}\frac{1}{(b-a)\int_a^b p(x)\,dx}\int_a^b p(x)\left[f_-'(b)(x-a)^2-f_+'(a)(b-x)^2\right]dx.$$

Proof. Using the convexity of f we get

$$f_-'(x)\int_a^x (t-a)\,dt \geq \int_a^x (t-a)\,f'(t)\,dt \geq f_+'(a)\int_a^x (t-a)\,dt$$

and

$$f_-'(b)\int_x^b (b-t)\,dt \geq \int_x^b (b-t)\,f'(t)\,dt \geq f_+'(x)\int_x^b (b-t)\,dt$$

namely

$$\frac{1}{2}f_-'(x)(x-a)^2 \geq \int_a^x (t-a)\,f'(t)\,dt \geq \frac{1}{2}f_+'(a)(x-a)^2$$

and

$$\frac{1}{2}f_-'(b)(b-x)^2 \geq \int_x^b (b-t)\,f'(t)\,dt \geq \frac{1}{2}f_+'(x)(b-x)^2$$

for $x \in (a,b)$.

These imply that

$$\frac{1}{2}\int_a^b p(x)\,f_-'(x)(x-a)^2\,dx$$
$$\geq \int_a^b p(x)\left(\int_a^x (t-a)\,f'(t)\,dt\right)dx \geq \frac{1}{2}f_+'(a)\int_a^b p(x)(x-a)^2\,dx$$

and

$$-\frac{1}{2}\int_a^b p(x)\,f_+'(x)(b-x)^2\,dx$$
$$\geq -\int_a^b p(x)\left(\int_x^b (b-t)\,f'(t)\,dt\right)dx \geq -\frac{1}{2}f_-'(b)\int_a^b p(x)(b-x)^2\,dx$$

and, by addition

$$\frac{1}{2}\int_a^b p(x)\,f_-'(x)(x-a)^2\,dx-\frac{1}{2}\int_a^b p(x)\,f_+'(x)(b-x)^2\,dx \tag{2.6}$$
$$\geq \int_a^b p(x)\left(\int_a^x (t-a)\,f'(t)\,dt\right)dx-\int_a^b p(x)\left(\int_x^b (b-t)\,f'(t)\,dt\right)dx$$
$$\geq \frac{1}{2}f_+'(a)\int_a^b p(x)(x-a)^2\,dx-\frac{1}{2}f_-'(b)\int_a^b p(x)(b-x)^2\,dx$$
$$=\frac{1}{2}\int_a^b p(x)\left\{f_+'(a)(x-a)^2\,dx-f_-'(b)(b-x)^2\right\}dx.$$

Since $f'_-(x) = f'_+(x)$ for every $x \in (a,b)$ except a countable number of points, we can write $f'(x)$ for either $f'_-(x)$ or $f'_+(x)$. Then

$$\begin{aligned}
&\frac{1}{2}\int_a^b p(x) f'_-(x)(x-a)^2 dx - \frac{1}{2}\int_a^b p(x) f'_+(x)(b-x)^2 dx \\
&= \frac{1}{2}\int_a^b p(x) f'(x)\left[(x-a)^2 - (b-x)^2\right] dx \\
&= (b-a)\int_a^b p(x) f'(x)\left(x - \frac{a+b}{2}\right) dx.
\end{aligned}$$

By making use of the first identity in (2.1) for $g = f$ and the inequality (2.6) we get the second inequality in (2.4).

By the convexity of f we also have

$$\int_a^b p(x) f'_-(x)(x-a)^2 dx \le f'_-(b)\int_a^b p(x)(x-a)^2 dx$$

and

$$f'_+(a)\int_a^b p(x)(b-x)^2 dx \le \int_a^b p(x) f'_+(x)(b-x)^2 dx.$$

These imply that

$$\begin{aligned}
&\frac{1}{2}\int_a^b p(x) f'_-(x)(x-a)^2 dx - \frac{1}{2}\int_a^b p(x) f'_+(x)(b-x)^2 dx \\
&\le \frac{1}{2} f'_-(b)\int_a^b p(x)(x-a)^2 dx - \frac{1}{2} f'_+(a)\int_a^b p(x)(b-x)^2 dx
\end{aligned}$$

and by (2.6) we get □

Corollary 1. *If $f : [a,b] \to \mathbb{R}$ is convex and $p : [a,b] \to [0,\infty)$ is Lebesgue integrable and symmetric, namely $p(a+b-x) = p(x)$ for all $x \in [a,b]$, then*

$$\begin{aligned}
&\frac{1}{2}\left[\frac{f'_+(a) - f'_-(b)}{b-a}\right]\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)(x-a)^2 dx \\
&\le \frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x) f(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \\
&\le \frac{1}{2}\left[f'_-(b) - f'_+(a)\right]\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\left|x - \frac{a+b}{2}\right| dx.
\end{aligned} \tag{2.7}$$

We also have

$$\begin{aligned}
&\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x) f(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \\
&\le \frac{1}{2}\left[\frac{f'_-(b) - f'_+(a)}{b-a}\right]\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)(x-a)^2 dx.
\end{aligned} \tag{2.8}$$

Proof. We have

$$\begin{aligned}
&\int_a^b p(x)\left\{f'_+(a)(x-a)^2 - f'_-(b)(b-x)^2\right\}dx \\
&= f'_+(a)\int_a^b p(x)(x-a)^2\,dx - f'_-(b)\int_a^b p(x)(b-x)^2\,dx \\
&= f'_+(a)\int_a^b p(x)(x-a)^2\,dx - f'_-(b)\int_a^b p(a+b-y)(y-a)^2\,dy \\
&= f'_+(a)\int_a^b p(x)(x-a)^2\,dx - f'_-(b)\int_a^b p(x)(x-a)^2\,dx \\
&= \left[f'_+(a) - f'_-(b)\right]\int_a^b p(x)(x-a)^2\,dx,
\end{aligned}$$

which proves the first inequality in (2.7).

Also,

$$\begin{aligned}
&\int_a^b p(x)f'(x)\left(x-\frac{a+b}{2}\right)dx \\
&= \frac{1}{2}\int_a^b p(x)f'(x)\left(x-\frac{a+b}{2}\right)dx + \frac{1}{2}\int_a^b p(x)f'(x)\left(x-\frac{a+b}{2}\right)dx \\
&= \frac{1}{2}\int_a^b p(x)f'(x)\left(x-\frac{a+b}{2}\right)dx \\
&+ \frac{1}{2}\int_a^b p(a+b-y)f'(a+b-y)\left(\frac{a+b}{2}-y\right)dy \\
&= \frac{1}{2}\int_a^b p(x)f'(x)\left(x-\frac{a+b}{2}\right)dx \\
&- \frac{1}{2}\int_a^b p(x)f'(a+b-x)\left(x-\frac{a+b}{2}\right)dx \\
&= \frac{1}{2}\int_a^b p(x)\left[f'(x)-f'(a+b-x)\right]\left(x-\frac{a+b}{2}\right)dx.
\end{aligned}$$

By the Čebyšev's weighted inequality for synchronous functions, since both $f'(x)$ and $g(x) := x - \frac{a+b}{2}$ are nondecreasing, hence

$$\begin{aligned}
&\int_a^b p(x)f'(x)\left(x-\frac{a+b}{2}\right)dx \\
&\geq \frac{1}{\int_a^b p(x)}\int_a^b p(x)f'(x)\,dx\int_a^b p(x)\left(x-\frac{a+b}{2}\right)dx = 0
\end{aligned}$$

since the function $p(x)\left(x-\frac{a+b}{2}\right)$ is asymmetric on $[a,b]$.

Therefore

$$
\begin{aligned}
0 &\leq \int_a^b p(x) f'(x) \left(x - \frac{a+b}{2}\right) dx = \left| \int_a^b p(x) f'(x) \left(x - \frac{a+b}{2}\right) dx \right| \\
&= \frac{1}{2} \left| \int_a^b p(x) \left[f'(x) - f'(a+b-x)\right] \left(x - \frac{a+b}{2}\right) dx \right| \\
&\leq \frac{1}{2} \int_a^b p(x) \left|f'(x) - f'(a+b-x)\right| \left|x - \frac{a+b}{2}\right| dx \\
&\leq \frac{1}{2} \left[f'_-(b) - f'_+(a)\right] \int_a^b p(x) \left|x - \frac{a+b}{2}\right| dx,
\end{aligned}
$$

which proves the second part of (2.7).

Now, observe that by the symmetry of p we have

$$
\int_a^b p(x) (b-x)^2 dx = \int_a^b p(a+b-y)(y-a)^2 dy = \int_a^b p(x)(x-a)^2 dx,
$$

which gives that

$$
\begin{aligned}
& f'_-(b) \int_a^b p(x)(x-a)^2 dx - f'_+(a) \int_a^b p(x)(b-x)^2 dx \\
&= f'_-(b) \int_a^b p(x)(x-a)^2 dx - f'_+(a) \int_a^b p(x)(x-a)^2 dx \\
&= \left[f'_-(b) - f'_+(a)\right] \int_a^b p(x)(x-a)^2 dx
\end{aligned}
$$

and by (2.5) we get (2.8). □

By utilising the first inequality in (2.7) and the inequality (2.8) we can state the following result as well:

Corollary 2. *If $f : [a,b] \to \mathbb{R}$ is convex and $p : [a,b] \to [0,\infty)$ is Lebesgue integrable and symmetric, then*

$$
\begin{aligned}
& \left| \frac{1}{\int_a^b p(x)\,dx} \int_a^b p(x) f(x)\,dx - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \\
& \leq \frac{1}{2} \left[\frac{f'_-(b) - f'_+(a)}{b-a}\right] \frac{1}{\int_a^b p(x)\,dx} \int_a^b p(x)(x-a)^2 dx.
\end{aligned} \tag{2.9}
$$

3. Some Examples

We consider the symmetric weight $p(x) = \left|x - \frac{a+b}{2}\right|$, $x \in [a,b]$. We have

$$
\int_a^b p(x)\,dx = \int_a^b \left|x - \frac{a+b}{2}\right| dx = \frac{1}{4}(b-a)^2,
$$

$$
\int_a^b p(x)(x-a)^2 dx = \int_a^b \left|x - \frac{a+b}{2}\right| (x-a)^2 dx = \frac{3}{32}(b-a)^4
$$

and

$$
\int_a^b p(x) \left|x - \frac{a+b}{2}\right| dx = \int_a^b \left(x - \frac{a+b}{2}\right)^2 dx = \frac{1}{12}(b-a)^3.
$$

By the inequality (2.7) for the convex function $f:[a,b]\to\mathbb{R}$ we have

$$-\frac{3}{16}(b-a)\left[f'_{-}(b)-f'_{+}(a)\right] \le \frac{4}{(b-a)^2}\int_a^b\left|x-\frac{a+b}{2}\right|f(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{1}{6}\left[f'_{-}(b)-f'_{+}(a)\right](b-a), \tag{3.1}$$

while from (2.9) we get

$$\left|\frac{4}{(b-a)^2}\int_a^b\left|x-\frac{a+b}{2}\right|f(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{3}{16}(b-a)\left[f'_{-}(b)-f'_{+}(a)\right]. \tag{3.2}$$

The second inequality in (3.1) is better than the corresponding inequality in (3.2).

Consider the symmetric weight $p(x)=(b-x)(x-a)$, $x\in[a,b]$. We have

$$\int_a^b p(x)\,dx=\int_a^b(b-x)(x-a)\,dx=\frac{1}{6}(b-a)^3,$$

$$\int_a^b p(x)(x-a)^2\,dx=\int_a^b(b-x)(x-a)^3\,dx=\frac{1}{20}(b-a)^5$$

and

$$\int_a^b p(x)\left|x-\frac{a+b}{2}\right|dx=\int_a^b(b-x)(x-a)\left|x-\frac{a+b}{2}\right|dx=\frac{1}{32}(b-a)^4.$$

By the inequality (2.7) for the convex function $f:[a,b]\to\mathbb{R}$ we have

$$-\frac{3}{20}\left[f'_{-}(b)-f'_{+}(a)\right](b-a) \le \frac{6}{(b-a)^3}\int_a^b(b-x)(x-a)f(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx \le \left[f'_{-}(b)-f'_{+}(a)\right]\frac{3}{32}(b-a), \tag{3.3}$$

while from (2.9) we get

$$\left|\frac{6}{(b-a)^3}\int_a^b(b-x)(x-a)f(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{3}{20}\left[f'_{-}(b)-f'_{+}(a)\right](b-a). \tag{3.4}$$

The second inequality in (3.3) is better than the corresponding inequality in (3.4).

References

[1] E. F. Beckenbach, Convex functions, *Bull. Amer. Math. Soc.* **54** (1948), 439–460.

[2] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.* **3** (2002), No. 2, Article 31. [Online https://www.emis.de/journals/JIPAM/article183.html?sid=183].

[3] S. S. Dragomir, An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.* **3** (2002), No. 3, Article 35. [Online https://www.emis.de/journals/JIPAM/article187.html?sid=187].

[4] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results. *Aust. J. Math. Anal. Appl.* **14** (2017), no. 1, Art. 1, 283 pp. [Online http://ajmaa.org/cgi-bin/paper.pl?string=v14n1/V14I1P1.tex].

[5] S. S. Dragomir. Reverses of Fejer's inequalities for convex functions, *RGMIA Res. Rep. Coll.* **22** (2019), Art. 88, 11pp. [Online http://rgmia.org/papers/v22/v22a88.pdf].

[6] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, 2000. [Online http://rgmia.org/monographs/hermite_hadamard.html].

[7] L. Féjer, Über die Fourierreihen, II, (In Hungarian). *Math. Naturwiss, Anz. Ungar. Akad. Wiss.*, **24** (1906), 369-390.

[8] D. S. Mitrinović and I. B. Lacković, Hermite and convexity, *Aequationes Math.* **28** (1985), 229–232.

[9] A. W. Roberts and D. E. Varberg, *Convex Functions,* Academic Press, 1973.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johanesburg, South Africa.