SOME INEQUALITIES FOR WEIGHTED AND INTEGRAL MEANS OF CONVEX FUNCTIONS

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Abstract. Let $f:[a,b]\to\mathbb{R}$ be convex and $p:[a,b]\to\mathbb{R}$ a Lebesgue integrable and symmetric function such that the condition

$$0\leq\int_a^x p(s)\,ds\leq\int_a^b p(s)\,ds \text{ for all } x\in[a,b]$$

holds. We show in this paper among others that

$$\begin{aligned}&\left|\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)f(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\leq\frac{f'_-(b)-f'_+(a)}{b-a}\frac{1}{\int_a^b p(x)\,dx}\int_a^b\left(\int_a^x p(s)\,ds\right)(b-x)\,dx\\&\leq\frac{1}{2}\left[f'_-(b)-f'_+(a)\right](b-a).\end{aligned}$$

Some examples are also given.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)dt\leq\frac{f(a)+f(b)}{2},\quad a,\ b\in\mathbb{R},\ a<b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [8]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [1]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [8]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the *Hermite-Hadamard inequality.* For a monograph devoted to this result see [6]. Reverses of the Hermite-Hadamard inequality are provided in [2] and [3]. The recent survey paper [4] provides other related results.

In 1906, Fejér [7], while studying trigonometric polynomials, obtained inequalities which generalize that of Hermite & Hadamard:

Theorem 1. *Consider the integral $\int_a^b f(t)\,p(t)\,dt$, where f is a convex function in the interval (a,b) and p is a positive function in the same interval such that*

$$p(a+t)=p(b-t),\quad 0\leq t\leq\frac{1}{2}(b-a),$$

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i.e., $y = p(t)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b), 0\right)$ and is normal to the t-axis. Under those conditions the following inequalities are valid:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{\int_a^b p(t)\,dt}\int_a^b f(t)\,p(t)\,dt \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

If f is concave on (a,b), then the inequalities reverse in (1.2)

In the recent paper [5] we obtained the following refinement and reverse of Féjer's first inequality:

Theorem 2. *Let f be a convex function on I and a, $b \in I$, with $a < b$. If $p : [a,b] \to [0,\infty)$ is Lebesgue integrable and symmetric, namely $p(b+a-t) = p(t)$ for all $t \in [a,b]$, then*

$$\begin{aligned} 0 &\leq \frac{1}{2}\frac{1}{\int_a^b p(t)\,dt}\int_a^b \left|t - \frac{a+b}{2}\right| p(t)\,dt \left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right] \\ &\leq \frac{1}{\int_a^b p(t)\,dt}\int_a^b p(t)\,f(t)\,dt - f\left(\frac{a+b}{2}\right) \\ &\leq \frac{1}{2}\frac{1}{\int_a^b p(t)\,dt}\int_a^b \left|t - \frac{a+b}{2}\right| p(t)\,dt \left[f'_-(b) - f'_+(a)\right]. \end{aligned} \tag{1.3}$$

In the same paper [5] we also obtained the corresponding result for the second Féjer's inequality:

Theorem 3. *Let f be a convex function on I and a, $b \in I$, with $a < b$. If $p : [a,b] \to [0,\infty)$ is Lebesgue integrable and symmetric, namely $p(b+a-t) = p(t)$ for all $t \in [a,b]$, then*

$$\begin{aligned} 0 &\leq \frac{1}{2}\frac{1}{\int_a^b p(t)\,dt}\int_a^b \left[\frac{1}{2}(b-a) - \left|t - \frac{a+b}{2}\right|\right] p(t)\,dt \\ &\quad\times \left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right] \\ &\leq \frac{f(a)+f(b)}{2} - \frac{1}{\int_a^b p(t)\,dt}\int_a^b p(t)\,f(t)\,dt \\ &\leq \frac{1}{2}\frac{1}{\int_a^b p(t)\,dt}\int_a^b \left[\frac{1}{2}(b-a) - \left|t - \frac{a+b}{2}\right|\right] p(t)\,dt \\ &\quad\times \left[f'_-(b) - f'_+(a)\right]. \end{aligned} \tag{1.4}$$

Motivated by the above results, in this paper we compare the weighted integral mean

$$\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\,f(x)\,dx$$

with the integral mean

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

in the case of convex functions $f : [a, b] \to \mathbb{R}$ and integrable wight p satisfying the condition

$$0 \le \int_a^x p(s)\, ds \le \int_a^b p(s)\, ds \text{ for all } x \in [a, b].$$

The case of symmetric weights p on $[a, b]$ is also analyzed. Some examples are given as well.

2. The Results

We start to the following identity that is of interest in itself as well:

Lemma 1. *Let $f : [a, b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$ and $g : [a, b] \to \mathbb{C}$ a Lebesgue integrable function, then we have the equalities*

$$\begin{aligned}(2.1)\quad & (b-a)\int_a^b g(x) f(x)\, dx - \int_a^b f(x)\, dx \int_a^b g(x)\, dx \\ &= \int_a^b \left(\int_x^b g(s)\, ds\right)(x-a) f'(x)\, dx + \int_a^b \left(\int_a^x g(s)\, ds\right)(x-b) f'(x)\, dx.\end{aligned}$$

Proof. We start to the Montgomery identity for an absolutely continuous function $f : [a, b] \to \mathbb{R}$

$$f(x)(b-a) - \int_a^b f(t)\, dt = \int_a^x (t-a) f'(t)\, dt + \int_x^b (t-b) f'(t)\, dt$$

that holds for all $x \in [a, b]$.

If we multiply this identity by $g(x)$ and integrate over x in $[a, b]$, then we get

$$\begin{aligned}(2.2)\quad & (b-a)\int_a^b g(x) f(x)\, dx - \int_a^b f(t)\, dt \int_a^b g(x)\, dx \\ &= \int_a^b g(x)\left(\int_a^x (t-a) f'(t)\, dt\right) dx + \int_a^b g(x)\left(\int_x^b (t-b) f'(t)\, dt\right) dx.\end{aligned}$$

Using integration by parts, we get

$$\begin{aligned}(2.3)\quad & \int_a^b g(x)\left(\int_a^x (t-a) f'(t)\, dt\right) dx \\ &= \int_a^b \left(\int_a^x (t-a) f'(t)\, dt\right) d\left(\int_a^x g(s)\, ds\right) \\ &= \left(\int_a^x (t-a) f'(t)\, dt\right)\left(\int_a^x g(s)\, ds\right)\Bigg|_a^b \\ &\quad - \int_a^b \left(\int_a^x g(s)\, ds\right)(x-a) f'(x)\, dx\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_a^b (t-a) f'(t)\, dt\right) \left(\int_a^b g(s)\, ds\right) \\
&- \int_a^b \left(\int_a^x g(s)\, ds\right) (x-a) f'(x)\, dx \\
&= \int_a^b \left(\int_a^b g(s)\, ds - \int_a^x g(s)\, ds\right) (x-a) f'(x)\, dx \\
&= \int_a^b \left(\int_x^b g(s)\, ds\right) (x-a) f'(x)\, dx
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_a^b g(x) \left(\int_x^b (t-b) f'(t)\, dt\right) dx \\
&= \int_a^b \left(\int_x^b (t-b) f'(t)\, dt\right) d\left(\int_a^x g(s)\, ds\right) \\
&= \left(\int_x^b (t-b) f'(t)\, dt\right) \left(\int_a^x g(s)\, ds\right)\Bigg|_a^b \\
&+ \int_a^b \left(\int_a^x g(s)\, ds\right) (x-b) f'(x)\, dx \\
&= \int_a^b \left(\int_a^x g(s)\, ds\right) (x-b) f'(x)\, dx,
\end{aligned} \tag{2.4}
$$

which proves the second identity on (2.1). □

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be convex and $p : [a,b] \to \mathbb{R}$ a Lebesgue integrable function such that*

$$
0 \leq \int_a^x p(s)\, ds \leq \int_a^b p(s)\, ds \text{ for all } x \in [a,b]. \tag{2.5}
$$

Then we have the inequalities

$$
\begin{aligned}
&f'_+(a) \int_a^b \left(\int_x^b p(s)\, ds\right) (x-a)\, dx - f'_-(b) \int_a^b \left(\int_a^x p(s)\, ds\right) (b-x)\, dx \\
&\leq (b-a) \int_a^b p(x) f(x)\, dx - \int_a^b f(x)\, dx \int_a^b p(x)\, dx \\
&\leq f'_-(b) \int_a^b \left(\int_x^b p(s)\, ds\right) (x-a)\, dx - f'_+(a) \int_a^b \left(\int_a^x p(s)\, ds\right) (b-x)
\end{aligned} \tag{2.6}
$$

or, equivalently,

$$\begin{aligned}(2.7)\qquad &\int_a^b \left(\int_a^x \left[f'_+(a)\,p(a+b-s) - f'_-(b)\,p(s)\right] ds\right)(b-x)\,dx \\ &\leq (b-a)\int_a^b p(x)\,f(x)\,dx - \int_a^b f(x)\,dx \int_a^b p(x)\,dx \\ &\leq \int_a^b \left(\int_a^x \left[f'_-(b)\,p(a+b-s) - f'_+(a)\,p(s)\right] ds\right)(b-x)\,dx.\end{aligned}$$

Proof. We have for f convex and $p : [a,b] \to \mathbb{R}$ a Lebesgue integrable function that

$$\begin{aligned}(2.8)\quad &(b-a)\int_a^b p(x)\,f(x)\,dx - \int_a^b f(x)\,dx \int_a^b p(x)\,dx \\ &= \int_a^b \left(\int_x^b p(s)\,ds\right)(x-a)\,f'(x)\,dx - \int_a^b \left(\int_a^x p(s)\,ds\right)(b-x)\,f'(x)\,dx.\end{aligned}$$

By the convexity of f we have that

$$(2.9)\qquad (x-a)\,f'_-(b) \geq (x-a)\,f'(x) \geq (x-a)\,f'_+(a)$$

and

$$(2.10)\qquad (b-x)\,f'_-(b) \geq (b-x)\,f'(x) \geq (b-x)\,f'_+(a)$$

for all $x \in (a,b)$.

From

$$\int_a^x p(s)\,ds \leq \int_a^b p(s)\,ds = \int_a^x p(s)\,ds + \int_x^b p(s)\,ds,$$

which implies that $\int_x^b p(s)\,ds \geq 0$ for all $x \in (a,b)$.

From (2.9) we get that

$$\begin{aligned}\left(\int_x^b p(s)\,ds\right)(x-a)\,f'_-(b) &\geq \left(\int_x^b p(s)\,ds\right)(x-a)\,f'(x) \\ &\geq \left(\int_x^b p(s)\,ds\right)(x-a)\,f'_+(a)\end{aligned}$$

and from (2.10) that

$$\begin{aligned}-\left(\int_a^x p(s)\,ds\right)(b-x)\,f'_+(a) &\leq -\left(\int_a^x p(s)\,ds\right)(b-x)\,f'(x) \\ &\leq -\left(\int_a^x p(s)\,ds\right)(b-x)\,f'_-(b)\end{aligned}$$

all $x \in (a,b)$.

If we integrate these inequalities over $x \in [a,b]$ and add the obtained results, we get

$$\begin{aligned}
&f'_{-}(b)\int_a^b\left(\int_x^b p(s)\,ds\right)(x-a)\,dx - f'_{+}(a)\int_a^b\left(\int_a^x p(s)\,ds\right)(b-x)\\
&\geq \int_a^b\left(\int_x^b p(s)\,ds\right)(x-a)\,f'(x)\,dx - \int_a^b\left(\int_a^x p(s)\,ds\right)(b-x)\,f'(x)\,dx\\
&\geq f'_{+}(a)\int_a^b\left(\int_x^b p(s)\,ds\right)(x-a)\,dx - f'_{-}(b)\int_a^b\left(\int_a^x p(s)\,ds\right)(b-x)\,dx.
\end{aligned}$$

By using the equality (2.1) we get

$$\begin{aligned}
&f'_{+}(a)\int_a^b\left(\int_x^b p(s)\,ds\right)(x-a)\,dx - f'_{-}(b)\int_a^b\left(\int_a^x p(s)\,ds\right)(b-x)\,dx\\
&\leq (b-a)\int_a^b p(x)\,f(x)\,dx - \int_a^b f(x)\,dx\int_a^b p(x)\,dx\\
&\leq f'_{-}(b)\int_a^b\left(\int_x^b p(s)\,ds\right)(x-a)\,dx - f'_{+}(a)\int_a^b\left(\int_a^x p(s)\,ds\right)(b-x),
\end{aligned}\tag{2.11}$$

namely (2.6).

If we change the variable $y = a+b-x$, then we have

$$\int_a^b\left(\int_x^b p(s)\,ds\right)(x-a)\,dx = \int_a^b\left(\int_{a+b-y}^b p(s)\,ds\right)(b-y)\,dy.$$

Also by the change of variable $u = a+b-s$, we get

$$\int_{a+b-y}^b p(s)\,ds = \int_a^y p(a+b-u)\,du,$$

which implies that

$$\int_a^b\left(\int_x^b p(s)\,ds\right)(x-a)\,dx = \int_a^b\left(\int_a^x p(a+b-s)\,ds\right)(b-x)\,dx.$$

Therefore

$$\begin{aligned}
&f'_{-}(b)\int_a^b\left(\int_x^b p(s)\,ds\right)(x-a)\,dx - f'_{+}(a)\int_a^b\left(\int_a^x p(s)\,ds\right)(b-x)\\
&= f'_{-}(b)\int_a^b\left(\int_a^x p(a+b-s)\,ds\right)(b-x)\,dx\\
&- f'_{+}(a)\int_a^b\left(\int_a^x p(s)\,ds\right)(b-x)\,dx\\
&= \int_a^b\left(\int_a^x\left[f'_{-}(b)\,p(a+b-s) - f'_{+}(a)\,p(s)\right]ds\right)(b-x)\,dx
\end{aligned}$$

and

$$
\begin{aligned}
& f'_{+}(a)\int_a^b\left(\int_x^b p(s)\,ds\right)(x-a)\,dx - f'_{-}(b)\int_a^b\left(\int_a^x p(s)\,ds\right)(b-x)\,dx \\
&= f'_{+}(a)\int_a^b\left(\int_a^x p(a+b-s)\,ds\right)(b-x)\,dx \\
&- f'_{-}(b)\int_a^b\left(\int_a^x p(s)\,ds\right)(b-x)\,dx \\
&= \int_a^b\left(\int_a^x \left[f'_{+}(a)\,p(a+b-s) - f'_{-}(b)\,p(s)\right]ds\right)(b-x)\,dx,
\end{aligned}
$$

and by (2.11) we get (2.7). □

We say that the function $p:[a,b]\to\mathbb{R}$ is symmetric on $[a,b]$ if

$$p(a+b-t) = p(t) \text{ for all } t\in[a,b].$$

Corollary 1. *Let $f:[a,b]\to\mathbb{R}$ be convex and $p:[a,b]\to\mathbb{R}$ a Lebesgue integrable and symmetric function such that the condition (2.5) holds. Then we have*

$$
\begin{aligned}
& \left|\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x) f(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\le \frac{f'_{-}(b)-f'_{+}(a)}{b-a}\,\frac{1}{\int_a^b p(x)\,dx}\int_a^b\left(\int_a^x p(s)\,ds\right)(b-x)\,dx \\
&\le \frac{1}{2}\left[f'_{-}(b)-f'_{+}(a)\right](b-a).
\end{aligned} \tag{2.12}
$$

Proof. Since p is symmetric, then $p(a+b-s)=p(s)$ for all $s\in[a,b]$ and by (2.7) we get

$$
\begin{aligned}
& \left[f'_{+}(a)-f'_{-}(b)\right]\int_a^b\left(\int_a^x p(s)\,ds\right)(b-x)\,dx \\
&\le (b-a)\int_a^b p(x) f(x)\,dx - \int_a^b f(x)\,dx\int_a^b p(x)\,dx \\
&\le \left[f'_{-}(b)-f'_{+}(a)\right]\int_a^b\left(\int_a^x p(s)\,ds\right)(b-x)\,dx,
\end{aligned}
$$

which is equivalent to the first part of (2.12).

Since $0\le\int_a^x p(s)\,ds\le\int_a^b p(x)\,dx$, hence

$$
\begin{aligned}
\int_a^b\left(\int_a^x p(s)\,ds\right)(b-x)\,dx &\le \int_a^b p(x)\,dx\int_a^b (b-x)\,dx \\
&= \frac{1}{2}(b-a)^2\int_a^b p(x)\,dx
\end{aligned}
$$

and the last part of (2.12) is proved. □

Remark 1. *If the function p is nonnegative and symmetric then the inequality (2.12) holds true.*

3. Some Examples

If we consider the weight $p:[a,b]\to[0,\infty)$, $p(x)=\left|x-\frac{a+b}{2}\right|$, then

$$\begin{aligned}
&\int_a^b\left(\int_a^x p(s)\,ds\right)(b-x)\,dx\\
&=\int_a^b\left(\int_a^x\left|s-\frac{a+b}{2}\right|ds\right)(b-x)\,dx\\
&=\int_a^{\frac{a+b}{2}}\left(\int_a^x\left|s-\frac{a+b}{2}\right|ds\right)(b-x)\,dx\\
&+\int_{\frac{a+b}{2}}^b\left(\int_a^x\left|s-\frac{a+b}{2}\right|ds\right)(b-x)\,dx\\
&=\int_a^{\frac{a+b}{2}}\left(\int_a^x\left(\frac{a+b}{2}-s\right)ds\right)(b-x)\,dx\\
&+\int_{\frac{a+b}{2}}^b\left(\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-s\right)ds+\int_{\frac{a+b}{2}}^x\left(s-\frac{a+b}{2}\right)\right)(b-x)\,dx\\
&=\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}(x-a)-\frac{x^2-a^2}{2}\right)(b-x)\,dx\\
&+\int_{\frac{a+b}{2}}^b\left(\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-s\right)ds+\int_{\frac{a+b}{2}}^x\left(s-\frac{a+b}{2}\right)ds\right)(b-x)\,dx.
\end{aligned}$$

We have

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}(x-a)-\frac{x^2-a^2}{2}\right)(b-x)\,dx\\
&=\frac{1}{2}\int_a^{\frac{a+b}{2}}(b-x)(x-a)(a+b-x-a)\,dx\\
&=\frac{1}{2}\int_a^{\frac{a+b}{2}}(b-x)^2(x-a)\,dx=\frac{11}{384}(b-a)^4
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{a+b}{2}}^b\left(\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-s\right)ds+\int_{\frac{a+b}{2}}^x\left(s-\frac{a+b}{2}\right)ds\right)(b-x)\,dx\\
&=\int_{\frac{a+b}{2}}^b\left(\frac{1}{8}(b-a)^2+\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2\right)(b-x)\,dx\\
&=\frac{1}{8}(b-a)^2\int_{\frac{a+b}{2}}^b(b-x)\,dx+\frac{1}{2}\int_{\frac{a+b}{2}}^b\left(x-\frac{a+b}{2}\right)^2(b-x)\,dx\\
&=\frac{7}{384}(b-a)^4.
\end{aligned}$$

Therefore

$$\int_a^b\left(\int_a^x p(s)\,ds\right)(b-x)\,dx=\frac{3}{64}(b-a)^4.$$

Since $\int_a^b \left|x - \frac{a+b}{2}\right| dx = \frac{1}{4}(b-a)^2$, hence

$$\frac{1}{\int_a^b p(x)\,dx}\int_a^b \left(\int_a^x p(s)\,ds\right)(b-x)\,dx = \frac{3}{16}(b-a)^2.$$

By utilising (2.12) we get

$$\left|\frac{4}{(b-a)^2}\int_a^b \left|x-\frac{a+b}{2}\right| f(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{3}{16}(b-a)\left[f'_-(b) - f'_+(a)\right], \tag{3.1}$$

where f is a convex function on $[a,b]$.

Consider now the symmetric function $p(x) = (b-x)(x-a)$, $x \in [a,b]$. Then

$$\int_a^x p(s)\,ds = \int_a^x (b-s)(s-a)\,ds = -\frac{1}{6}(x-a)^2(2x-3b+a),\ \ x \in [a,b]$$

and

$$\begin{aligned}\int_a^b \left(\int_a^x p(s)\,ds\right)(b-x)\,dx &= -\frac{1}{6}\int_a^b (x-a)^2(2x-3b+a)(b-x)\,dx \\ &= \frac{1}{40}(b-a)^5.\end{aligned}$$

Also

$$\int_a^b p(x)\,dx = \int_a^b (b-x)(x-a)\,dx = \frac{1}{6}(b-a)^3$$

and

$$\frac{1}{\int_a^b p(x)\,dx}\int_a^b \left(\int_a^x p(s)\,ds\right)(b-x)\,dx = \frac{3}{20}(b-a)^2$$

and by (2.12) we obtain

$$\left|\frac{6}{(b-a)^3}\int_a^b (b-x)(x-a)f(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{3}{20}(b-a)\left[f'_-(b) - f'_+(a)\right], \tag{3.2}$$

where f is a convex function on $[a,b]$.

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