SOME INEQUALITIES FOR WEIGHTED AND INTEGRAL MEANS OF OPERATOR CONVEX FUNCTIONS

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Abstract. Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, the convex set of selfadjoint operators with spectra in I. If $A \neq B$ and f, as an operator function, is Gâteaux differentiable on

$$[A, B] := \{(1-t)A + tB \mid t \in [0,1]\},$$

while $p : [0,1] \to [0, \infty)$ is Lebesgue integrable satisfying the condition

$$0 \leq \int_0^\tau p(s)\,ds \leq \int_0^1 p(s)\,ds \text{ for all } \tau \in [0,1]$$

and symmetric, namely $p(1-t) = p(t)$ for all $t \in [0,1]$, then

$$-\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\,[\nabla f_B(B-A) - \nabla f_A(B-A)]$$
$$\leq \frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau) f((1-\tau)A + \tau B)\,d\tau - \int_0^1 f((1-\tau)A + \tau B)\,d\tau$$
$$\leq \frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\,[\nabla f_B(B-A) - \nabla f_A(B-A)].$$

Some particular examples of interest are also given.

1. Introduction

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda) f(A) + \lambda f(B) \tag{1.1}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [8] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0, \infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave

1991 *Mathematics Subject Classification.* 47A63; 47A99.

Key words and phrases. Operator convex functions, Integral inequalities, Hermite-Hadamard inequality, Féjer's inequalities.

RGMIA Res. Rep. Coll. 22 (2019), Art. 96, 15 pp. Received 30/09/19

on $(0,\infty)$. The entropy function $f(t)=-t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t)=e^{t}$ is neither operator convex nor operator monotone.

In [5] we obtained among others the following Hermite-Hadamard type inequalities for operator convex functions $f:I\to\mathbb{R}$

$$f\left(\frac{A+B}{2}\right)\leq\int_{0}^{1}f\left((1-s)A+sB\right)ds\leq\frac{f(A)+f(B)}{2}, \tag{1.2}$$

where A, B are selfadjoint operators with spectra included in I.

For two distinct operators $A,B\in\mathcal{SA}_{I}(H)$ we consider the segment of selfadjoint operators

$$[A,B]:=\{(1-t)A+tB\mid t\in[0,1]\}.$$

We observe that $A,B\in[A,B]$ and $[A,B]\subset\mathcal{SA}_{I}(H)$.

A continuous function $g:\mathcal{SA}_{I}(H)\to\mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A\in\mathcal{SA}_{I}(H)$ along the direction $B\in\mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla g_{A}(B):=\lim_{s\to0}\frac{g(A+sB)-g(A)}{s}\in\mathcal{B}(H). \tag{1.3}$$

If the limit (1.3) exists for all $B\in\mathcal{B}(H)$, then we say that f is *Gâteaux differentiable* in A and we can write $g\in\mathcal{G}(A)$. If this is true for any A in a subset $\mathcal{S}$ from $\mathcal{SA}_{I}(H)$ we write that $g\in\mathcal{G}(\mathcal{S})$.

In the recent paper [7], we obtained the following operator *Féjer's type inequalities:*

Theorem 1. *Let f be an operator convex function on I and $A,B\in\mathcal{SA}_{I}(H)$, with $A\neq B$. If $f\in\mathcal{G}([A,B])$ and $p:[0,1]\to[0,\infty)$ is Lebesgue integrable and symmetric, namely $p(1-t)=p(t)$ for all $t\in[0,1]$, then*

$$\begin{aligned}0&\leq\int_{0}^{1}p(t)f((1-t)A+tB)dt-\left(\int_{0}^{1}p(t)dt\right)f\left(\frac{A+B}{2}\right)\\&\leq\frac{1}{2}\left(\int_{0}^{1}\left|t-\frac{1}{2}\right|p(t)dt\right)[\nabla f_{B}(B-A)-\nabla f_{A}(B-A)].\end{aligned} \tag{1.4}$$

In particular, for $p\equiv1$ we get

$$\begin{aligned}0&\leq\int_{0}^{1}f((1-t)A+tB)dt-f\left(\frac{A+B}{2}\right)\\&\leq\frac{1}{8}[\nabla f_{B}(B-A)-\nabla f_{A}(B-A)].\end{aligned} \tag{1.5}$$

We also have:

Theorem 2. *Let f be an operator convex function on I and $A,B\in\mathcal{SA}_{I}(H)$, with $A\neq B$. If $f\in\mathcal{G}([A,B])$ and $p:[0,1]\to[0,\infty)$ is Lebesgue integrable and symmetric, namely $p(1-t)=p(t)$ for all $t\in[0,1]$, then*

$$\begin{aligned}0&\leq\left(\int_{0}^{1}p(t)dt\right)\frac{f(A)+f(B)}{2}-\int_{0}^{1}p(t)f((1-t)A+tB)dt\\&\leq\frac{1}{2}\int_{0}^{1}\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)dt[\nabla f_{B}(B-A)-\nabla f_{A}(B-A)].\end{aligned} \tag{1.6}$$

In particular, for $p \equiv 1$ we get

$$0 \le \frac{f(A)+f(B)}{2} - \int_0^1 f((1-t)A+tB)\,dt \le \frac{1}{8}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right]. \tag{1.7}$$

For recent inequalities for operator convex functions see [1]-[6] and [9]-[18].

Motivated by the above results, we establish in this paper some upper and lower bounds in the operator order for the difference

$$\int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)A+\tau B)\,d\tau$$

in the case when the operator convex function f is Gâteaux differentiable as a function of selfadjoint operators. Two particular examples of interest for $f(x) = -\ln x$ and $f(x) = x^{-1}$ are also given.

2. Some Preliminary Facts

Let f be an operator convex function on I. For A, $B \in \mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in I, we consider the auxiliary function $\varphi_{(A,B)} : [0,1] \to \mathcal{SA}_I(H)$ defined by

$$\varphi_{(A,B)}(t) := f((1-t)A+tB). \tag{2.1}$$

For $x \in H$ we can also consider the auxiliary function $\varphi_{(A,B);x} : [0,1] \to \mathbb{R}$ defined by

$$\varphi_{(A,B);x}(t) := \left\langle \varphi_{(A,B)}(t)x, x\right\rangle = \langle f((1-t)A+tB)x, x\rangle. \tag{2.2}$$

We have the following basic fact:

Lemma 1. *Let f be an operator convex function on I. For any A, $B \in \mathcal{SA}_I(H)$, $\varphi_{(A,B)}$ is well defined and convex in the operator order. For any $(A,B) \in \mathcal{SA}_I(H)$ and $x \in H$ the function $\varphi_{(A,B);x}$ is convex in the usual sense on $[0,1]$.*

Proof. If $(A,B) \in \mathcal{SA}_I(H)$ and $t \in [0,1]$ the convex combination $(1-t)A+tB$ is a selfadjoint operator with the spectrum in I showing that $\mathcal{SA}_I(H)$ is convex in the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators on H. By the continuous functional calculus of selfadjoint operator we also conclude that $f((1-t)A+tB)$ is a selfadjoint operator with spectrum in I.

Let A, $B \in \mathcal{SA}_I(H)$ and t_1, $t_2 \in [0,1]$. If α, $\beta > 0$ with $\alpha+\beta = 1$, then

$$\begin{aligned}\varphi_{(A,B)}(\alpha t_1+\beta t_2) &:= f((1-\alpha t_1-\beta t_2)A+(\alpha t_1+\beta t_2)B)\\ &= f((\alpha+\beta-\alpha t_1-\beta t_2)A+(\alpha t_1+\beta t_2)B)\\ &= f(\alpha[(1-t_1)A+t_1B]+\beta[(1-t_2)A+t_2B])\\ &\le \alpha f((1-t_1)A+t_1B)+\beta f((1-t_2)A+t_2B)\\ &= \alpha\varphi_{(A,B)}(t_1)+\beta\varphi_{(A,B)}(t_2),\end{aligned}$$

which proves the convexity $\varphi_{(A,B)}$ in the operator order.

Ley $A,\ B\in\mathcal{SA}_I(H)$ and $x\in H$. If $t_1,\ t_2\in[0,1]$ and $\alpha,\ \beta>0$ with $\alpha+\beta=1$, then

$$\begin{aligned}\varphi_{(A,B);x}(\alpha t_1+\beta t_2)&=\left\langle\varphi_{(A,B)}(\alpha t_1+\beta t_2)x,x\right\rangle\\&\le\left\langle\left[\alpha\varphi_{(A,B)}(t_1)+\beta\varphi_{(A,B)}(t_2)\right]x,x\right\rangle\\&=\alpha\left\langle\varphi_{(A,B)}(t_1)x,x\right\rangle+\beta\left\langle\varphi_{(A,B)}(t_2)x,x\right\rangle\\&=\alpha\varphi_{(A,B);x}(t_1)+\beta\varphi_{(A,B);x}(t_2),\end{aligned}$$

which proves the convexity of $\varphi_{(A,B);x}$ on $[0,1]$. □

Lemma 2. *Let f be an operator convex function on I and $A,\ B\in\mathcal{SA}_I(H)$, with $A\neq B$. If $f\in\mathcal{G}([A,B])$, then the auxiliary function $\varphi_{(A,B)}$ is differentiable on $(0,1)$ and*

$$\varphi'_{(A,B)}(t)=\nabla f_{(1-t)A+tB}(B-A). \tag{2.3}$$

Also we have for the lateral derivative that

$$\varphi'_{(A,B)}(0+)=\nabla f_A(B-A) \tag{2.4}$$

and

$$\varphi'_{(A,B)}(1-)=\nabla f_B(B-A). \tag{2.5}$$

Proof. Let $t\in(0,1)$ and $h\neq0$ small enough such that $t+h\in(0,1)$. Then

$$\begin{aligned}&\frac{\varphi_{(A,B)}(t+h)-\varphi_{(A,B)}(t)}{h}\\&=\frac{f((1-t-h)A+(t+h)B)-f((1-t)A+tB)}{h}\\&=\frac{f((1-t)A+tB+h(B-A))-f((1-t)A+tB)}{h}.\end{aligned} \tag{2.6}$$

Since $f\in\mathcal{G}([A,B])$, hence by taking the limit over $h\to0$ in (2.6) we get

$$\begin{aligned}\varphi'_{(A,B)}(t)&=\lim_{h\to0}\frac{\varphi_{(A,B)}(t+h)-\varphi_{(A,B)}(t)}{h}\\&=\lim_{h\to0}\frac{f((1-t)A+tB+h(B-A))-f((1-t)A+tB)}{h}\\&=\nabla g_{(1-t)A+tB}(B-A),\end{aligned}$$

which proves (2.7).

Also, we have

$$\begin{aligned}\varphi'_{(A,B)}(0+)&=\lim_{h\to0+}\frac{\varphi_{(A,B)}(h)-\varphi_{(A,B)}(0)}{h}\\&=\lim_{h\to0+}\frac{f((1-h)A+hB)-f(A)}{h}\\&=\lim_{h\to0+}\frac{f(A+h(B-A))-f(A)}{h}\\&=\nabla f_A(B-A)\end{aligned}$$

since f is assumed to be Gâteaux differentiable in A. This proves (2.4).

The equality (2.5) follows in a similar way. □

Lemma 3. *Let f be an operator convex function on I and $A,\ B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then for $0 < t_1 < t_2 < 1$ we have*

$$\nabla g_{(1-t_1)A+t_1B}(B-A) \leq \nabla g_{(1-t_2)A+t_2B}(B-A) \tag{2.7}$$

in the operator order.

We also have

$$\nabla f_A(B-A) \leq \nabla g_{(1-t_1)A+t_1B}(B-A) \tag{2.8}$$

and

$$\nabla g_{(1-t_2)A+t_2B}(B-A) \leq \nabla f_B(B-A). \tag{2.9}$$

Proof. Let $x \in H$. The auxiliary function $\varphi_{(A,B);x}$ is convex in the usual sense on $[0,1]$ and differentiable on $(0,1)$ and for $t \in (0,1)$

$$\begin{aligned}\varphi'_{(A,B),x}(t) &= \lim_{h\to 0}\frac{\varphi_{(A,B),x}(t+h)-\varphi_{(A,B),x}(t)}{h}\\ &= \lim_{h\to 0}\left\langle\frac{\varphi_{(A,B)}(t+h)-\varphi_{(A,B)}(t)}{h}x,x\right\rangle\\ &= \left\langle\lim_{h\to 0}\frac{\varphi_{(A,B)}(t+h)-\varphi_{(A,B)}(t)}{h}x,x\right\rangle\\ &= \left\langle\nabla g_{(1-t)A+tB}(B-A)x,x\right\rangle.\end{aligned}$$

Since for $0 < t_1 < t_2 < 1$ we have by the gadient inequality for scalar convex functions that

$$\varphi'_{(A,B),x}(t_1) \leq \varphi'_{(A,B),x}(t_2),$$

then we get

$$\left\langle\nabla g_{(1-t_1)A+t_1B}(B-A)x,x\right\rangle \leq \left\langle\nabla g_{(1-t_2)A+t_2B}(B-A)x,x\right\rangle \tag{2.10}$$

for all $x \in H$, which is equivalent to the inequality (2.7) in the operator order.

Let $0 < t_1 < 1$. By the gadient inequality for scalar convex functions we also have

$$\varphi'_{(A,B),x}(0+) \leq \varphi'_{(A,B),x}(t_1),$$

which, as above, implies that

$$\langle\nabla f_A(B-A)x,x\rangle \leq \left\langle\nabla g_{(1-t_1)A+t_1B}(B-A)x,x\right\rangle$$

for all $x \in H$, that is equivalent to the operator inequality (2.8).

The inequality (2.9) follows in a similar way. □

Corollary 1. *Let f be an operator convex function on I and $A,\ B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then for all $t \in (0,1)$ we have*

$$\nabla f_A(B-A) \leq \nabla f_{(1-t)A+tB}(B-A) \leq \nabla f_B(B-A). \tag{2.11}$$

3. Main Results

We start to the following identity that is of interest in itself as well:

Lemma 4. *Let f be an operator convex function on I and $A,\, B\in \mathcal{SA}_I(H)$, with $A\neq B$. If $f\in \mathcal{G}([A,B])$ and $g:[0,1]\to\mathbb{C}$ is a Lebesgue integrable function, then we have the equality*

$$\begin{aligned}(3.1)\qquad &\int_0^1 g(\tau)\varphi_{(A,B)}(\tau)\,d\tau-\int_0^1 g(\tau)\,d\tau\int_0^1\varphi_{(A,B)}(\tau)\,d\tau\\ &=\int_0^1\left(\int_\tau^1 g(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau\\ &+\int_0^1\left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,\varphi'_{(A,B)}(\tau)\,d\tau.\end{aligned}$$

Proof. Integrating by parts in the Bochner's integral, we have

$$\begin{aligned}&\int_0^\tau t\varphi'_{(A,B)}(t)\,dt+\int_\tau^1 (t-1)\,\varphi'_{(A,B)}(t)\,dt\\ &=\tau\varphi_{(A,B)}(\tau)-\int_0^\tau\varphi_{(A,B)}(t)\,dt-(\tau-1)\,\varphi_{(A,B)}(\tau)-\int_\tau^1\varphi_{(A,B)}(t)\,dt\\ &=\varphi_{(A,B)}(\tau)-\int_0^1\varphi_{(A,B)}(t)\,dt\end{aligned}$$

that holds for all $\tau\in[0,1]$.

If we multiply this identity by $g(\tau)$ and integrate over τ in $[0,1]$, then we get

$$\begin{aligned}(3.2)\quad &\int_0^1 g(\tau)\varphi_{(A,B)}(\tau)\,d\tau-\int_0^1 g(\tau)\,d\tau\int_0^1\varphi_{(A,B)}(t)\,dt\\ &=\int_0^1 g(\tau)\left(\int_0^\tau t\varphi'_{(A,B)}(t)\,dt\right)d\tau+\int_0^1 g(\tau)\left(\int_\tau^1 (t-1)\,\varphi'_{(A,B)}(t)\,dt\right)d\tau.\end{aligned}$$

Using integration by parts, we get

$$\begin{aligned}(3.3)\qquad &\int_0^1 g(\tau)\left(\int_0^\tau t\varphi'_{(A,B)}(t)\,dt\right)d\tau\\ &=\int_0^1\left(\int_0^\tau t\varphi'_{(A,B)}(t)\,dt\right)d\left(\int_0^\tau g(s)\,ds\right)\\ &=\left(\int_0^\tau g(s)\,ds\right)\left(\int_0^\tau t\varphi'_{(A,B)}(t)\,dt\right)\Bigg|_0^1\\ &-\int_0^1\left(\int_0^\tau g(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_0^1 g(s)\,ds\right)\left(\int_0^1 t\varphi'_{(A,B)}(t)\,dt\right) \\
&- \int_0^1 \left(\int_0^\tau g(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau \\
&= \int_0^1 \left(\int_0^1 g(s)\,ds - \int_0^\tau g(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau \\
&= \int_0^1 \left(\int_\tau^1 g(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 g(\tau)\left(\int_\tau^1 (t-1)\,\varphi'_{(A,B)}(t)\,dt\right)d\tau \\
&= \int_0^1 \left(\int_\tau^1 (t-1)\,\varphi'_{(A,B)}(t)\,dt\right)d\left(\int_0^\tau g(s)\,ds\right) \\
&= \left(\int_\tau^1 (t-1)\,\varphi'_{(A,B)}(t)\,dt\right)\left(\int_0^\tau g(s)\,ds\right)\Bigg|_0^1 \\
&+ \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,\varphi'_{(A,B)}(\tau)\,d\tau \\
&= \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,\varphi'_{(A,B)}(\tau)\,d\tau,
\end{aligned} \tag{3.4}
$$

which proves the identity in (3.1). □

Theorem 3. *Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$ and $p : [0,1] \to \mathbb{R}$ is a Lebesgue integrable function such that*

$$
0 \le \int_0^\tau p(s)\,ds \le \int_0^1 p(s)\,ds \text{ for all } \tau \in [0,1], \tag{3.5}
$$

then we have the inequalities

$$
\begin{aligned}
&\int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau d\tau \nabla f_A(B-A) \\
&- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \nabla f_B(B-A) \\
&\le \int_0^1 p(\tau)\,f((1-\tau)A+\tau B)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)A+\tau B)\,d\tau \\
&\le \int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau d\tau \nabla f_B(B-A) \\
&- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \nabla f_A(B-A)
\end{aligned} \tag{3.6}
$$

or, equivalently,

$$\int_0^1 (1-\tau)\left(\int_0^\tau [p(1-s)\nabla f_B(B-A) - p(s)\nabla f_A(B-A)]\,ds\right)d\tau \tag{3.7}$$
$$\leq \int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)A+\tau B)\,d\tau$$
$$\leq \int_0^1 (1-\tau)\left(\int_0^\tau [p(1-s)\nabla f_A(B-A) - p(s)\nabla f_B(B-A)]\,ds\right)d\tau.$$

Proof. We have for $\varphi_{(A,B)}$ and $p:[0,1]\to\mathbb{R}$ a Lebesgue integrable function that

$$\int_0^1 p(\tau)\varphi_{(A,B)}(\tau)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 \varphi_{(A,B)}(\tau)\,d\tau \tag{3.8}$$
$$= \int_0^1 \left(\int_\tau^1 p(s)\,ds\right)(\tau)\,\varphi'_{(A,B)}(\tau)\,d\tau$$
$$- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,\varphi'_{(A,B)}(\tau)\,d\tau.$$

By the properties of $\varphi_{(A,B)}$ from the above section, we have in the operator order that

$$\tau\varphi'_{(A,B)}(1-) \geq \tau\varphi'_{(A,B)}(\tau) \geq \tau\varphi'_{(A,B)}(0+) \tag{3.9}$$

and

$$(1-\tau)\varphi'_{(A,B)}(1-) \geq (1-\tau)\varphi'_{(A,B)}(\tau) \geq (1-\tau)\varphi'_{(A,B)}(0+) \tag{3.10}$$

for all $\tau\in(0,1)$.

From

$$\int_0^\tau p(s)\,ds \leq \int_0^1 p(s)\,ds = \int_0^\tau p(s)\,ds + \int_\tau^1 p(s)\,ds,$$

we get that $\int_\tau^1 p(s)\,ds \geq 0$ for all $\tau\in(0,1)$.

From (3.9) we get that

$$\left(\int_\tau^1 p(s)\,ds\right)\tau\varphi'_{(A,B)}(1-) \geq \left(\int_\tau^1 p(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)$$
$$\geq \left(\int_\tau^1 p(s)\,ds\right)\tau\varphi'_{(A,B)}(0+)$$

and from (3.10) that

$$-\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\varphi'_{(A,B)}(0+) \leq -\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\varphi'_{(A,B)}(\tau)$$
$$\leq -\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\varphi'_{(A,B)}(1-)$$

all $\tau\in(0,1)$.

If we integrate these inequalities over $\tau \in [0,1]$ and add the obtained results, then we get

$$\begin{aligned}&\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau \varphi'_{(A,B)}(1-) - \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \varphi'_{(A,B)+}(0)\\ &\geq \int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau \varphi'_{(A,B)}(\tau)\,d\tau - \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,\varphi'_{(A,B)}(\tau)\,d\tau\\ &\geq \int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau \varphi'_{(A,B)}(0+) - \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \varphi'_{(A,B)}(1-)\,.\end{aligned}$$

By using the equality (2.1) we get

$$\begin{aligned}&\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau \varphi'_{(A,B)}(0+)\\ &- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \varphi'_{(A,B)}(1-)\\ &\leq \int_0^1 p(\tau)\,\varphi_{(A,B)}(\tau)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 \varphi_{(A,B)}(\tau)\,d\tau\\ &\leq \int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau \varphi'_{(A,B)}(1-)\\ &- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \varphi'_{(A,B)}(0+)\,,\end{aligned} \tag{3.11}$$

and since $\varphi'_{(A,B)}(1-) = \nabla f_B(B-A)$ and $\varphi'_{(A,B)}(0+) = \nabla f_B(B-A)$ hence we obtain (3.6).

If we change the variable $y = 1-\tau$, then we have

$$\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau = \int_0^1 \left(\int_{1-y}^1 p(s)\,ds\right)(1-y)\,dy.$$

Also by the change of variable $u = 1-s$, we get

$$\int_{1-y}^1 p(s)\,ds = \int_0^y p(1-u)\,du,$$

which implies that

$$\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau = \int_0^1 \left(\int_0^\tau p(1-s)\,ds\right)(1-\tau)\,d\tau.$$

Therefore

$$\begin{aligned}&\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau \varphi'_{(A,B)}(1-) - \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \varphi'_{(A,B)}(0+)\\ &= \int_0^1 \left(\int_0^\tau p(1-s)\,ds\right)(1-\tau)\,d\tau \varphi'_{(A,B)}(1-)\\ &- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \varphi'_{(A,B)}(0+)\\ &= \int_0^1 (1-\tau)\left(\int_0^\tau \left[p(1-s)\,\varphi'_{(A,B)}(1-) - p(s)\,\varphi'_{(A,B)+}(0+)\right] ds\right) d\tau\end{aligned}$$

and

$$\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau \varphi'_{(A,B)}(0+) - \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \varphi'_{(A,B)}(1-)$$
$$= \int_0^1 \left(\int_0^\tau p(1-s)\,ds\right)(1-\tau)\,d\tau\varphi'_{(A,B)}(0+)$$
$$- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\varphi'_{(A,B)}(1-)$$
$$= \int_0^1 (1-\tau)\left(\int_0^\tau \left[p(1-s)\varphi'_{(A,B)}(0+) - p(s)\varphi'_{(A,B)}(1-)\right] ds\right) d\tau,$$

and by (3.11) we get (3.7). □

We say that the function $p : [0,1] \to \mathbb{R}$ is symmetric on $[0,1]$ if

$$p(1-t) = p(t) \text{ for all } t \in [0,1].$$

Corollary 2. *Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$ and $p : [0,1] \to \mathbb{R}$ a Lebesgue integrable and symmetric function such that the condition (3.5) holds, then we have*

$$(3.12) \quad -\frac{1}{2}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right]$$
$$\le -\frac{1}{\int_0^1 p(\tau)\,d\tau} \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau$$
$$\times \left[\nabla f_B(B-A) - \nabla f_A(B-A)\right]$$
$$\le \frac{1}{\int_0^1 p(\tau)\,d\tau} \int_0^1 p(\tau) f((1-\tau)A + \tau B)\,d\tau - \int_0^1 f((1-\tau)A+\tau B)\,d\tau$$
$$\le \frac{1}{\int_0^1 p(\tau)\,d\tau} \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau$$
$$\times \left[\nabla f_B(B-A) - \nabla f_A(B-A)\right]$$
$$\le \frac{1}{2}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right].$$

Proof. Since p is symmetric, then $p(1-s) = p(s)$ for all $s \in [0,1]$ and by (3.7) we get

$$\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\left[\varphi'_{(A,B)}(0+) - \varphi'_{(A,B)}(1-)\right]$$
$$\le \int_0^1 p(\tau)\varphi_{(A,B)}(\tau)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 \varphi_{(A,B)}(\tau)\,d\tau$$
$$\le \left[\varphi'_{(A,B)}(1-) - \varphi'_{(A,B)}(0+)\right] \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau,$$

which is equivalent to the second and third inequalities (3.12).

Since $0 \le \int_0^\tau p(s)\,ds \le \int_0^1 p(\tau)\,d\tau$, hence

$$\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \le \int_0^1 p(\tau)\,d\tau \int_0^1 (1-\tau)\,d\tau = \frac{1}{2}\int_0^1 p(\tau)\,d\tau$$

and the last part of (3.12) is proved. □

Remark 1. *If the function p is nonnegative and symmetric then the inequality (3.12) holds true.*

Remark 2. *It is well known that, if f is a C^1-function defined on an open interval, then the operator function $f(X)$ is Fréchet differentiable and the derivative $Df(A)(B)$ equals the Gâteaux derivative $\nabla f_A(B)$. So for operator convex functions f that are of class C^1 on I and $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable and symmetric weight on $[0,1]$ such that the condition (3.5) holds, we have the inequalities*

$$
\begin{aligned}
(3.13)\quad & -\frac{1}{2}\left[Df(B)(B-A)-Df(A)(B-A)\right] \\
&\le -\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \\
&\times\left[Df(B)(B-A)-Df(A)(B-A)\right] \\
&\le \frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)\,f((1-\tau)A+\tau B)\,d\tau-\int_0^1 f((1-\tau)A+\tau B)\,d\tau \\
&\le \frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \\
&\times\left[Df(B)(B-A)-Df(A)(B-A)\right] \\
&\le \frac{1}{2}\left[Df(B)(B-A)-Df(A)(B-A)\right]
\end{aligned}
$$

for $A,\ B\in\mathcal{SA}_I(H)$

If we consider the weight $p:[0,1]\to[0,\infty)$, $p(s)=\left|s-\frac{1}{2}\right|$, then

$$
\begin{aligned}
&\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \\
&=\int_0^1\left(\int_0^\tau\left|s-\frac{1}{2}\right|ds\right)(1-\tau)\,d\tau \\
&=\int_0^{\frac{1}{2}}\left(\int_0^\tau\left|s-\frac{1}{2}\right|ds\right)(1-\tau)\,d\tau \\
&+\int_{\frac{1}{2}}^1\left(\int_0^\tau\left|s-\frac{1}{2}\right|ds\right)(1-\tau)\,d\tau \\
&=\int_0^{\frac{1}{2}}\left(\int_0^\tau\left(\frac{1}{2}-s\right)ds\right)(1-\tau)\,d\tau \\
&+\int_{\frac{1}{2}}^1\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-s\right)ds+\int_{\frac{1}{2}}^\tau\left(s-\frac{1}{2}\right)\right)(1-\tau)\,d\tau \\
&=\int_0^{\frac{1}{2}}\left(\frac{1}{2}\tau-\frac{\tau^2}{2}\right)(1-\tau)\,d\tau \\
&+\int_{\frac{1}{2}}^1\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-s\right)ds+\int_{\frac{1}{2}}^\tau\left(s-\frac{1}{2}\right)ds\right)(1-\tau)\,d\tau.
\end{aligned}
$$

We have

$$\int_0^{\frac{1}{2}}\left(\frac{1}{2}\tau-\frac{\tau^2}{2}\right)(1-\tau)\,d\tau=\frac{1}{2}\int_0^{\frac{1}{2}}(1-\tau)\,\tau\,(1-\tau)\,d\tau$$
$$=\frac{1}{2}\int_0^{\frac{1}{2}}(1-\tau)^2\,\tau d\tau=\frac{11}{384}$$

and

$$\int_{\frac{1}{2}}^1\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-s\right)ds+\int_{\frac{1}{2}}^{\tau}\left(s-\frac{1}{2}\right)ds\right)(1-\tau)\,d\tau$$
$$=\int_{\frac{1}{2}}^1\left(\frac{1}{8}+\frac{1}{2}\left(\tau-\frac{1}{2}\right)^2\right)(1-\tau)\,d\tau$$
$$=\frac{1}{8}\int_{\frac{1}{2}}^1(1-\tau)\,d\tau+\frac{1}{2}\int_{\frac{1}{2}}^1\left(\tau-\frac{1}{2}\right)^2(1-\tau)\,d\tau=\frac{7}{384}.$$

Therefore

$$\int_0^1\left(\int_0^{\tau}p(s)\,ds\right)(1-\tau)\,d\tau=\frac{3}{64}.$$

Since $\int_0^1\left|\tau-\frac{1}{2}\right|d\tau=\frac{1}{4}$, hence

$$\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^{\tau}p(s)\,ds\right)(1-\tau)\,d\tau=\frac{3}{16}.$$

Utilising (3.12) for symmetric weight $p:[0,1]\to[0,\infty)$, $p(s)=\left|s-\frac{1}{2}\right|$, we get

$$\begin{aligned}(3.14)\qquad &-\frac{3}{16}\left[\nabla f_B(B-A)-\nabla f_A(B-A)\right]\\ &\leq 4\int_0^1\left|\tau-\frac{1}{2}\right|f((1-\tau)A+\tau B)\,d\tau-\int_0^1 f((1-\tau)A+\tau B)\,d\tau\\ &\leq\frac{3}{16}\left[\nabla f_B(B-A)-\nabla f_A(B-A)\right],\end{aligned}$$

where f is an operator convex function on I, A, $B\in\mathcal{SA}_I(H)$, with $A\neq B$ and $f\in\mathcal{G}([A,B])$.

Consider now the symmetric function $p(s)=(1-s)\,s$, $x\in[0,1]$. Then

$$\int_0^{\tau}p(s)\,ds=\int_a^{\tau}(1-s)\,sds=-\frac{1}{6}\tau^2(2\tau-3)\,,\ \ \tau\in[0,1]$$

and

$$\int_0^1\left(\int_0^{\tau}p(s)\,ds\right)(1-\tau)\,d\tau=-\frac{1}{6}\int_0^1\tau^2(2\tau-3)(1-\tau)\,d\tau=\frac{1}{40}.$$

Also

$$\int_0^1 p(\tau)\,d\tau=\int_0^1(1-\tau)\,\tau d\tau=\frac{1}{6}$$

and

$$\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^{\tau}p(s)\,ds\right)(1-\tau)\,d\tau=\frac{3}{20}$$

and by (3.12) we obtain

$$-\frac{3}{20}\left[\nabla f_B(B-A)-\nabla f_A(B-A)\right] \tag{3.15}$$
$$\leq 6\int_0^1 (1-\tau)\tau f((1-\tau)A+\tau B)\,d\tau-\int_0^1 f((1-\tau)A+\tau B)\,d\tau$$
$$\leq \frac{3}{20}\left[\nabla f_B(B-A)-\nabla f_A(B-A)\right],$$

where f is an operator convex function on I, A, $B\in\mathcal{SA}_I(H)$, with $A\neq B$ and $f\in\mathcal{G}([A,B])$.

4. Some Examples

The function $f(x)=x^{-1}$ is operator convex on $(0,\infty)$, operator Gâteaux differentiable and

$$\nabla f_T(S)=-T^{-1}ST^{-1}$$

for T, $S>0$.

If $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable and symmetric function such that the condition (3.5) holds, then we have

$$-\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \tag{4.1}$$
$$\times\left[A^{-1}(B-A)A^{-1}-B^{-1}(B-A)B^{-1}\right]$$
$$\leq\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)((1-\tau)A+\tau B)^{-1}\,d\tau-\int_0^1((1-\tau)A+\tau B)^{-1}\,d\tau$$
$$\leq\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau$$
$$\times\left[A^{-1}(B-A)A^{-1}-B^{-1}(B-A)B^{-1}\right]$$

for all A, $B>0$.

In particular,

$$-\frac{3}{16}\left[A^{-1}(B-A)A^{-1}-B^{-1}(B-A)B^{-1}\right] \tag{4.2}$$
$$\leq 4\int_0^1\left|\tau-\frac{1}{2}\right|((1-\tau)A+\tau B)^{-1}\,d\tau-\int_0^1((1-\tau)A+\tau B)^{-1}\,d\tau$$
$$\leq\frac{3}{16}\left[A^{-1}(B-A)A^{-1}-B^{-1}(B-A)B^{-1}\right],$$

and

$$-\frac{3}{20}\left[A^{-1}(B-A)A^{-1}-B^{-1}(B-A)B^{-1}\right] \tag{4.3}$$
$$\leq 6\int_0^1(1-\tau)\tau((1-\tau)A+\tau B)^{-1}\,d\tau-\int_0^1((1-\tau)A+\tau B)^{-1}\,d\tau$$
$$\leq\frac{3}{20}\left[A^{-1}(B-A)A^{-1}-B^{-1}(B-A)B^{-1}\right]$$

for all A, $B>0$.

We note that the function $f(x)=-\ln x$ is operator convex on $(0,\infty)$. The ln function is operator Gâteaux differentiable with the following explicit formula for the derivative (cf. Pedersen [13, p. 155]):

$$\nabla \ln_T(S)=\int_0^\infty (s1_H+T)^{-1} S (s1_H+T)^{-1} ds \tag{4.4}$$

for T, $S>0$.

If we write the inequality (3.5) for $-\ln$ and $p:[0,1]\to\mathbb{R}$ is Lebesgue integrable and symmetric function such that the condition (3.5) holds, then we get

$$\begin{aligned}
&-\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \\
&\times\left[\int_0^\infty (s1_H+A)^{-1}(B-A)(s1_H+A)^{-1}ds\right. \\
&\left.-\int_0^\infty (s1_H+B)^{-1}(B-A)(s1_H+B)^{-1}ds\right] \\
&\le \int_0^1 \ln((1-\tau)A+\tau B)\,d\tau \\
&-\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)\ln((1-\tau)A+\tau B)\,d\tau \\
&\le \frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \\
&\times\left[\int_0^\infty (s1_H+A)^{-1}(B-A)(s1_H+A)^{-1}ds\right. \\
&\left.-\int_0^\infty (s1_H+B)^{-1}(B-A)(s1_H+B)^{-1}ds\right]
\end{aligned}\tag{4.5}$$

for all A, $B>0$.

If we take in (4.5) $p(\tau)=\left|\tau-\frac{1}{2}\right|$, $\tau\in[0,1]$, then we get

$$\begin{aligned}
&-\frac{3}{16}\left[\int_0^\infty (s1_H+A)^{-1}(B-A)(s1_H+A)^{-1}ds\right. \\
&\left.-\int_0^\infty (s1_H+B)^{-1}(B-A)(s1_H+B)^{-1}ds\right] \\
&\le \int_0^1 \ln((1-\tau)A+\tau B)\,d\tau \\
&-4\int_0^1\left|\tau-\frac{1}{2}\right|\ln((1-\tau)A+\tau B)\,d\tau \\
&\le \frac{3}{16}\left[\int_0^\infty (s1_H+A)^{-1}(B-A)(s1_H+A)^{-1}ds\right. \\
&\left.-\int_0^\infty (s1_H+B)^{-1}(B-A)(s1_H+B)^{-1}ds\right]
\end{aligned}\tag{4.6}$$

for all A, $B>0$.

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