BOUNDS FOR THE DIFFERENCE BETWEEN WEIGHTED AND INTEGRAL MEANS OF CONVEX FUNCTIONS

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Abstract. Let $f:[a,b]\to\mathbb{R}$ be convex and $p:[a,b]\to\mathbb{R}$ a Lebesgue integrable function such that

$$\frac{1}{x-a}\int_a^x p(s)\,ds\le\frac{1}{b-x}\int_x^b p(s)\,ds \text{ for all } x\in(a,b).$$

Then we have the inequalities

$$\begin{aligned}&f'_+(a)\left[\int_a^b xp(x)\,dx-\frac{a+b}{2}\int_a^b p(x)\,dx\right]\\&\le\int_a^b p(x)f(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\int_a^b p(x)\,dx\\&\le f'_-(b)\left[\int_a^b xp(x)\,dx-\frac{a+b}{2}\int_a^b p(x)\,dx\right].\end{aligned}$$

Some examples are also given.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(t)dt\le\frac{f(a)+f(b)}{2},\quad a,\ b\in\mathbb{R},\ a<b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [8]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [1]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [8]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the *Hermite-Hadamard inequality*. For a monograph devoted to this result see [6]. Reverses of the Hermite-Hadamard inequality are provided in [2] and [3]. The recent survey paper [4] provides other related results.

In 1906, Fejér [7], while studying trigonometric polynomials, obtained inequalities which generalize that of Hermite & Hadamard:

Theorem 1. *Consider the integral $\int_a^b f(t)\,p(t)\,dt$, where f is a convex function in the interval (a,b) and p is a positive function in the same interval such that*

$$p(a+t)=p(b-t),\ \ 0\le t\le\frac{1}{2}(b-a),$$

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i.e., $y = p(t)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b), 0\right)$ and is normal to the t-axis. Under those conditions the following inequalities are valid:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{\int_a^b p(t)\,dt} \int_a^b f(t)\,p(t)\,dt \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

If f is concave on (a,b), then the inequalities reverse in (1.2)

In the recent paper [5] we obtained the following refinement and reverse of Féjer's first inequality:

Theorem 2. *Let f be a convex function on I and a, $b \in I$, with $a < b$. If $p : [a,b] \to [0,\infty)$ is Lebesgue integrable and symmetric, namely $p(b+a-t) = p(t)$ for all $t \in [a,b]$, then*

$$\begin{aligned} 0 &\leq \frac{1}{2}\frac{1}{\int_a^b p(t)\,dt} \int_a^b \left|t - \frac{a+b}{2}\right| p(t)\,dt \left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right] \\ &\leq \frac{1}{\int_a^b p(t)\,dt} \int_a^b p(t)\,f(t)\,dt - f\left(\frac{a+b}{2}\right) \\ &\leq \frac{1}{2}\frac{1}{\int_a^b p(t)\,dt} \int_a^b \left|t - \frac{a+b}{2}\right| p(t)\,dt \left[f'_-(b) - f'_+(a)\right]. \end{aligned} \tag{1.3}$$

In the same paper [5] we also obtained the corresponding result for the second Féjer's inequality:

Theorem 3. *Let f be a convex function on I and a, $b \in I$, with $a < b$. If $p : [a,b] \to [0,\infty)$ is Lebesgue integrable and symmetric, namely $p(b+a-t) = p(t)$ for all $t \in [a,b]$, then*

$$\begin{aligned} 0 &\leq \frac{1}{2}\frac{1}{\int_a^b p(t)\,dt} \int_a^b \left[\frac{1}{2}(b-a) - \left|t - \frac{a+b}{2}\right|\right] p(t)\,dt \\ &\times \left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right] \\ &\leq \frac{f(a)+f(b)}{2} - \frac{1}{\int_a^b p(t)\,dt} \int_a^b p(t)\,f(t)\,dt \\ &\leq \frac{1}{2}\frac{1}{\int_a^b p(t)\,dt} \int_a^b \left[\frac{1}{2}(b-a) - \left|t - \frac{a+b}{2}\right|\right] p(t)\,dt \\ &\times \left[f'_-(b) - f'_+(a)\right]. \end{aligned} \tag{1.4}$$

Motivated by the above results, in this paper we establish upper and lower bounds for the difference

$$\int_a^b p(x)\,f(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \int_a^b p(x)\,dx$$

in the case of convex functions $f : [a,b] \to \mathbb{R}$ and integrable wight p satisfying the condition

$$\frac{1}{x-a}\int_a^x p(s)\,ds \leq \frac{1}{b-x}\int_x^b p(s)\,ds \text{ for all } x \in (a,b).$$

The case of monotonic nondecreasing weights p on $[a,b]$ is also analyzed. Some examples are given as well.

2. Main Results

We start with the following identity:

Lemma 1. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ and $g:[a,b]\to\mathbb{C}$ a Lebesgue integrable function, then we have the equality*

$$(b-a)\int_a^b g(x)\,f(x)\,dx-\int_a^b f(x)\,dx\int_a^b g(x)\,dx = \int_a^b (x-a)(b-x)\left(\frac{\int_x^b g(s)\,ds}{b-x}-\frac{\int_a^x g(s)\,ds}{x-a}\right)f'(x)\,dx. \tag{2.1}$$

Proof. We start to the Montgomery identity for an absolutely continuous function $f:[a,b]\to\mathbb{R}$

$$f(x)(b-a)-\int_a^b f(t)\,dt=\int_a^x (t-a)\,f'(t)\,dt+\int_x^b (t-b)\,f'(t)\,dt$$

that holds for all $x\in[a,b]$.

If we multiply this identity by $g(x)$ and integrate over x in $[a,b]$, then we get

$$(b-a)\int_a^b g(x)\,f(x)\,dx-\int_a^b f(t)\,dt\int_a^b g(x)\,dx = \int_a^b g(x)\left(\int_a^x (t-a)\,f'(t)\,dt\right)dx+\int_a^b g(x)\left(\int_x^b (t-b)\,f'(t)\,dt\right)dx. \tag{2.2}$$

Using integration by parts, we get

$$\begin{aligned}
&\int_a^b g(x)\left(\int_a^x (t-a)\,f'(t)\,dt\right)dx \\
&= \int_a^b \left(\int_a^x (t-a)\,f'(t)\,dt\right)d\left(\int_a^x g(s)\,ds\right) \\
&= \left(\int_a^x (t-a)\,f'(t)\,dt\right)\left(\int_a^x g(s)\,ds\right)\Big|_a^b \\
&\quad -\int_a^b \left(\int_a^x g(s)\,ds\right)(x-a)\,f'(x)\,dx \\
&= \left(\int_a^b (t-a)\,f'(t)\,dt\right)\left(\int_a^b g(s)\,ds\right) \\
&\quad -\int_a^b \left(\int_a^x g(s)\,ds\right)(x-a)\,f'(x)\,dx \\
&= \int_a^b \left(\int_a^b g(s)\,ds-\int_a^x g(s)\,ds\right)(x-a)\,f'(x)\,dx \\
&= \int_a^b \left(\int_x^b g(s)\,ds\right)(x-a)\,f'(x)\,dx
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
(2.4)\qquad & \int_a^b g(x)\left(\int_x^b (t-b) f'(t)\,dt\right)dx \\
&= \int_a^b \left(\int_x^b (t-b) f'(t)\,dt\right) d\left(\int_a^x g(s)\,ds\right) \\
&= \left(\int_x^b (t-b) f'(t)\,dt\right)\left(\int_a^x g(s)\,ds\right)\Bigg|_a^b \\
&+ \int_a^b \left(\int_a^x g(s)\,ds\right)(x-b) f'(x)\,dx \\
&= \int_a^b \left(\int_a^x g(s)\,ds\right)(x-b) f'(x)\,dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
&(b-a)\int_a^b g(x) f(x)\,dx - \int_a^b f(t)\,dt \int_a^b g(x)\,dx \\
&= \int_a^b \left(\int_x^b g(s)\,ds\right)(x-a) f'(x)\,dx - \int_a^b \left(\int_a^x g(s)\,ds\right)(b-x) f'(x)\,dx \\
&= \int_a^b (x-a)(b-x)\left(\frac{\int_x^b g(s)\,ds}{b-x} - \frac{\int_a^x g(s)\,ds}{x-a}\right) f'(x)\,dx
\end{aligned}$$

and the identity (2.1) is proved. □

We have:

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be convex and $p:[a,b]\to\mathbb{R}$ a Lebesgue integrable function such that*

$$(2.5)\qquad \frac{1}{x-a}\int_a^x p(s)\,ds \le \frac{1}{b-x}\int_x^b p(s)\,ds \text{ for all } x\in(a,b).$$

Then we have the inequalities

$$\begin{aligned}
(2.6)\qquad & f'_+(a)\left[\int_a^b xp(x)\,dx - \frac{a+b}{2}\int_a^b p(x)\,dx\right] \\
&\le \int_a^b p(x) f(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \int_a^b p(x)\,dx \\
&\le f'_-(b)\left[\int_a^b xp(x)\,dx - \frac{a+b}{2}\int_a^b p(x)\,dx\right].
\end{aligned}$$

Proof. Since f is convex, then $f_+'(a) \le f'(x) \le f_-'(b)$ for almost every $x \in [a,b]$. By the condition (2.5) we get

$$\begin{aligned}(2.7)\qquad & f_+'(a) \int_a^b (x-a)(b-x)\left(\frac{\int_x^b p(s)\,ds}{b-x} - \frac{\int_a^x p(s)\,ds}{x-a}\right) dx \\ & \le \int_a^b (x-a)(b-x)\left(\frac{\int_x^b p(s)\,ds}{b-x} - \frac{\int_a^x p(s)\,ds}{x-a}\right) f'(x)\,dx \\ & \le f_-'(b) \int_a^b (x-a)(b-x)\left(\frac{\int_x^b p(s)\,ds}{b-x} - \frac{\int_a^x p(s)\,ds}{x-a}\right) dx.\end{aligned}$$

Observe that, for $f(x) = x$ in Lemma 1 we have

$$\begin{aligned}& \int_a^b (x-a)(b-x)\left(\frac{\int_x^b p(s)\,ds}{b-x} - \frac{\int_a^x p(s)\,ds}{x-a}\right) dx \\ & = (b-a)\int_a^b p(x)\,x dx - \int_a^b x dx \int_a^b p(x)\,dx \\ & = (b-a)\left[\int_a^b p(x)\,x dx - \frac{a+b}{2}\int_a^b p(x)\,dx\right],\end{aligned}$$

while for $g = p$ we get

$$\begin{aligned}& \int_a^b (x-a)(b-x)\left(\frac{\int_x^b p(s)\,ds}{b-x} - \frac{\int_a^x p(s)\,ds}{x-a}\right) f'(x)\,dx \\ & = (b-a)\int_a^b p(x)\,f(x)\,dx - \int_a^b f(x)\,dx \int_a^b p(x)\,dx.\end{aligned}$$

By (2.7) we then get (2.6). □

Corollary 1. *Let $f : [a,b] \to \mathbb{R}$ be convex and $p : [a,b] \to \mathbb{R}$ a monotonic nondecreasing function, then we have the inequalities*

$$\begin{aligned}(2.8)\qquad & f_+'(a)\left[\int_a^b xp(x)\,dx - \frac{a+b}{2}\int_a^b p(x)\,dx\right] \\ & \le \int_a^b p(x)\,f(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \int_a^b p(x)\,dx \\ & \le f_-'(b)\left[\int_a^b xp(x)\,dx - \frac{a+b}{2}\int_a^b p(x)\,dx\right].\end{aligned}$$

Proof. If $p : [a,b] \to \mathbb{R}$ is a monotonic nondecreasing function, then

$$\frac{1}{x-a}\int_a^x p(s)\,ds \le p(x) \le \frac{1}{b-x}\int_x^b p(s)\,ds$$

for $x \in (a,b)$. Then by applying Theorem 4 we get the desired result (2.8). □

Corollary 2. *Let $f : [a,b] \to \mathbb{R}$ be convex and monotonic nondecreasing and $p : [a,b] \to \mathbb{R}$ a monotonic nondecreasing function, then we have the inequalities*

$$0 \le f'_+(a)\left[\int_a^b xp(x)\,dx - \frac{a+b}{2}\int_a^b p(x)\,dx\right] \le \int_a^b p(x) f(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\int_a^b p(x)\,dx \le f'_-(b)\left[\int_a^b xp(x)\,dx - \frac{a+b}{2}\int_a^b p(x)\,dx\right]. \tag{2.9}$$

If $\int_a^b p(x)\,dx > 0$, *then*

$$0 \le f'_+(a)\left[\frac{1}{\int_a^b p(x)\,dx}\int_a^b xp(x)\,dx - \frac{a+b}{2}\right] \le \frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x) f(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \le f'_-(b)\left[\frac{1}{\int_a^b p(x)\,dx}\int_a^b xp(x)\,dx - \frac{a+b}{2}\right]. \tag{2.10}$$

Proof. Since f is nondecreasing convex, hence $f'_+(a) \ge 0$. Also, by the Čebyšev' s inequality for synchronous functions we have

$$\int_a^b xp(x)\,dx - \frac{a+b}{2}\int_a^b p(x)\,dx \ge 0.$$

By employing (2.8) we derive (2.9). □

We say that the function $p : [a,b] \to \mathbb{R}$ is asymmetric if

$$p(a+b-x) = -p(x) \text{ for all } x \in [a,b].$$

If $p : [a,b] \to \mathbb{R}$ is asymmetric and Lebesgue integrable, then $\int_a^b p(s)\,ds = 0$. If $x \in [a,b]$ then $\int_a^x p(s)\,ds + \int_x^b p(s)\,ds = 0$, which implies that $\int_x^b p(s)\,ds = -\int_a^x p(s)\,ds$.

Corollary 3. *Let $f : [a,b] \to \mathbb{R}$ be convex and $p : [a,b] \to \mathbb{R}$ an asymmetric Lebesgue integrable function such that*

$$\int_a^x p(s)\,ds \le 0 \text{ for all } x \in [a,b], \tag{2.11}$$

or, equivalently

$$0 \le \int_x^b p(s)\,ds \text{ for all } x \in [a,b], \tag{2.12}$$

then we have the inequalities

$$f'_+(a)\int_a^b xp(x)\,dx \le \int_a^b p(x) f(x)\,dx \le f'_-(b)\int_a^b xp(x)\,dx. \tag{2.13}$$

Proof. The condition

$$\frac{1}{x-a}\int_a^x p(s)\,ds \le \frac{1}{b-x}\int_x^b p(s)\,ds \text{ for all } x \in (a,b)$$

is equivalent to
$$\frac{1}{x-a}\int_a^x p(s)\,ds \le -\frac{1}{b-x}\int_a^x p(s)\,ds$$
namely
$$\frac{1}{x-a}\int_a^x p(s)\,ds + \frac{1}{b-x}\int_a^x p(s)\,ds \le 0,$$
which is equivalent to (2.11).

By utilising (2.6) we derive the desired result (2.13). □

If $q:[a,b]\to\mathbb{R}$ is integrable, then the function $p(s)=q(s)-q(a+b-s)$ is asymmetric. By the condition (2.11) we have
$$\int_a^x [q(s)-q(a+b-s)]\,ds \le 0$$
namely
$$\int_a^x q(s)\,ds \le \int_a^x q(a+b-s)\,ds,\ \ x\in[a,b]. \tag{2.14}$$
If we put $u=a+b-s$, then
$$\int_a^x q(a+b-s)\,ds = \int_{a+b-x}^b q(s)\,ds$$
and we obtain
$$\int_a^x q(s)\,ds \le \int_{a+b-x}^b q(s)\,ds,\ \ x\in[a,b]. \tag{2.15}$$

We also have
$$\begin{aligned}\int_a^b xp(x)\,dx &= \int_a^b s[q(s)-q(a+b-s)]\,ds\\ &= \int_a^b sq(s)\,ds - \int_a^b (a+b-s)\,q(s)\,ds\\ &= \int_a^b [2s-(a+b)]\,q(s)\,ds = 2\int_a^b\left(s-\frac{a+b}{2}\right)q(s)\,ds\end{aligned}$$
and
$$\begin{aligned}\int_a^b p(s)\,f(s)\,ds &= \int_a^b [q(s)-q(a+b-s)]\,f(s)\,ds\\ &= \int_a^b q(s)\,f(s)\,ds - \int_a^b q(a+b-s)\,f(s)\,ds\\ &= \int_a^b q(s)\,f(s)\,ds - \int_a^b q(s)\,f(a+b-s)\,ds\\ &= \int_a^b q(s)\,[f(s)-f(a+b-s)]\,ds.\end{aligned}$$
We can state:

Corollary 4. *Let $f:[a,b]\to\mathbb{R}$ be convex and $q:[a,b]\to\mathbb{R}$ a Lebesgue integrable function such that (2.14) holds, then we have the inequalities*

$$f'_{+}(a)\int_a^b\left(s-\frac{a+b}{2}\right)q(s)\,ds\le\int_a^b q(x)\tilde{f}(x)\,dx \le f'_{-}(b)\int_a^b\left(s-\frac{a+b}{2}\right)q(s)\,ds, \tag{2.16}$$

where

$$\tilde{f}(x):=\frac{1}{2}\left[f(x)-f(a+b-x)\right],\ x\in[a,b].$$

3. Some Examples

We consider the function $p(x)=x$, $x\in[a,b]$. Observe that

$$\begin{aligned}\int_a^b xp(x)\,dx-\frac{a+b}{2}\int_a^b p(x)\,dx&=\int_a^b x^2dx-\frac{a+b}{2}\int_a^b xdx\\&=(b-a)\left[\frac{b^2+ab+a^2}{3}-\left(\frac{a+b}{2}\right)^2\right]\\&=\frac{1}{12}(b-a)^3.\end{aligned}$$

Let $f:[a,b]\to\mathbb{R}$ be convex, then by (2.9) we get

$$\frac{1}{12}(b-a)^3f'_{+}(a)\le\int_a^b xf(x)\,dx-\frac{a+b}{2}\int_a^b f(x)\,dx\le\frac{1}{12}(b-a)^3f'_{-}(b). \tag{3.1}$$

For n a natural number, the function $p(x)=\left(x-\frac{a+b}{2}\right)^{2n+1}$, is increasing, then for $f:[a,b]\to\mathbb{R}$ a convex function, we have by (2.9)

$$\begin{aligned}0&\le f'_{+}(a)\left[\int_a^b x\left(x-\frac{a+b}{2}\right)^{2n+1}dx-\frac{a+b}{2}\int_a^b\left(x-\frac{a+b}{2}\right)^{2n+1}dx\right]\\&\le\int_a^b\left(x-\frac{a+b}{2}\right)^{2n+1}f(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\int_a^b\left(x-\frac{a+b}{2}\right)^{2n+1}dx\\&\le f'_{-}(b)\left[\int_a^b x\left(x-\frac{a+b}{2}\right)^{2n+1}dx-\frac{a+b}{2}\int_a^b\left(x-\frac{a+b}{2}\right)^{2n+1}dx\right].\end{aligned}$$

Observe that

$$\begin{aligned}&\int_a^b x\left(x-\frac{a+b}{2}\right)^{2n+1}dx-\frac{a+b}{2}\int_a^b\left(x-\frac{a+b}{2}\right)^{2n+1}dx\\&=\int_a^b\left(x-\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)^{2n+1}dx=\int_a^b\left(x-\frac{a+b}{2}\right)^{2n+2}dx\\&=\frac{2}{2n+3}\left(\frac{b-a}{2}\right)^{2n+3}=\frac{(b-a)^{2n+3}}{(2n+3)2^{2n+2}}\end{aligned}$$

and

$$\int_a^b\left(x-\frac{a+b}{2}\right)^{2n+1}dx=0,$$

which gives

$$
\begin{aligned}
(3.2) \qquad 0 &\le f'_{+}(a)\frac{(b-a)^{2n+3}}{(2n+3)\,2^{2n+2}} \le \int_a^b \left(x-\frac{a+b}{2}\right)^{2n+1} f(x)\,dx \\
&\le f'_{-}(b)\frac{(b-a)^{2n+3}}{(2n+3)\,2^{2n+2}}
\end{aligned}
$$

for $f:[a,b]\to\mathbb{R}$ a convex function and n a natural number.

Consider the function $p(x)=-\frac{1}{x}$ for $x\in[a,b]\subset(0,\infty)$. Then p is increasing on $[a,b]$ and by (2.9) we get

$$
\begin{aligned}
& f'_{+}(a)\left[\frac{a+b}{2}\int_a^b \frac{dx}{x}-\int_a^b dx\right] \\
& \le \frac{1}{b-a}\int_a^b f(x)\,dx\int_a^b \frac{dx}{x}-\int_a^b \frac{f(x)}{x}dx \\
& \le f'_{-}(b)\left[\frac{a+b}{2}\int_a^b \frac{dx}{x}-\int_a^b dx\right],
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
& f'_{+}(a)\left[\frac{a+b}{2}(\ln b-\ln a)-(b-a)\right] \\
& \le \frac{1}{b-a}\int_a^b f(x)\,dx\,(\ln b-\ln a)-\int_a^b \frac{f(x)}{x}dx \\
& \le f'_{-}(b)\left[\frac{a+b}{2}(\ln b-\ln a)-(b-a)\right],
\end{aligned}
$$

namely

$$
\begin{aligned}
(3.3) \qquad & f'_{+}(a)\left[A(a,b)-L(a,b)\right] \\
& \le \frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx \\
& \le f'_{-}(b)\left[A(a,b)-L(a,b)\right],
\end{aligned}
$$

where $A(a,b)=\frac{a+b}{2}$ is the *arithmetic mean* and $L(a,b)=\frac{b-a}{\ln b-\ln a}$ is the *logarithmic mean* of the positive numbers $a<b$.

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