SOME INEQUALITIES FOR WEIGHTED AND INTEGRAL MEANS OF CONVEX FUNCTIONS ON LINEAR SPACES

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Abstract. Let f be a convex function on a convex subset C of a linear space and x, $y\in C$, with $x\neq y$. If $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable and symmetric function, namely $p(1-t)=p(t)$ for all $t\in[0,1]$ and such that the condition

$$0\leq\int_0^\tau p(s)\,ds\leq\int_0^1 p(s)\,ds \text{ for all } \tau\in[0,1]$$

holds, then we have

$$\begin{aligned}&\left|\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)\,f((1-\tau)x+\tau y)\,d\tau-\int_0^1 f((1-\tau)x+\tau y)\,d\tau\right|\\&\leq\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\left[\nabla_-f_y(y-x)-\nabla_+f_x(y-x)\right]\\&\leq\frac{1}{2}\left[\nabla_-f_y(y-x)-\nabla_+f_x(y-x)\right].\end{aligned}$$

Some applications for norms and semi-inner products are also provided.

1. Introduction

Let X be a real linear space, $x,y\in X$, $x\neq y$ and let $[x,y]:=\{(1-\lambda)x+\lambda y,\ \lambda\in[0,1]\}$ be the *segment* generated by x and y. We consider the function $f:[x,y]\to\mathbb{R}$ and the attached function $\varphi_{(x,y)}:[0,1]\to\mathbb{R}$, $\varphi_{(x,y)}(t):=f[(1-t)x+ty]$, $t\in[0,1]$.

It is well known that f is convex on $[x,y]$ iff $\varphi(x,y)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy

(i) $\varphi'_{\pm(x,y)}(s)=\nabla_\pm f_{(1-s)x+sy}(y-x)$, $s\in[0,1)$,
(ii) $\varphi'_{+(x,y)}(0)=\nabla_+f_x(y-x)$,
(iii) $\varphi'_{-(x,y)}(1)=\nabla_-f_y(y-x)$,

where $\nabla_\pm f_x(y)$ are the *Gâteaux lateral derivatives,* we recall that

$$\begin{aligned}\nabla_+f_x(y)&:=\lim_{h\to0+}\frac{f(x+hy)-f(x)}{h},\\\nabla_-f_x(y)&:=\lim_{k\to0-}\frac{f(x+ky)-f(x)}{k},\quad x,\ y\in X.\end{aligned}$$

The following inequality is the well-known Hermite-Hadamard integral inequality for convex functions defined on a segment $[x,y]\subset X$:

$$f\left(\frac{x+y}{2}\right)\leq\int_0^1 f[(1-t)x+ty]\,dt\leq\frac{f(x)+f(y)}{2},\tag{HH}$$

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which easily follows by the classical Hermite-Hadamard inequality for the convex function $\varphi(x,y) : [0,1] \to \mathbb{R}$

$$\varphi_{(x,y)}\left(\frac{1}{2}\right) \le \int_0^1 \varphi_{(x,y)}(t)\,dt \le \frac{\varphi_{(x,y)}(0) + \varphi_{(x,y)}(1)}{2}.$$

For other related results see the monograph on line [8]. For some recent results in linear spaces see [1], [2] and [9]-[12].

In the recent paper we established the following refinements and reverses of Féjer's inequality for functions defined on linear spaces:

Theorem 1. *Let f be an convex function on C and $x,\ y \in C$ with $x \neq y$. If $p : [0,1] \to [0,\infty)$ is Lebesgue integrable and symmetric, namely $p(1-t) = p(t)$ for all $t \in [0,1]$, then*

$$\begin{aligned} 0 &\le \frac{1}{2}\left[\nabla_+ f_{\frac{x+y}{2}}(y-x) - \nabla_- f_{\frac{x+y}{2}}(y-x)\right]\int_0^1 \left|t-\frac{1}{2}\right| p(t)\,dt \\ &\le \int_0^1 f((1-t)x+ty)\,p(t)\,dt - f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt \\ &\le \frac{1}{2}\left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x)\right]\left(\int_0^1 \left|t-\frac{1}{2}\right| p(t)\,dt\right) \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} 0 &\le \frac{1}{2}\left[\nabla_+ f_{\frac{x+y}{2}}(y-x) - \nabla_- f_{\frac{x+y}{2}}(y-x)\right]\int_0^1 \left(\frac{1}{2} - \left|t-\frac{1}{2}\right|\right) p(t)\,dt \\ &\le \frac{f(x)+f(y)}{2}\int_0^1 p(t)\,dt - \int_0^1 f((1-t)x+ty)\,p(t)\,dt \\ &\le \frac{1}{2}\left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x)\right]\int_0^1 \left(\frac{1}{2} - \left|t-\frac{1}{2}\right|\right) p(t)\,dt. \end{aligned} \tag{1.2}$$

If we take $p \equiv 1$ in (1.1), then we get

$$\begin{aligned} 0 &\le \frac{1}{8}\left[\nabla_+ f_{\frac{x+y}{2}}(y-x) - \nabla_- f_{\frac{x+y}{2}}(y-x)\right] \\ &\le \int_0^1 f[(1-t)x+ty]\,dt - f\left(\frac{x+y}{2}\right) \\ &\le \frac{1}{8}\left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x)\right] \end{aligned} \tag{1.3}$$

that was firstly obtained in [4], while from (1.2) we recapture the result obtained in [5]

$$\begin{aligned} 0 &\le \frac{1}{8}\left[\nabla_+ f_{\frac{x+y}{2}}(y-x) - \nabla_- f_{\frac{x+y}{2}}(y-x)\right] \\ &\le \frac{f(x)+f(y)}{2} - \int_0^1 f[(1-t)x+ty]\,dt \\ &\le \frac{1}{8}\left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x)\right]. \end{aligned} \tag{1.4}$$

Motivated by the above results, we establish in this paper some upper and lower bounds for the difference

$$\int_0^1 p(\tau)\,f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau$$

where f is a convex function on C and $x, y \in C$, with $x \neq y$ while $p : [0,1] \to \mathbb{R}$ is a Lebesgue integrable function such that

$$0 \leq \int_0^\tau p(s)\,ds \leq \int_0^1 p(s)\,ds \text{ for all } \tau \in [0,1].$$

Some applications for norms and semi-inner products are also provided.

2. Main Results

We start to the following identity that is of interest in itself as well:

Lemma 1. *Let f be a convex function on C and $x, y \in C$, with $x \neq y$. If $g : [0,1] \to \mathbb{C}$ is a Lebesgue integrable function, then we have the equality*

$$\begin{aligned}\int_0^1 g(\tau)\varphi_{(x,y)}(\tau)\,d\tau - \int_0^1 g(\tau)\,d\tau \int_0^1 \varphi_{(x,y)}(\tau)\,d\tau \\ = \int_0^1 \left(\int_\tau^1 g(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau \\ + \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\varphi'_{(x,y)}(\tau)\,d\tau.\end{aligned} \tag{2.1}$$

Proof. Integrating by parts in the Lebesgue integral, we have

$$\begin{aligned}&\int_0^\tau t\varphi'_{(x,y)}(t)\,dt + \int_\tau^1 (t-1)\varphi'_{(x,y)}(t)\,dt \\ &= \tau\varphi_{(x,y)}(\tau) - \int_0^\tau \varphi_{(x,y)}(t)\,dt - (\tau-1)\varphi_{(x,y)}(\tau) - \int_\tau^1 \varphi_{(x,y)}(t)\,dt \\ &= \varphi_{(x,y)}(\tau) - \int_0^1 \varphi_{(x,y)}(t)\,dt\end{aligned}$$

that holds for all $\tau \in [0,1]$.

If we multiply this identity by $g(\tau)$ and integrate over τ in $[0,1]$, then we get

$$\begin{aligned}\int_0^1 g(\tau)\varphi_{(x,y)}(\tau)\,d\tau - \int_0^1 g(\tau)\,d\tau \int_0^1 \varphi_{(x,y)}(t)\,dt \\ = \int_0^1 g(\tau)\left(\int_0^\tau t\varphi'_{(x,y)}(t)\,dt\right)d\tau + \int_0^1 g(\tau)\left(\int_\tau^1 (t-1)\varphi'_{(x,y)}(t)\,dt\right)d\tau.\end{aligned} \tag{2.2}$$

Using integration by parts, we get

$$\begin{aligned}&\int_0^1 g(\tau)\left(\int_0^\tau t\varphi'_{(x,y)}(t)\,dt\right)d\tau \\ &= \int_0^1 \left(\int_0^\tau t\varphi'_{(x,y)}(t)\,dt\right)d\left(\int_0^\tau g(s)\,ds\right) \\ &= \left(\int_0^\tau g(s)\,ds\right)\left(\int_0^\tau t\varphi'_{(x,y)}(t)\,dt\right)\Bigg|_0^1 \\ &- \int_0^1 \left(\int_0^\tau g(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau\end{aligned} \tag{2.3}$$

$$
\begin{aligned}
&= \left(\int_0^1 g(s)\,ds\right)\left(\int_0^1 t\varphi'_{(x,y)}(t)\,dt\right) \\
&- \int_0^1 \left(\int_0^\tau g(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau \\
&= \int_0^1 \left(\int_0^1 g(s)\,ds - \int_0^\tau g(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau \\
&= \int_0^1 \left(\int_\tau^1 g(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 g(\tau)\left(\int_\tau^1 (t-1)\,\varphi'_{(x,y)}(t)\,dt\right)d\tau \\
&= \int_0^1 \left(\int_\tau^1 (t-1)\,\varphi'_{(x,y)}(t)\,dt\right)d\left(\int_0^\tau g(s)\,ds\right) \\
&= \left(\int_\tau^1 (t-1)\,\varphi'_{(x,y)}(t)\,dt\right)\left(\int_0^\tau g(s)\,ds\right)\Bigg|_0^1 \\
&+ \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,\varphi'_{(x,y)}(\tau)\,d\tau \\
&= \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,\varphi'_{(x,y)}(\tau)\,d\tau,
\end{aligned}
\tag{2.4}
$$

which proves the identity in (2.1). □

Theorem 2. *Let f be an operator convex function on C and $x, y \in C$, with $x \neq y$. If $p : [0,1] \to \mathbb{R}$ is a Lebesgue integrable function such that*

$$
0 \le \int_0^\tau p(s)\,ds \le \int_0^1 p(s)\,ds \text{ for all } \tau \in [0,1], \tag{2.5}
$$

then we have the inequalities

$$
\begin{aligned}
&\int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau d\tau \nabla_+ f_x(y-x) \\
&- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \nabla_- f_y(y-x) \\
&\le \int_0^1 p(\tau)\,f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau \\
&\le \int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau d\tau \nabla_- f_y(y-x) \\
&- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \nabla_+ f_x(y-x)
\end{aligned}
\tag{2.6}
$$

or, equivalently,

$$\begin{aligned}(2.7)\quad &\int_0^1 (1-\tau)\left(\int_0^\tau \left[p(1-s)\nabla_- f_y(y-x) - p(s)\nabla_+ f_x(y-x)\right] ds\right) d\tau \\ &\le \int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau \\ &\le \int_0^1 (1-\tau)\left(\int_0^\tau \left[p(1-s)\nabla_+ f_x(y-x) - p(s)\nabla_- f_y(y-x)\right] ds\right) d\tau.\end{aligned}$$

Proof. We have for $\varphi_{(x,y)}$ and $p:[0,1]\to\mathbb{R}$ a Lebesgue integrable function that

$$\begin{aligned}(2.8)\quad &\int_0^1 p(\tau)\varphi_{(x,y)}(\tau)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 \varphi_{(x,y)}(\tau)\,d\tau \\ &= \int_0^1 \left(\int_\tau^1 p(s)\,ds\right)(\tau)\,\varphi'_{(x,y)}(\tau)\,d\tau \\ &- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,\varphi'_{(x,y)}(\tau)\,d\tau.\end{aligned}$$

By the gradient inequalities for $\varphi_{(x,y)}$ we have

$$(2.9)\quad \tau\nabla_- f_y(y-x) \ge \tau\varphi'_{(x,y)}(\tau) \ge \tau\nabla_+ f_x(y-x)$$

and

$$(2.10)\quad (1-\tau)\nabla_- f_y(y-x) \ge (1-\tau)\varphi'_{(x,y)}(\tau) \ge (1-\tau)\nabla_+ f_x(y-x)$$

for all $\tau\in(0,1)$.

From

$$\int_0^\tau p(s)\,ds \le \int_0^1 p(s)\,ds = \int_0^\tau p(s)\,ds + \int_\tau^1 p(s)\,ds,$$

we get that $\int_\tau^1 p(s)\,ds \ge 0$ for all $\tau\in(0,1)$.

From (2.9) we derive that

$$\begin{aligned}\left(\int_\tau^1 p(s)\,ds\right)\tau\nabla_- f_y(y-x) &\ge \left(\int_\tau^1 p(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau) \\ &\ge \left(\int_\tau^1 p(s)\,ds\right)\tau\nabla_+ f_x(y-x)\end{aligned}$$

and from (2.10) that

$$\begin{aligned}-\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\nabla_+ f_x(y-x) &\le -\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\varphi'_{(x,y)}(\tau) \\ &\le -\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\nabla_- f_y(y-x)\end{aligned}$$

all $\tau\in(0,1)$.

If we integrate these inequalities over $\tau \in [0,1]$ and add the obtained results, then we get

$$\begin{aligned}
&\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau \nabla_- f_y(y-x) - \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \nabla_+ f_x(y-x) \\
&\geq \int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau \varphi'_{(x,y)}(\tau)\,d\tau - \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,\varphi'_{(x,y)}(\tau)\,d\tau \\
&\geq \int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau \nabla_+ f_x(y-x) - \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \nabla_- f_y(y-x).
\end{aligned}$$

By using the equality (2.1) we obtain

$$\begin{aligned}
&\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau \nabla_+ f_x(y-x) \\
&- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \nabla_- f_y(y-x) \\
&\leq \int_0^1 p(\tau)\,\varphi_{(x,y)}(\tau)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 \varphi_{(x,y)}(\tau)\,d\tau \\
&\leq \int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau \nabla_- f_y(y-x) \\
&- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \nabla_+ f_x(y-x),
\end{aligned} \tag{2.11}$$

namely (2.6).

If we change the variable $y = 1-\tau$, then we have

$$\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau = \int_0^1 \left(\int_{1-y}^1 p(s)\,ds\right)(1-y)\,dy.$$

Also by the change of variable $u = 1-s$, we get

$$\int_{1-y}^1 p(s)\,ds = \int_0^y p(1-u)\,du,$$

which implies that

$$\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau = \int_0^1 \left(\int_0^\tau p(1-s)\,ds\right)(1-\tau)\,d\tau.$$

Therefore

$$\begin{aligned}
&\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau \nabla_- f_y(y-x) - \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \nabla_+ f_x(y-x) \\
&= \int_0^1 \left(\int_0^\tau p(1-s)\,ds\right)(1-\tau)\,d\tau \nabla_- f_y(y-x) \\
&- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \nabla_+ f_x(y-x) \\
&= \int_0^1 (1-\tau)\left(\int_0^\tau \left[p(1-s)\,\nabla_- f_y(y-x) - p(s)\,\nabla_+ f_x(y-x)\right] ds\right) d\tau
\end{aligned}$$

and

$$\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau \nabla_+ f_x(y-x) - \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \nabla_- f_y(y-x)$$
$$= \int_0^1 \left(\int_0^\tau p(1-s)\,ds\right)(1-\tau)\,d\tau \nabla_+ f_x(y-x)$$
$$- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \nabla_- f_y(y-x)$$
$$= \int_0^1 (1-\tau)\left(\int_0^\tau \left[p(1-s)\nabla_+ f_x(y-x) - p(s)\nabla_- f_y(y-x)\right]ds\right)d\tau,$$

and by (2.11) we get (2.7). □

We say that the function $p:[0,1]\to\mathbb{R}$ is symmetric on $[0,1]$ if

$$p(1-t) = p(t) \text{ for all } t\in[0,1].$$

Corollary 1. *Let f be a convex function on C and x, $y\in C$, with $x\neq y$. If $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable and symmetric function such that the condition (2.5) holds, then we have*

$$\left|\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 f((1-\tau)x+\tau y)\,d\tau\right| \tag{2.12}$$
$$\leq \frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x)\right]$$
$$\leq \frac{1}{2}\left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x)\right].$$

Proof. Since p is symmetric, then $p(1-s)=p(s)$ for all $s\in[0,1]$ and by (2.7) we get

$$\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\left[\nabla_+ f_x(y-x) - \nabla_- f_y(y-x)\right]$$
$$\leq \int_0^1 p(\tau)\varphi_{(x,y)}(\tau)\,d\tau - \int_0^1 p(\tau)\,d\tau\int_0^1 \varphi_{(x,y)}(\tau)\,d\tau$$
$$\leq \left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x)\right]\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau,$$

which is equivalent to the first inequality in (2.12).

Since $0\leq \int_0^\tau p(s)\,ds \leq \int_0^1 p(\tau)\,d\tau$, hence

$$\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \leq \int_0^1 p(\tau)\,d\tau\int_0^1(1-\tau)\,d\tau = \frac{1}{2}\int_0^1 p(\tau)\,d\tau$$

and the last part of (2.12) is proved. □

Remark 1. *If the function p is nonnegative and symmetric then the inequality (2.12) holds true.*

If we consider the weight $p:[0,1]\to[0,\infty)$, $p(s)=\left|s-\frac{1}{2}\right|$, then

$$\begin{aligned}
&\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\\
&=\int_0^1\left(\int_0^\tau\left|s-\frac{1}{2}\right|ds\right)(1-\tau)\,d\tau\\
&=\int_0^{\frac{1}{2}}\left(\int_0^\tau\left|s-\frac{1}{2}\right|ds\right)(1-\tau)\,d\tau\\
&+\int_{\frac{1}{2}}^1\left(\int_0^\tau\left|s-\frac{1}{2}\right|ds\right)(1-\tau)\,d\tau\\
&=\int_0^{\frac{1}{2}}\left(\int_0^\tau\left(\frac{1}{2}-s\right)ds\right)(1-\tau)\,d\tau\\
&+\int_{\frac{1}{2}}^1\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-s\right)ds+\int_{\frac{1}{2}}^\tau\left(s-\frac{1}{2}\right)\right)(1-\tau)\,d\tau\\
&=\int_0^{\frac{1}{2}}\left(\frac{1}{2}\tau-\frac{\tau^2}{2}\right)(1-\tau)\,d\tau\\
&+\int_{\frac{1}{2}}^1\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-s\right)ds+\int_{\frac{1}{2}}^\tau\left(s-\frac{1}{2}\right)ds\right)(1-\tau)\,d\tau.
\end{aligned}$$

We have

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left(\frac{1}{2}\tau-\frac{\tau^2}{2}\right)(1-\tau)\,d\tau=\frac{1}{2}\int_0^{\frac{1}{2}}(1-\tau)\,\tau\,(1-\tau)\,d\tau\\
&=\frac{1}{2}\int_0^{\frac{1}{2}}(1-\tau)^2\,\tau d\tau=\frac{11}{384}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^1\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-s\right)ds+\int_{\frac{1}{2}}^\tau\left(s-\frac{1}{2}\right)ds\right)(1-\tau)\,d\tau\\
&=\int_{\frac{1}{2}}^1\left(\frac{1}{8}+\frac{1}{2}\left(\tau-\frac{1}{2}\right)^2\right)(1-\tau)\,d\tau\\
&=\frac{1}{8}\int_{\frac{1}{2}}^1(1-\tau)\,d\tau+\frac{1}{2}\int_{\frac{1}{2}}^1\left(\tau-\frac{1}{2}\right)^2(1-\tau)\,d\tau=\frac{7}{384}.
\end{aligned}$$

Therefore

$$\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau=\frac{3}{64}.$$

Since $\int_0^1\left|\tau-\frac{1}{2}\right|d\tau=\frac{1}{4}$, hence

$$\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau=\frac{3}{16}.$$

Utilising (2.12) for symmetric weight $p:[0,1]\to[0,\infty)$, $p(s)=\left|s-\frac{1}{2}\right|$, we get

$$\left|4\int_0^1\left|\tau-\frac{1}{2}\right|f\left((1-\tau)x+\tau y\right)d\tau-\int_0^1 f\left((1-\tau)x+\tau y\right)d\tau\right| \le \frac{3}{16}\left[\nabla_- f_y(y-x)-\nabla_+ f_x(y-x)\right] \tag{2.13}$$

where f is a convex function on C and $x,\ y\in C$, with $x\neq y$.

Consider now the symmetric function $p(s)=(1-s)s$, $x\in[0,1]$. Then

$$\int_0^\tau p(s)\,ds=\int_a^\tau(1-s)\,sds=-\frac{1}{6}\tau^2(2\tau-3),\ \ \tau\in[0,1]$$

and

$$\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau=-\frac{1}{6}\int_0^1\tau^2(2\tau-3)(1-\tau)\,d\tau=\frac{1}{40}.$$

Also

$$\int_0^1 p(\tau)\,d\tau=\int_0^1(1-\tau)\,\tau d\tau=\frac{1}{6}$$

and

$$\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau=\frac{3}{20}$$

and by (2.12) we obtain

$$\left|6\int_0^1(1-\tau)\,\tau f\left((1-\tau)x+\tau y\right)d\tau-\int_0^1 f\left((1-\tau)x+\tau y\right)d\tau\right| \le \frac{3}{20}\left[\nabla_- f_y(y-x)-\nabla_+ f_x(y-x)\right], \tag{2.14}$$

where f is a convex function on C and $x,\ y\in C$, with $x\neq y$.

3. Examples for Norms

Now, assume that $(X,\|\cdot\|)$ is a normed linear space. The function $f_0(s)=\frac{1}{2}\|x\|^2$, $x\in X$ is convex and thus the following limits exist

(iv) $\langle x,y\rangle_s:=\nabla_+ f_{0,y}(x)=\lim\limits_{t\to0+}\frac{\|y+tx\|^2-\|y\|^2}{2t}$;

(v) $\langle x,y\rangle_i:=\nabla_- f_{0,y}(x)=\lim\limits_{s\to0-}\frac{\|y+sx\|^2-\|y\|^2}{2s}$;

for any $x,\ y\in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [2] or [6]), assuming that $p,\ q\in\{s,i\}$ and $p\neq q$:

(a) $\langle x,x\rangle_p=\|x\|^2$ for all $x\in X$;

(aa) $\langle\alpha x,\beta y\rangle_p=\alpha\beta\langle x,y\rangle_p$ if $\alpha,\ \beta\ge0$ and $x,\ y\in X$;

(aaa) $\left|\langle x,y\rangle_p\right|\le\|x\|\,\|y\|$ for all $x,\ y\in X$;

(av) $\langle\alpha x+y,x\rangle_p=\alpha\langle x,x\rangle_p+\langle y,x\rangle_p$ if $x,\ y\in X$ and $\alpha\in\mathbb{R}$;

(v) $\langle -x,y\rangle_p=-\langle x,y\rangle_q$ for all $x,\ y\in X$;

(va) $\langle x+y,z\rangle_p\le\|x\|\,\|z\|+\langle y,z\rangle_p$ for all $x,\ y,\ z\in X$;

(vaa) The mapping $\langle\cdot,\cdot\rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p = s$ (or $p = i$);

(vaaa) The normed linear space $(X, \|\cdot\|)$ is smooth at the point $x_0 \in X\setminus\{0\}$ if and only if $\langle y, x_0\rangle_s = \langle y, x_0\rangle_i$ for all $y \in X$; in general $\langle y, x\rangle_i \le \langle y, x\rangle_s$ for all x, $y \in X$;

(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$, then $\langle y, x\rangle_i = \langle y, x\rangle = \langle y, x\rangle_s$ for all x, $y \in X$.

The function $f_r(x) = \|x\|^r$ ($x \in X$ and $1 \le r < \infty$) is also convex. Therefore, the following limits, which are related to the superior (inferior) semi-inner products,

$$\begin{aligned}\nabla_{\pm} f_{r,y}(x) &:= \lim_{t\to 0\pm} \frac{\|y+tx\|^r - \|y\|^r}{t}\\ &= r\|y\|^{r-1} \lim_{t\to 0\pm} \frac{\|y+tx\| - \|y\|}{t} = r\|y\|^{r-2}\langle x, y\rangle_{s(i)}\end{aligned}$$

exist for all x, $y \in X$ whenever $r \ge 2$; otherwise, they exist for any $x \in X$ and nonzero $y \in X$. In particular, if $r = 1$, then the following limits

$$\nabla_{\pm} f_{1,y}(x) := \lim_{t\to 0\pm} \frac{\|y+tx\| - \|y\|}{t} = \frac{\langle x, y\rangle_{s(i)}}{\|y\|}$$

exist for x, $y \in X$ and $y \ne 0$.

If $p : [0,1] \to \mathbb{R}$ is a Lebesgue integrable and symmetric function such that the condition

$$0 \le \int_0^\tau p(s)\,ds \le \int_0^1 p(s)\,ds \text{ for all } \tau \in [0,1],$$

is valid, then by (2.12) we get

$$\begin{aligned}&\left|\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)\|(1-\tau)x+\tau y\|^r d\tau - \int_0^1 \|(1-\tau)x+\tau y\|^r d\tau\right|\\ &\le \frac{r}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\\ &\times\left[\|y\|^{r-2}\langle y-x, y\rangle_i - \|x\|^{r-2}\langle y-x, x\rangle_s\right].\end{aligned} \tag{3.1}$$

If $r \ge 2$, then the inequality (3.1) holds for all x, $y \in X$. If $r \in [1,2)$, then the inequality (3.1) holds for all x, $y \in X$ with x, $y \ne 0$.

For $r = 2$ we get

$$\begin{aligned}&\left|\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)\|(1-\tau)x+\tau y\|^2 d\tau - \int_0^1 \|(1-\tau)x+\tau y\|^2 d\tau\right|\\ &\le \frac{2}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\left[\langle y-x, y\rangle_i - \langle y-x, x\rangle_s\right]\end{aligned} \tag{3.2}$$

for all x, $y \in X$.

If we take $p(\tau) = \left|\tau - \frac{1}{2}\right|$, $\tau \in [0,1]$ in (3.1), then we obtain

$$\begin{aligned}&\left|4\int_0^1\left|\tau - \frac{1}{2}\right|\|(1-\tau)x+\tau y\|^r d\tau - \int_0^1 \|(1-\tau)x+\tau y\|^r d\tau\right|\\ &\le \frac{3r}{16}\left[\|y\|^{r-2}\langle y-x, y\rangle_i - \|x\|^{r-2}\langle y-x, x\rangle_s\right].\end{aligned} \tag{3.3}$$

If $X = H$ a real inner product space, then from (3.2) we get

$$\left|\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)\left\|(1-\tau)x+\tau y\right\|^2 d\tau - \int_0^1 \left\|(1-\tau)x+\tau y\right\|^2 d\tau\right| \tag{3.4}$$
$$\leq \frac{2}{\int_0^1 p(\tau)\,d\tau}\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \left\|y-x\right\|^2$$

for all x, $y \in H$.

4. Examples for Functions of Several Variables

Now, let $\Omega \subset \mathbb{R}^n$ be an open convex set in $\mathbb{R}^n$. If $F : \Omega \to \mathbb{R}$ is a differentiable convex function on Ω, then, obviously, for any $\bar{c} \in \Omega$ we have

$$\nabla F_{\bar{c}}(\bar{y}) = \sum_{i=1}^n \frac{\partial F(\bar{c})}{\partial x_i}\cdot y_i,\ \ \bar{y} = (y_1, ..., y_n) \in \mathbb{R}^n,$$

where $\frac{\partial F}{\partial x_i}$ are the partial derivatives of F with respect to the variable x_i ($i = 1, ..., n$).

If $p : [0,1] \to \mathbb{R}$ is a Lebesgue integrable and symmetric function such that the condition (2.5) holds, then we have for all $\bar{a}$, $\bar{b} \in \Omega$ that

$$\left|\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)\,F\left((1-\tau)\bar{a}+\tau\bar{b}\right)d\tau - \int_0^1 f\left((1-\tau)\bar{a}+\tau\bar{b}\right)d\tau\right| \tag{4.1}$$
$$\leq \frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau$$
$$\times \sum_{i=1}^n \left(\frac{\partial F(\bar{b})}{\partial x_i} - \frac{\partial F(\bar{a})}{\partial x_i}\right)(b_i - a_i)$$
$$\leq \frac{1}{2}\sum_{i=1}^n \left(\frac{\partial F(\bar{b})}{\partial x_i} - \frac{\partial F(\bar{a})}{\partial x_i}\right)(b_i - a_i).$$

If we take $p(\tau) = \left|\tau - \frac{1}{2}\right|$, $\tau \in [0,1]$ in (4.1), then we get

$$\left|4\int_0^1 \left|\tau-\frac{1}{2}\right| F\left((1-\tau)\bar{a}+\tau\bar{b}\right)d\tau - \int_0^1 f\left((1-\tau)\bar{a}+\tau\bar{b}\right)d\tau\right| \tag{4.2}$$
$$\leq \frac{3}{16}\sum_{i=1}^n \left(\frac{\partial F(\bar{b})}{\partial x_i} - \frac{\partial F(\bar{a})}{\partial x_i}\right)(b_i - a_i)$$

for all $\bar{a}$, $\bar{b} \in \Omega$.

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