BOUNDS FOR THE DIFFERENCE BETWEEN WEIGHTED AND INTEGRAL MEANS OF OPERATOR CONVEX FUNCTIONS

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Abstract. Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, the convex set of selfadjoint operators with spectra in I. If $A \neq B$ and f, as an operator function, is Gâteaux differentiable on

$$[A, B] := \{(1-t)A + tB \mid t \in [0, 1]\},$$

while $p : [0, 1] \to \mathbb{R}$ is Lebesgue integrable satisfying the condition

$$\frac{1}{\tau}\int_0^\tau g(s)\,ds \leq \frac{1}{1-\tau}\int_\tau^1 g(s)\,ds \text{ for all } \tau \in (0, 1),$$

then we have the inequalities

$$\begin{aligned} &\left[\int_0^1 \tau p(\tau)\,d\tau - \frac{1}{2}\int_0^1 p(\tau)\,d\tau\right]\nabla f_A(B-A) \\ &\leq \int_0^1 p(\tau) f((1-\tau)A + \tau B)\,d\tau \\ &- \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)A + \tau B)\,d\tau \\ &\leq \left[\int_0^1 \tau p(\tau)\,d\tau - \frac{1}{2}\int_0^1 p(\tau)\,d\tau\right]\nabla f_B(B-A). \end{aligned}$$

Some particular examples of interest are also given.

1. Introduction

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda) f(A) + \lambda f(B) \tag{1.1}$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $\mathrm{Sp}(A), \mathrm{Sp}(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [8] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0, \infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave

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on $(0,\infty)$. The entropy function $f(t)=-t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t)=e^{t}$ is neither operator convex nor operator monotone.

In [5] we obtained among others the following Hermite-Hadamard type inequalities for operator convex functions $f:I\to\mathbb{R}$

$$f\left(\frac{A+B}{2}\right)\leq\int_{0}^{1}f\left((1-s)A+sB\right)ds\leq\frac{f(A)+f(B)}{2}, \tag{1.2}$$

where A, B are selfadjoint operators with spectra included in I.

For two distinct operators $A,\,B\in\mathcal{SA}_{I}(H)$ we consider the segment of selfadjoint operators

$$[A,B]:=\{(1-t)A+tB\mid t\in[0,1]\}.$$

We observe that $A,\,B\in[A,B]$ and $[A,B]\subset\mathcal{SA}_{I}(H)$.

A continuous function $g:\mathcal{SA}_{I}(H)\to\mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A\in\mathcal{SA}_{I}(H)$ along the direction $B\in\mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla g_{A}(B):=\lim_{s\to0}\frac{g(A+sB)-g(A)}{s}\in\mathcal{B}(H). \tag{1.3}$$

If the limit (1.3) exists for all $B\in\mathcal{B}(H)$, then we say that f is *Gâteaux differentiable* in A and we can write $g\in\mathcal{G}(A)$. If this is true for any A in a subset $\mathcal{S}$ from $\mathcal{SA}_{I}(H)$ we write that $g\in\mathcal{G}(\mathcal{S})$.

Let f be an operator convex function on I. For $A,\,B\in\mathcal{SA}_{I}(H)$, the class of all selfadjoint operators with spectra in I, we consider the auxiliary function $\varphi_{(A,B)}:[0,1]\to\mathcal{SA}_{I}(H)$ defined by

$$\varphi_{(A,B)}(t):=f\left((1-t)A+tB\right). \tag{1.4}$$

For $x\in H$ we can also consider the auxiliary function $\varphi_{(A,B);x}:[0,1]\to\mathbb{R}$ defined by

$$\varphi_{(A,B);x}(t):=\left\langle\varphi_{(A,B)}(t)\,x,x\right\rangle=\left\langle f\left((1-t)A+tB\right)x,x\right\rangle. \tag{1.5}$$

We have the following basic facts, see for instance :

Lemma 1. *Let f be an operator convex function on I. For any $A,\,B\in\mathcal{SA}_{I}(H)$, $\varphi_{(A,B)}$ is well defined and convex in the operator order. For any $(A,B)\in\mathcal{SA}_{I}(H)$ and $x\in H$ the function $\varphi_{(A,B);x}$ is convex in the usual sense on $[0,1]$.*

Lemma 2. *Let f be an operator convex function on I and $A,\,B\in\mathcal{SA}_{I}(H)$, with $A\neq B$. If $f\in\mathcal{G}([A,B])$, then the auxiliary function $\varphi_{(A,B)}$ is differentiable on $(0,1)$ and*

$$\varphi'_{(A,B)}(t)=\nabla f_{(1-t)A+tB}(B-A). \tag{1.6}$$

Also we have for the lateral derivative that

$$\varphi'_{(A,B)}(0+)=\nabla f_{A}(B-A) \tag{1.7}$$

and

$$\varphi'_{(A,B)}(1-)=\nabla f_{B}(B-A). \tag{1.8}$$

We also have:

Lemma 3. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then for $0 < t_1 < t_2 < 1$ we have*

$$\nabla g_{(1-t_1)A+t_1B}(B-A) \leq \nabla g_{(1-t_2)A+t_2B}(B-A) \tag{1.9}$$

in the operator order.

We also have

$$\nabla f_A(B-A) \leq \nabla g_{(1-t_1)A+t_1B}(B-A) \tag{1.10}$$

and

$$\nabla g_{(1-t_2)A+t_2B}(B-A) \leq \nabla f_B(B-A). \tag{1.11}$$

In the recent paper [7], we obtained the following operator *Féjer's type inequalities:*

Theorem 1. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$ and $p : [0,1] \to [0,\infty)$ is Lebesgue integrable and symmetric, namely $p(1-t) = p(t)$ for all $t \in [0,1]$, then*

$$\begin{aligned} 0 &\leq \int_0^1 p(t) f((1-t)A+tB)\,dt - \left(\int_0^1 p(t)\,dt\right) f\left(\frac{A+B}{2}\right) \\ &\leq \frac{1}{2}\left(\int_0^1 \left|t-\frac{1}{2}\right| p(t)\,dt\right)\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right]. \end{aligned} \tag{1.12}$$

In particular, for $p \equiv 1$ we get

$$\begin{aligned} 0 &\leq \int_0^1 f((1-t)A+tB)\,dt - f\left(\frac{A+B}{2}\right) \\ &\leq \frac{1}{8}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right]. \end{aligned} \tag{1.13}$$

We also have:

Theorem 2. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$ and $p : [0,1] \to [0,\infty)$ is Lebesgue integrable and symmetric, namely $p(1-t) = p(t)$ for all $t \in [0,1]$, then*

$$\begin{aligned} 0 &\leq \left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2} - \int_0^1 p(t) f((1-t)A+tB)\,dt \\ &\leq \frac{1}{2}\int_0^1 \left(\frac{1}{2} - \left|t-\frac{1}{2}\right|\right) p(t)\,dt\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right]. \end{aligned} \tag{1.14}$$

In particular, for $p \equiv 1$ we get

$$\begin{aligned} 0 &\leq \frac{f(A)+f(B)}{2} - \int_0^1 f((1-t)A+tB)\,dt \\ &\leq \frac{1}{8}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right]. \end{aligned} \tag{1.15}$$

For recent inequalities for operator convex functions see [1]-[6] and [9]-[18].

Motivated by the above results, we establish in this paper some upper and lower bounds in the operator order for the difference

$$\int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)A+\tau B)\,d\tau$$

in the case when the operator convex function f is Gâteaux differentiable as a function of selfadjoint operators and $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable function such that

$$\frac{1}{\tau}\int_0^\tau p(s)\,ds\le\frac{1}{1-\tau}\int_\tau^1 p(s)\,ds \text{ for all } \tau\in(0,1).$$

Two particular examples of interest for $f(x)=-\ln x$ and $f(x)=x^{-1}$ are also given.

2. Main Results

We start to the following identity that is of interest in itself as well:

Lemma 4. *Let f be an operator convex function on I and A, $B\in\mathcal{SA}_I(H)$, with $A\neq B$. If $f\in\mathcal{G}([A,B])$ and $g:[0,1]\to\mathbb{C}$ is a Lebesgue integrable function, then we have the equality*

$$\begin{aligned}(2.1)\quad &\int_0^1 g(\tau)\varphi_{(A,B)}(\tau)\,d\tau-\int_0^1 g(\tau)\,d\tau\int_0^1\varphi_{(A,B)}(\tau)\,d\tau\\&=\int_0^1\tau(1-\tau)\left(\frac{\int_\tau^1 g(s)\,ds}{1-\tau}-\frac{\int_0^\tau g(s)\,ds}{\tau}\right)\varphi'_{(A,B)}(\tau)\,d\tau.\end{aligned}$$

Proof. Integrating by parts in the Bochner's integral, we have

$$\begin{aligned}&\int_0^\tau t\varphi'_{(A,B)}(t)\,dt+\int_\tau^1(t-1)\varphi'_{(A,B)}(t)\,dt\\&=\tau\varphi_{(A,B)}(\tau)-\int_0^\tau\varphi_{(A,B)}(t)\,dt-(\tau-1)\varphi_{(A,B)}(\tau)-\int_\tau^1\varphi_{(A,B)}(t)\,dt\\&=\varphi_{(A,B)}(\tau)-\int_0^1\varphi_{(A,B)}(t)\,dt\end{aligned}$$

that holds for all $\tau\in[0,1]$.

If we multiply this identity by $g(\tau)$ and integrate over τ in $[0,1]$, then we get

$$\begin{aligned}(2.2)\quad &\int_0^1 g(\tau)\varphi_{(A,B)}(\tau)\,d\tau-\int_0^1 g(\tau)\,d\tau\int_0^1\varphi_{(A,B)}(t)\,dt\\&=\int_0^1 g(\tau)\left(\int_0^\tau t\varphi'_{(A,B)}(t)\,dt\right)d\tau+\int_0^1 g(\tau)\left(\int_\tau^1(t-1)\varphi'_{(A,B)}(t)\,dt\right)d\tau.\end{aligned}$$

Using integration by parts, we get

$$\begin{aligned}(2.3)\quad &\int_0^1 g(\tau)\left(\int_0^\tau t\varphi'_{(A,B)}(t)\,dt\right)d\tau\\&=\int_0^1\left(\int_0^\tau t\varphi'_{(A,B)}(t)\,dt\right)d\left(\int_0^\tau g(s)\,ds\right)\\&=\left(\int_0^\tau g(s)\,ds\right)\left(\int_0^\tau t\varphi'_{(A,B)}(t)\,dt\right)\Bigg|_0^1\\&-\int_0^1\left(\int_0^\tau g(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_0^1 g(s)\,ds\right)\left(\int_0^1 t\varphi'_{(A,B)}(t)\,dt\right) \\
&- \int_0^1 \left(\int_0^\tau g(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau \\
&= \int_0^1 \left(\int_0^1 g(s)\,ds - \int_0^\tau g(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau \\
&= \int_0^1 \left(\int_\tau^1 g(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 g(\tau)\left(\int_\tau^1 (t-1)\,\varphi'_{(A,B)}(t)\,dt\right)d\tau \\
&= \int_0^1 \left(\int_\tau^1 (t-1)\,\varphi'_{(A,B)}(t)\,dt\right)d\left(\int_0^\tau g(s)\,ds\right) \\
&= \left(\int_\tau^1 (t-1)\,\varphi'_{(A,B)}(t)\,dt\right)\left(\int_0^\tau g(s)\,ds\right)\Bigg|_0^1 \\
&+ \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,\varphi'_{(A,B)}(\tau)\,d\tau \\
&= \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,\varphi'_{(A,B)}(\tau)\,d\tau,
\end{aligned} \tag{2.4}
$$

which proves the identity

$$
\begin{aligned}
&\int_0^1 g(\tau)\,\varphi_{(A,B)}(\tau)\,d\tau - \int_0^1 g(\tau)\,d\tau \int_0^1 \varphi_{(A,B)}(\tau)\,d\tau \\
&= \int_0^1 \left(\int_\tau^1 g(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau \\
&+ \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,\varphi'_{(A,B)}(\tau)\,d\tau.
\end{aligned} \tag{2.5}
$$

Now, observe that

$$
\begin{aligned}
&\int_0^1 \left(\int_\tau^1 g(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau + \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,\varphi'_{(A,B)}(\tau)\,d\tau \\
&= \int_0^1 \tau\left(\int_\tau^1 g(s)\,ds\right)\varphi'_{(A,B)}(\tau)\,d\tau - \int_0^1 (1-\tau)\left(\int_0^\tau g(s)\,ds\right)\varphi'_{(A,B)}(\tau)\,d\tau \\
&= \int_0^1 \tau(1-\tau)\left(\frac{\int_\tau^1 g(s)\,ds}{1-\tau} - \frac{\int_0^\tau g(s)\,ds}{t}\right)\varphi'_{(A,B)}(\tau)\,d\tau
\end{aligned}
$$

and by (2.5) we obtain the desired equality (2.1). □

We have the following result:

Theorem 3. *Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$ and $p : [0,1] \to \mathbb{R}$ is a Lebesgue integrable function such that*

$$
\frac{1}{\tau}\int_0^\tau p(s)\,ds \leq \frac{1}{1-\tau}\int_\tau^1 p(s)\,ds \text{ for all } \tau \in (0,1), \tag{2.6}
$$

then we have the inequalities

$$\begin{aligned}(2.7)\qquad &\left[\int_0^1 \tau p(\tau)\,d\tau - \frac{1}{2}\int_0^1 p(\tau)\,d\tau\right]\nabla f_A(B-A)\\ &\le \int_0^1 p(\tau)\,f((1-\tau)A+\tau B)\,d\tau - \int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\\ &\le \left[\int_0^1 \tau p(\tau)\,d\tau - \frac{1}{2}\int_0^1 p(\tau)\,d\tau\right]\nabla f_B(B-A).\end{aligned}$$

Proof. By the properties of $\varphi_{(A,B)}$ from the above section, we have in the operator order that

$$(2.8)\qquad \varphi'_{(A,B)}(1-) \ge \varphi'_{(A,B)}(\tau) \ge \varphi'_{(A,B)}(0+)$$

for all $\tau \in (0,1)$.

Since

$$\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{t} \ge 0$$

for all $\tau \in (0,1)$, hence

$$\begin{aligned}&\tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{t}\right)\nabla f_B(B-A)\\ &\ge \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{t}\right)\varphi'_{(A,B)}(\tau)\\ &\ge \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{t}\right)\nabla f_A(B-A)\end{aligned}$$

for all $\tau \in (0,1)$.

By taking the integral in this inequality, we get

$$\begin{aligned}(2.9)\qquad &\int_0^1 \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{t}\right)d\tau\nabla f_B(B-A)\\ &\ge \int_0^1 \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{t}\right)\varphi'_{(A,B)}(\tau)\,d\tau\\ &\ge \int_0^1 \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{t}\right)d\tau\nabla f_A(B-A).\end{aligned}$$

By the scalar version of the identity (2.1) we also have

$$\begin{aligned}&\int_0^1 \tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{t}\right)d\tau\\ &= \int_0^1 g(\tau)\,\tau d\tau - \int_0^1 g(\tau)\,d\tau\int_0^1 \tau d\tau = \int_0^1 \tau p(\tau)\,d\tau - \frac{1}{2}\int_0^1 p(\tau)\,d\tau\end{aligned}$$

and by employing Lemma 4 and the inequality (2.9) we obtain (2.7). $\square$

Corollary 1. *Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$ and $p : [0,1] \to \mathbb{R}$ a monotonic nondecreasing function, then we have the inequalities (2.7).*

Proof. If $p:[0,1]\to\mathbb{R}$ is a monotonic nondecreasing function, then

$$\frac{1}{x}\int_0^x p(s)\,ds \le p(x) \le \frac{1}{1-x}\int_x^1 p(s)\,ds$$

for $x\in(0,1)$. Then by applying Theorem 3 we get the desired result. □

If $p:[0,1]\to\mathbb{R}$ is asymmetric and Lebesgue integrable, then $\int_0^1 p(s)\,ds=0$. If $\tau\in[0,1]$ then $\int_0^\tau p(s)\,ds+\int_\tau^1 p(s)\,ds=0$, which implies that $\int_\tau^1 p(s)\,ds=-\int_0^\tau p(s)\,ds$.

Corollary 2. *Let f be an operator convex function on I and A, $B\in\mathcal{SA}_I(H)$, with $A\ne B$. If $f\in\mathcal{G}([A,B])$ and $p:[0,1]\to\mathbb{R}$ an asymmetric Lebesgue integrable function such that*

$$\int_0^\tau p(s)\,ds\le 0 \text{ for all } \tau\in[0,1], \tag{2.10}$$

or, equivalently,

$$0\le\int_\tau^1 p(s)\,ds \text{ for all } \tau\in[0,1], \tag{2.11}$$

then we have the inequalities

$$\begin{aligned}\int_0^1 \tau p(\tau)\,d\tau\nabla f_A(B-A) &\le \int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau \\ &\le \int_0^1 \tau p(\tau)\,d\tau\nabla f_B(B-A).\end{aligned} \tag{2.12}$$

Proof. The condition

$$\frac{1}{\tau}\int_0^\tau p(s)\,ds\le\frac{1}{1-\tau}\int_\tau^1 p(s)\,ds \text{ for all } \tau\in(0,1)$$

is equivalent to

$$\frac{1}{\tau}\int_0^\tau p(s)\,ds\le-\frac{1}{1-\tau}\int_0^\tau p(s)\,ds$$

namely

$$\frac{1}{\tau}\int_0^\tau p(s)\,ds+\frac{1}{1-\tau}\int_0^\tau p(s)\,ds\le 0,$$

which is equivalent to (2.10).

By utilising (2.7) we derive the desired result (2.12). □

If $q:[0,1]\to\mathbb{R}$ is integrable, then the function $p(s)=q(s)-q(1-s)$ is asymmetric. By the condition (2.10) we have

$$\int_0^\tau [q(s)-q(1-s)]\,ds\le 0$$

namely

$$\int_0^\tau q(s)\,ds\le\int_0^\tau q(1-s)\,ds,\ \ \tau\in[0,1]. \tag{2.13}$$

If we put $u=1-s$, then

$$\int_0^\tau q(1-s)\,ds=\int_{1-\tau}^1 q(s)\,ds$$

and we obtain

$$\int_0^{\tau} q(s)\,ds \le \int_{1-\tau}^{1} q(s)\,ds,\ \ \tau \in [0,1]. \tag{2.14}$$

We also have

$$\begin{aligned}\int_0^1 \tau p(\tau)\,d\tau &= \int_0^1 s\left[q(s) - q(1-s)\right] ds \\ &= \int_0^1 sq(s)\,ds - \int_0^1 (1-s)\,q(s)\,ds \\ &= \int_0^1 [2s-1]\,q(s)\,ds = 2\int_0^1 \left(s - \frac{1}{2}\right) q(s)\,ds\end{aligned}$$

and, for an integrable function $f : [0,1] \to \mathcal{SA}_I(H)$ we have

$$\begin{aligned}\int_0^1 p(s)\,f(s)\,ds &= \int_0^1 \left[q(s) - q(1-s)\right] f(s)\,ds \\ &= \int_0^1 q(s)\,f(s)\,ds - \int_0^1 q(1-s)\,f(s)\,ds \\ &= \int_0^1 q(s)\,f(s)\,ds - \int_0^1 q(s)\,f(1-s)\,ds \\ &= \int_0^1 q(s)\left[f(s) - f(1-s)\right] ds.\end{aligned}$$

We can state:

Corollary 3. *Let f be an operator convex function on I and $A,\ B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$ and $q : [0,1] \to \mathbb{R}$ a Lebesgue integrable function such that (2.13) holds, then we have the inequalities*

$$\begin{aligned}&\int_0^1 \left(\tau - \frac{1}{2}\right) q(\tau)\,d\tau \nabla f_A(B-A) \\ &\le \frac{1}{2}\int_0^1 q(\tau)\left[f((1-\tau)A + \tau B) - f(\tau A + (1-\tau)B)\right] d\tau \\ &\le \int_0^1 \left(\tau - \frac{1}{2}\right) q(\tau)\,d\tau \nabla f_B(B-A).\end{aligned} \tag{2.15}$$

3. Some Examples

We consider the function $p(\tau) = \tau$, $\tau \in [0,1]$. Observe that

$$\int_0^1 \tau p(\tau)\,d\tau - \frac{1}{2}\int_0^1 p(\tau)\,d\tau = \int_0^1 \tau^2 d\tau - \frac{1}{2}\int_0^1 \tau d\tau = \frac{1}{12}.$$

Let f be an operator convex function on I and $A,\, B\in\mathcal{SA}_I(H)$, with $A\neq B$. If $f\in\mathcal{G}([A,B])$, then by (2.7) we get

$$\begin{aligned}&\frac{1}{12}\nabla f_A(B-A)\\ &\leq\int_0^1\tau f((1-\tau)A+\tau B)\,d\tau-\frac{1}{2}\int_0^1 f((1-\tau)A+\tau B)\,d\tau\\ &\leq\frac{1}{12}\nabla f_B(B-A).\end{aligned}\tag{3.1}$$

For n a natural number, the function $p(\tau)=\left(\tau-\frac{1}{2}\right)^{2n+1}$, is increasing, then for f an operator convex function on I and $A,\, B\in\mathcal{SA}_I(H)$, with $A\neq B$ and $f\in\mathcal{G}([A,B])$, we have by (2.7)

$$\begin{aligned}&\left[\int_0^1\tau\left(\tau-\frac{1}{2}\right)^{2n+1}d\tau-\frac{1}{2}\int_0^1\left(\tau-\frac{1}{2}\right)^{2n+1}d\tau\right]\nabla f_A(B-A)\\ &\leq\int_0^1\left(\tau-\frac{1}{2}\right)^{2n+1}f((1-\tau)A+\tau B)\,d\tau\\ &-\int_0^1\left(\tau-\frac{1}{2}\right)^{2n+1}d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\\ &\leq\left[\int_0^1\tau\left(\tau-\frac{1}{2}\right)^{2n+1}d\tau-\frac{1}{2}\int_0^1\left(\tau-\frac{1}{2}\right)^{2n+1}d\tau\right]\nabla f_B(B-A).\end{aligned}$$

Observe that

$$\begin{aligned}&\int_0^1\tau\left(\tau-\frac{1}{2}\right)^{2n+1}d\tau-\frac{1}{2}\int_0^1\left(\tau-\frac{1}{2}\right)^{2n+1}d\tau\\ &=\int_0^1\left(\tau-\frac{1}{2}\right)\left(\tau-\frac{1}{2}\right)^{2n+1}d\tau=\int_0^1\left(\tau-\frac{1}{2}\right)^{2n+2}d\tau\\ &=\frac{2}{2n+3}\left(\frac{1}{2}\right)^{2n+3}=\frac{1}{(2n+3)\,2^{2n+2}}\end{aligned}$$

and

$$\int_0^1\left(\tau-\frac{1}{2}\right)^{2n+1}d\tau=0,$$

which gives

$$\begin{aligned}\frac{1}{(2n+3)\,2^{2n+2}}\nabla f_A(B-A)&\leq\int_0^1\left(\tau-\frac{1}{2}\right)^{2n+1}f((1-\tau)A+\tau B)\,d\tau\\ &\leq\frac{1}{(2n+3)\,2^{2n+2}}\nabla f_B(B-A)\end{aligned}\tag{3.2}$$

for f an operator convex function on I, $A,\, B\in\mathcal{SA}_I(H)$, with $A\neq B$ and $f\in\mathcal{G}([A,B])$ while n is a natural number.

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