REFINING HÖLDER INTEGRAL INEQUALITY FOR PARTITIONS OF WEIGHTS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we establish a refinement and some reverses for Hölder inequality for the general Lebesgue integral on measurable spaces and partitions of weights. Applications for discrete inequalities and weighted means of positive numbers are also given.

1. Introduction

The Hölder inequality plays an important role in different branches of Modern Mathematics including Hilbert Spaces Theory, Probability & Statistics, Classical Real and Complex Analysis, Numerical Analysis, Qualitative Theory of Differential Equations and their applications.

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a σ-algebra of subsets of Ω denoted by Σ and μ a countably additive and positive measure on Σ with values in $\mathbb{R}\cup\{\infty\}$. Denote by $L_w^r(\Omega,\mu)$, $r \geq 1$ the Banach space of all $\mathbb{C}$-valued functions f defined on Ω that are r-w-integrable on Ω, i.e., $\int_\Omega w(x)|f(x)|^r d\mu(x) < \infty$, where $w : \Omega \to [0,\infty)$ is a given μ-measurable function on Ω. We write for simplicity $\int_\Omega w|f|^r d\mu$ instead of $\int_\Omega w(x)|f(x)|^r d\mu(x)$.

The following inequality is well known in the literature as the *integral Hölder inequality*:

$$\left|\int_\Omega wfgd\mu\right| \leq \left(\int_\Omega w|f|^p d\mu\right)^{1/p}\left(\int_\Omega w|g|^q d\mu\right)^{1/q} \tag{1.1}$$

provided that $f \in L_w^p(\Omega,\mu)$, $g \in L_w^q(\Omega,\mu)$ with $p,\, q > 1$ and $\frac{1}{p}+\frac{1}{q}=1$.

For some recent results related to Hölder inequality, see [1]-[2] and [6]-[14].

For the μ-integrable positive μ-a.e. weight w and a given $n \geq 2$ we consider the set $\mathfrak{P}_n(w)$ all possible n-tuples of μ-integrable positive weights $\overline{w} = (w_1, ..., w_n)$ with the property that

$$\sum_{i=1}^n w_i = w.$$

This is called a partition of w.

It is clear that $\sum_{i=1}^n \int_\Omega w_i d\mu = \int_\Omega wd\mu$ for any $\overline{w} = (w_1, ..., w_n) \in \mathfrak{P}_n(w)$ and $\int_\Omega w_i d\mu > 0$.

1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 94A17.

Key words and phrases. Hölder-Inequality, Measurable spaces, Lebesgue integral, Weighted means.

RGMIA Res. Rep. Coll. 23 (2020), Art. 1, 11 pp. Received 12/01/20

For $f\in L_{w}^{p}(\Omega,\mu)$, $g\in L_{w}^{q}(\Omega,\mu)$ with p, $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$ we consider the functional $\beta_{p,q}(|f|,|g|,\cdot):\mathfrak{P}_{n}(w)\rightarrow[0,\infty)$ defined by

$$\beta_{p,q}(|f|,|g|,\overline{w}):=\sum_{i=1}^{n}\left(\int_{\Omega}w_{i}|f|^{p}d\mu\right)^{1/p}\left(\int_{\Omega}w_{i}|g|^{q}d\mu\right)^{1/q}. \tag{1.2}$$

In this paper we establish some inequalities concerning the functional $\beta_{p,q}(|f|,|g|,\cdot)$ that provide refinements and reverses for the Hölder integral inequality (1.1). Applications for discrete inequalities and weighted means of positive numbers are also given.

2. Some General Facts

The following refinement of the Hölder inequality holds:

Theorem 1. *For $f\in L_{w}^{p}(\Omega,\mu)$, $g\in L_{w}^{q}(\Omega,\mu)$ with p, $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$ and $\overline{w}=(w_{1},....,w_{n})\in\mathfrak{P}_{n}(w)$ we have*

$$\left|\int_{\Omega}wfgd\mu\right|\leq\beta_{p,q}(|f|,|g|,\overline{w})\leq\left(\int_{\Omega}w|f|^{p}d\mu\right)^{1/p}\left(\int_{\Omega}w|g|^{q}d\mu\right)^{1/q}. \tag{2.1}$$

Proof. We have by Hölder integral inequality in that

$$\begin{aligned}\left|\int_{\Omega}wfgd\mu\right|&\leq\int_{\Omega}w|fg|d\mu=\int_{\Omega}\sum_{i=1}^{n}w_{i}|fg|d\mu=\sum_{i=1}^{n}\int_{\Omega}w_{i}|fg|d\mu\\&\leq\sum_{i=1}^{n}\left(\int_{\Omega}w_{i}|f|^{p}d\mu\right)^{1/p}\left(\int_{\Omega}w_{i}|g|^{q}d\mu\right)^{1/q}=\beta_{p,q}(|f|,|g|,\overline{w}),\end{aligned}$$

which proves the first inequality in (2.1).

By the Hölder discrete inequality we also have

$$\begin{aligned}\beta_{p,q}(|f|,|g|,\overline{w})&=\sum_{i=1}^{n}\left(\int_{\Omega}w_{i}|f|^{p}d\mu\right)^{1/p}\left(\int_{\Omega}w_{i}|g|^{q}d\mu\right)^{1/q}\\&\leq\left[\sum_{i=1}^{n}\left(\left(\int_{\Omega}w_{i}|f|^{p}d\mu\right)^{1/p}\right)^{p}\right]^{1/p}\left[\sum_{i=1}^{n}\left(\left(\int_{\Omega}w_{i}|g|^{q}d\mu\right)^{1/q}\right)^{q}\right]^{1/q}\\&=\left[\sum_{i=1}^{n}\int_{\Omega}w_{i}|f|^{p}d\mu\right]^{1/p}\left[\sum_{i=1}^{n}\int_{\Omega}w_{i}|g|^{q}d\mu\right]^{1/q}\\&=\left[\int_{\Omega}\left(\sum_{i=1}^{n}w_{i}\right)|f|^{p}d\mu\right]^{1/p}\left[\int_{\Omega}\left(\sum_{i=1}^{n}w_{i}\right)|g|^{q}d\mu\right]^{1/q}\\&=\left[\int_{\Omega}w|f|^{p}d\mu\right]^{1/p}\left[\int_{\Omega}w|g|^{q}d\mu\right]^{1/q},\end{aligned}$$

which proves the second inequality in (2.1). □

Remark 1. *For $p=q=2$ we recapture the result from* [3] *related to the Cauchy-Bunyakovsky-Schwarz inequality.*

We consider the n-tuple of integrable functions on Ω, $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$ with $\alpha_i(x) \geq 0$, $i \in \{1, ..., n\}$ and $\sum_{i=1}^n \alpha_i(x) = 1$ for μ-almost every $x \in \Omega$. We define

$$\beta_{p,q}(|f|, |g|, w, \boldsymbol{\alpha}) := \sum_{i=1}^n \left(\int_\Omega \alpha_i w |f|^p d\mu \right)^{1/p} \left(\int_\Omega \alpha_i w |g|^q d\mu \right)^{1/q}. \tag{2.2}$$

Observe that

$$\sum_{i=1}^n \alpha_i w = w$$

and by (2.1) we get

$$\left| \int_\Omega wfg d\mu \right| \leq \beta_{p,q}(|f|, |g|, w, \boldsymbol{\alpha}) \leq \left(\int_\Omega w |f|^p d\mu \right)^{1/p} \left(\int_\Omega w |g|^q d\mu \right)^{1/q}. \tag{2.3}$$

This inequality obviously hold if $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$ is a probability, namely α_i are real numbers with $\alpha_i \geq 0$, $i \in \{1, ..., n\}$ and $\sum_{i=1}^n \alpha_i = 1$.

For $t \in (0, 1)$ we define

$$\begin{aligned}\beta_{p,q}(|f|, |g|, w, \mathbf{t}) := &\left(\int_\Omega tw |f|^p d\mu \right)^{1/p} \left(\int_\Omega tw |g|^q d\mu \right)^{1/q} \\ &+ \left(\int_\Omega (1-t) w |f|^p d\mu \right)^{1/p} \left(\int_\Omega (1-t) w |g|^q d\mu \right)^{1/q},\end{aligned} \tag{2.4}$$

that is derived from (2.2) for $\alpha_1 = t$ and $\alpha_2 = t$, and by (2.1) we get

$$\left| \int_\Omega wfg d\mu \right| \leq \beta_{p,q}(|f|, |g|, w, \mathbf{t}) \leq \left(\int_\Omega w |f|^p d\mu \right)^{1/p} \left(\int_\Omega w |g|^q d\mu \right)^{1/q}. \tag{2.5}$$

If $\Omega = [a, b] \subset \mathbb{R}$ and if we take $\alpha_1(x) = \frac{x-a}{b-a}$ and $\alpha_2(x) = \frac{b-x}{b-a}$, $x \in [a, b]$, then we get the refinement of Hölder's univariate integral inequality

$$\begin{aligned}&\left| \int_a^b w(x) f(x) g(x) dx \right| \\ &\leq \frac{1}{b-a} \left(\int_a^b (x-a) w(x) |f(x)|^p dx \right)^{1/p} \left(\int_a^b (x-a) w(x) |g(x)|^q dx \right)^{1/q} \\ &+ \frac{1}{b-a} \left(\int_a^b (b-x) w(x) |f(x)|^p dx \right)^{1/p} \left(\int_a^b (b-x) w(x) |g(x)|^q dx \right)^{1/q} \\ &\leq \left(\int_a^b w(x) |f(x)|^p dx \right)^{1/p} \left(\int_a^b w(x) |g(x)|^q dx \right)^{1/q},\end{aligned} \tag{2.6}$$

obtained by İşcan in [5] in the case of uniform weight $w \equiv 1$.

Let $\Omega = [0, 2\pi] \times [0, \pi]$ and consider $\alpha_1(\theta, \phi) := \cos^2\theta \sin^2\phi$, $\alpha_2(\theta, \phi) := \sin^2\theta \sin^2\phi$, $\alpha_3(\theta, \phi) := \cos^2\phi$. Then

$$\alpha_1(\theta, \phi) + \alpha_2(\theta, \phi) + \alpha_3(\theta, \phi) = 1 \text{ for } (\theta, \phi) \in [0, 2\pi] \times [0, \pi]$$

and by (2.2) we get

$$\left|\int_0^{2\pi}\int_0^{\pi} w(\theta,\phi) f(\theta,\phi) g(\theta,\phi) \, d\theta d\phi\right| \tag{2.7}$$
$$\leq \left(\int_0^{2\pi}\int_0^{\pi} \cos^2\theta \sin^2\phi w(\theta,\phi) |f(\theta,\phi)|^p \, d\theta d\phi\right)^{1/p}$$
$$\times \left(\int_0^{2\pi}\int_0^{\pi} \cos^2\theta \sin^2\phi w(\theta,\phi) |g(\theta,\phi)|^q \, d\theta d\phi\right)^{1/q}$$
$$+ \left(\int_0^{2\pi}\int_0^{\pi} \sin^2\theta \sin^2\phi w(\theta,\phi) |f(\theta,\phi)|^p \, d\theta d\phi\right)^{1/p}$$
$$\times \left(\int_0^{2\pi}\int_0^{\pi} \sin^2\theta \sin^2\phi w(\theta,\phi) |g(\theta,\phi)|^q \, d\theta d\phi\right)^{1/q}$$
$$+ \left(\int_0^{2\pi}\int_0^{\pi} \cos^2\phi w(\theta,\phi) |f(\theta,\phi)|^p \, d\theta d\phi\right)^{1/p}$$
$$\times \left(\int_0^{2\pi}\int_0^{\pi} \cos^2\phi w(\theta,\phi) |g(\theta,\phi)|^q \, d\theta d\phi\right)^{1/q}$$
$$\leq \left(\int_0^{2\pi}\int_0^{\pi} w(\theta,\phi) |f(\theta,\phi)|^p \, d\theta d\phi\right)^{1/p}$$
$$\times \left(\int_0^{2\pi}\int_0^{\pi} w(\theta,\phi) |g(\theta,\phi)|^q \, d\theta d\phi\right)^{1/q}.$$

We observe that $\mathfrak{P}_n(w)$ is a convex set. Indeed if $\overline{w} = (w_1, ..., w_n)$, $\overline{p} = (p_1, ..., p_n) \in \mathfrak{P}_n(w)$ then for $t \in [0,1]$ we have

$$\sum_{i=1}^{n} [(1-t) w_i + t p_i] = (1-t) \sum_{i=1}^{n} w_i + t \sum_{i=1}^{n} p_i = (1-t) w + t w = w,$$

which shows that $(1-t)\overline{w} + t\overline{p} \in \mathfrak{P}_n(w)$.

Theorem 2. *For $f \in L_w^p(\Omega,\mu)$, $g \in L_w^q(\Omega,\mu)$ with p, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have that $\beta_{p,q}(|f|, |g|, \cdot)$ is a concave mapping on $\mathfrak{P}_n(w)$.*

Proof. Let $\overline{w} = (w_1, ..., w_n)$, $\overline{p} = (p_1, ..., p_n) \in \mathfrak{P}_n(w)$ and $t \in [0,1]$. Then

$$\beta_{p,q}(|f|, |g|, (1-t)\overline{w} + t\overline{p}) \tag{2.8}$$
$$= \sum_{i=1}^{n} \left(\int_\Omega ((1-t) w_i + t p_i) |f|^p \, d\mu\right)^{1/p} \left(\int_\Omega ((1-t) w_i + t p_i) |g|^q \, d\mu\right)^{1/q}$$
$$= \sum_{i=1}^{n} \left(\left((1-t)\int_\Omega w_i |f|^p \, d\mu + t\int_\Omega p_i |f|^p \, d\mu\right)\right)^{1/p}$$
$$\times \left(\left((1-t)\int_\Omega w_i |g|^q \, d\mu + t\int_\Omega p_i |g|^q \, d\mu\right)\right)^{1/q}.$$

By the elementary inequality

$$[(1-t) a^p + t b^p]^{1/p} [(1-t) c^q + t d^q]^{1/q} \geq (1-t) ac + t bd$$

that holds for the nonnegative numbers $a,\ b,\ c,\ d,\ t\in[0,1]$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have

$$
\begin{aligned}
&\left(\left((1-t)\int_\Omega w_i\left|f\right|^p d\mu+t\int_\Omega p_i\left|f\right|^p d\mu\right)\right)^{1/p}\\
&\times\left(\left((1-t)\int_\Omega w_i\left|g\right|^q d\mu+t\int_\Omega p_i\left|g\right|^q d\mu\right)\right)^{1/q}\\
&=\left(\left((1-t)\left[\left(\int_\Omega w_i\left|f\right|^p d\mu\right)^{1/p}\right]^p+t\left[\left(\int_\Omega p_i\left|f\right|^p d\mu\right)^{1/p}\right]^p\right)\right)^{1/p}\\
&\times\left(\left((1-t)\left[\left(\int_\Omega w_i\left|g\right|^q d\mu\right)^{1/q}\right]^q+t\left[\left(\int_\Omega p_i\left|g\right|^2 d\mu\right)^{1/q}\right]^q\right)\right)^{1/q}\\
&\geq(1-t)\left(\int_\Omega w_i\left|f\right|^p d\mu\right)^{1/p}\left(\int_\Omega w_i\left|g\right|^q d\mu\right)^{1/q}\\
&+t\left(\int_\Omega p_i\left|f\right|^p d\mu\right)^{1/p}\left(\int_\Omega p_i\left|g\right|^q d\mu\right)^{1/q}
\end{aligned}
\tag{2.9}
$$

for any $i\in\{1,...,n\}$ and $t\in[0,1]$.

If we sum over i from 1 to n in the inequality (2.9) and use (2.8), then we get

$$
\begin{aligned}
&\beta_{p,q}\left(\left|f\right|,\left|g\right|,(1-t)\,\overline{w}+t\overline{p}\right)\\
&\geq(1-t)\sum_{i=1}^n\left(\int_\Omega w_i\left|f\right|^p d\mu\right)^{1/p}\left(\int_\Omega w_i\left|g\right|^q d\mu\right)^{1/q}\\
&+t\sum_{i=1}^n\left(\int_\Omega p_i\left|f\right|^p d\mu\right)^{1/p}\left(\int_\Omega p_i\left|g\right|^q d\mu\right)^{1/q}\\
&=(1-t)\,\beta_{p,q}\left(\left|f\right|,\left|g\right|,\overline{w}\right)+t\beta_{p,q}\left(\left|f\right|,\left|g\right|,\overline{p}\right)
\end{aligned}
$$

and the concavity of the mapping $\beta_{p,q}\left(\left|f\right|,\left|g\right|,\cdot\right)$ is proven. □

Remark 2. *For $p=q=2$ we recapture the result from* [3] *related to the Cauchy-Bunyakovsky-Schwarz inequality.*

We have:

Corollary 1. *If $\overline{w}_k\in\mathfrak{P}_n(w)$ for $k\in\{1,...,m\}$ and $p_k\geq0$ with $\sum_{k=1}^n p_k=1$, then by Jensen's inequality we have*

$$
\sum_{k=1}^n p_k\beta_{p,q}\left(\left|f\right|,\left|g\right|,\overline{w}_k\right)\leq\beta_{p,q}\left(\left|f\right|,\left|g\right|,\sum_{k=1}^n p_k\overline{w}_k\right)
\tag{2.10}
$$

for $f\in L_w^p(\Omega,\mu)$, $g\in L_w^q(\Omega,\mu)$ with $p,\ q>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

3. Lower Bounds for $\beta_{p,q}\left(\left|f\right|,\left|g\right|,\overline{w}\right)$

If there exists the constants $m_1,\ M_1,\ m_2,\ M_2$ such that

$$
0<m_1\leq a_i\leq M_1<\infty,\ 0<m_2\leq b_i\leq M_2<\infty\text{ for any }i\in\{1,...,n\}
\tag{3.1}
$$

and $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have the following reverse of Hölder's inequality [4]

$$(1\le)\,\frac{\left(\sum_{i=1}^{n}p_i a_i^p\right)^{1/p}\left(\sum_{i=1}^{n}p_i b_i^q\right)^{1/q}}{\sum_{i=1}^{n}p_i a_i b_i} \le \max\left\{g_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right), g_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)\right\}, \tag{3.2}$$

where $g_{\frac{1}{p}}$ is defined by

$$g_{\frac{1}{p}}(x)=\frac{1}{q}x^{-\frac{1}{p}}+\frac{1}{p}x^{\frac{1}{q}},\ x>0. \tag{3.3}$$

We recall that *Specht's ratio* is defined by [11]

$$S(h):=\begin{cases}\frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h\in(0,1)\cup(1,\infty),\\ \\ 1 \text{ if } h=1.\end{cases} \tag{3.4}$$

It is well known that $\lim_{h\to 1}S(h)=1,\ S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0,\ h\neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In terms of the *Specht's ratio* we have the following reverse of Hölder's inequality [4]

$$(1\le)\,\frac{\left(\sum_{i=1}^{n}p_i a_i^p\right)^{1/p}\left(\sum_{i=1}^{n}p_i b_i^q\right)^{1/q}}{\sum_{i=1}^{n}p_i a_i b_i}\le S\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right). \tag{3.5}$$

We consider the *Kantorovich's constant* defined by

$$U(h):=\frac{(h+1)^2}{4h},\ h>0. \tag{3.6}$$

The function U is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $U(h)\ge 1$ for any $h>0$ and $U(h)=U\left(\frac{1}{h}\right)$ for any $h>0$.

In [4] we also obtained the reverse of Hölder's inequality

$$(1\le)\,\frac{\left(\sum_{i=1}^{n}p_i a_i^p\right)^{1/p}\left(\sum_{i=1}^{n}p_i b_i^q\right)^{1/q}}{\sum_{i=1}^{n}p_i a_i b_i}\le U^T\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right), \tag{3.7}$$

where $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

We assume that, in general, we have the following reverse of Hölder's inequality

$$(1\le)\,\frac{\left(\sum_{i=1}^{n}p_i a_i^p\right)^{1/p}\left(\sum_{i=1}^{n}p_i b_i^q\right)^{1/q}}{\sum_{i=1}^{n}p_i a_i b_i}\le B_{p,q}(m_1,M_1,m_2,M_2), \tag{3.8}$$

provided that $a_i,\ b_i,\ i\in\{1,...,n\}$ satisfy the condition (3.1), where $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Here $B_{p,q}(m_1,M_1,m_2,M_2)$ is a function depending on the parameters m_1,M_1,m_2,M_2 with the property (3.1) and $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

The following result holds:

Theorem 3. *Let $f \in L_w^p(\Omega,\mu)$, $g \in L_w^q(\Omega,\mu)$ with $p,\ q>1$ and $\frac{1}{p}+\frac{1}{q}=1$ and $\overline{w}=(w_1,....,w_n)\in \mathfrak{P}_n(w)$ be such that there exists $k,\ K,\ l,\ L>0$ with the property*

$$0<k\le\left(\int_\Omega w_i\left|f\right|^p d\mu\right)^{1/p}\le K<\infty \tag{3.9}$$

and

$$0<l\le\left(\int_\Omega w_i\left|g\right|^q d\mu\right)^{1/q}\le L<\infty \tag{3.10}$$

for each $i\in\{1,...,n\}$. Then

$$\frac{\left(\int_\Omega w\left|f\right|^p d\mu\right)^{1/p}\left(\int_\Omega w\left|g\right|^q d\mu\right)^{1/q}}{B_{p,q}(k,K,l,L)}\le\beta_{p,q}\left(\left|f\right|,\left|g\right|,\overline{w}\right). \tag{3.11}$$

Proof. Now, if we take $a_i=\left(\int_\Omega w_i\left|f\right|^p d\mu\right)^{1/p}$, $b_i=\left(\int_\Omega w_i\left|g\right|^q d\mu\right)^{1/q}$, $p_i=1$, $i\in\{1,...,n\}$, $m_1=k$, $M_1=K$, $m_2=l$ and $M_2=L$, then by (3.8) we get

$$\frac{\left(\sum_{i=1}^n\left(\left(\int_\Omega w_i\left|f\right|^p d\mu\right)^{1/p}\right)^p\right)^{1/p}\left(\sum_{i=1}^n\left(\left(\int_\Omega w_i\left|g\right|^q d\mu\right)^{1/q}\right)^q\right)^{1/q}}{\sum_{i=1}^n\left(\int_\Omega w_i\left|f\right|^p d\mu\right)^{1/p}\left(\int_\Omega w_i\left|g\right|^q d\mu\right)^{1/q}}$$
$$\le B_{p,q}(m_1,M_1,m_2,M_2),$$

namely

$$\frac{\left(\sum_{i=1}^n\int_\Omega w_i\left|f\right|^p d\mu\right)^{1/p}\left(\sum_{i=1}^n\int_\Omega w_i\left|g\right|^q d\mu\right)^{1/q}}{\sum_{i=1}^n\left(\int_\Omega w_i\left|f\right|^p d\mu\right)^{1/p}\left(\int_\Omega w_i\left|g\right|^q d\mu\right)^{1/q}}$$
$$\le B_{p,q}(m_1,M_1,m_2,M_2),$$

which is equivalent to

$$\frac{\left(\int_\Omega w\left|f\right|^p d\mu\right)^{1/p}\left(\int_\Omega w\left|g\right|^q d\mu\right)^{1/q}}{\sum_{i=1}^n\left(\int_\Omega w_i\left|f\right|^p d\mu\right)^{1/p}\left(\int_\Omega w_i\left|g\right|^q d\mu\right)^{1/q}}$$
$$\le B_{p,q}(m_1,M_1,m_2,M_2),$$

and the inequality (3.11) is obtained. □

Corollary 2. *With the assumptions of Theorem 3 we have the inequality (3.11) with*

$$B_{p,q}(k,K,l,L)=\max\left\{g_{\frac{1}{p}}\left(\left[\left(\frac{K}{k}\right)^p\left(\frac{L}{l}\right)^q\right]^{-1}\right),g_{\frac{1}{p}}\left(\left(\frac{K}{k}\right)^p\left(\frac{L}{l}\right)^q\right)\right\}$$

or

$$B_{p,q}(k,K,l,L)=S\left(\left(\frac{K}{k}\right)^p\left(\frac{L}{l}\right)^q\right)$$

or

$$B_{p,q}(k,K,l,L)=U^T\left(\left(\frac{K}{k}\right)^p\left(\frac{L}{l}\right)^q\right),$$

where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

4. Upper Bounds for $\beta_{p,q}\left(\left|f\right|,\left|g\right|,\overline{w}\right)$

Let f, g be μ-measurable functions with the property that there exists the constants m_1, M_1, m_2, M_2 such that

$$0<m_1\leq f\leq M_1<\infty,\ 0<m_2\leq g\leq M_2<\infty \ \mu\text{-a.e. on } \Omega. \tag{4.1}$$

Let p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have the following reverse of Hölder's integral inequality [4]

$$(1\leq)\ \frac{\left(\int_\Omega wf^p d\mu\right)^{1/p}\left(\int_\Omega wg^q d\mu\right)^{1/q}}{\int_\Omega wfgd\mu} \leq \max\left\{g_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right), g_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)\right\}, \tag{4.2}$$

where $g_{\frac{1}{p}}$ is defined by (3.3).

We also we have [4]

$$(1\leq)\ \frac{\left(\int_\Omega wf^p d\mu\right)^{1/p}\left(\int_\Omega wg^q d\mu\right)^{1/q}}{\int_\Omega wfgd\mu}\leq S\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right) \tag{4.3}$$

and

$$(1\leq)\ \frac{\left(\int_\Omega wf^p d\mu\right)^{1/p}\left(\int_\Omega wg^q d\mu\right)^{1/q}}{\int_\Omega wfgd\mu}\leq U^T\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right), \tag{4.4}$$

where p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

We assume that, in general, we have the following reverse of Hölder's inequality

$$(1\leq)\ \frac{\left(\int_\Omega wf^p d\mu\right)^{1/p}\left(\int_\Omega wg^q d\mu\right)^{1/q}}{\int_\Omega wfgd\mu}\leq M_{p,q}\left(m_1,M_1,m_2,M_2\right) \tag{4.5}$$

provided that f, g be μ-measurable functions with the property that there exists the constants m_1, M_1, m_2, M_2 such that (4.1) is valid.

Theorem 4. *Let $f\in L_w^p\left(\Omega,\mu\right)$, $g\in L_w^q\left(\Omega,\mu\right)$ with p, $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$ and $\overline{w}=\left(w_1,....,w_n\right)\in\mathfrak{P}_n\left(w\right)$ be such that there exists the constants m_1, M_1, m_2, M_2 such that (4.1) is valid. Then*

$$\beta_{p,q}\left(f,g,\overline{w}\right)\leq M_{p,q}\left(m_1,M_1,m_2,M_2\right)\int_\Omega wfgd\mu. \tag{4.6}$$

Proof. By the inequality (4.5) we get

$$\left(\int_\Omega w_i f^p d\mu\right)^{1/p}\left(\int_\Omega w_i g^q d\mu\right)^{1/q}\leq M_{p,q}\left(m_1,M_1,m_2,M_2\right)\int_\Omega w_i fgd\mu$$

for all $i\in\left\{1,...,n\right\}$.

If we sum over i from 1 to n, then we get

$$\sum_{i=1}^n\left(\int_\Omega w_i f^p d\mu\right)^{1/p}\left(\int_\Omega w_i g^q d\mu\right)^{1/q} \leq M_{p,q}\left(m_1,M_1,m_2,M_2\right)\sum_{i=1}^n\int_\Omega w_i fgd\mu = M_{p,q}\left(m_1,M_1,m_2,M_2\right)\int_\Omega wfgd\mu$$

and the inequality (4.6) is proved. □

Corollary 3. *With the assumptions of Theorem 4 we have the inequality (4.6) with*

$$M_{p,q}(m_1, M_1, m_2, M_2) = \max\left\{ g_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right), g_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right)\right\}$$

or

$$M_{p,q}(m_1, M_1, m_2, M_2) = S\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right)$$

or

$$M_{p,q}(m_1, M_1, m_2, M_2) = U^T\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right),$$

where $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, $T = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

5. Discrete Inequalities

When μ is the discrete measure on $\Omega = \{1, \ldots, n\}$, then the corresponding discrete inequalities for complex (real) numbers can be stated as well. We give here some examples.

Consider the sequences of complex numbers $\overline{x} = (x_1, ..., x_m)$, $\overline{y} = (y_1, ..., y_m) \in \mathbb{C}^m$, $\overline{w} = (w_1, ..., w_m) \in \mathbb{R}^n$, with $w_k > 0$, $k \in \{1, ..., m\}$. Let $w_{ki} > 0$ for $k \in \{1, ..., m\}$, $i \in \{1, ..., n\}$ with $m,\ n \geq 2$ and

$$\sum_{i=1}^{n} w_{ki} = w_k \text{ for any } k \in \{1, ..., m\}. \tag{5.1}$$

We consider the functional associated with the matrix $W := \{w_{ki}\}_{k\in\{1,...,m\}, i\in\{1,...,n\}}$ that satisfy (5.1),

$$\beta_{p,q}(|\overline{x}|, |\overline{y}|, W) := \sum_{i=1}^{n} \left(\sum_{k=1}^{m} w_{ki} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{m} w_{ki} |y_k|^q\right)^{1/q}, \tag{5.2}$$

where $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Using the inequality (2.1) we have the following refinement of the discrete Hölder inequality

$$\left|\sum_{k=1}^{m} w_k x_k y_k\right| \leq \sum_{i=1}^{n} \left(\sum_{k=1}^{m} w_{ki} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{m} w_{ki} |y_k|^q\right)^{1/q} \leq \left(\sum_{k=1}^{m} w_k |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{m} w_k |y_k|^q\right)^{1/q}, \tag{5.3}$$

where $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

If there exists $k,\ K,\ l,\ L > 0$ with the property that

$$0 < k \leq \left(\sum_{j=1}^{m} w_{ji} |x_j|^p\right)^{1/p} \leq K < \infty$$

and

$$0 < l \leq \left(\sum_{j=1}^{m} w_{ji} w_i |y_j|^q \, d\mu \right)^{1/q} \leq L < \infty$$

for each $i \in \{1, ..., n\}$, then

$$\frac{\left(\sum_{j=1}^{m} w_j |x_j|^p\right)^{1/p} \left(\sum_{j=1}^{m} w_j |y_j|^p\right)^{1/p}}{B_{p,q}(k, K, l, L)} \leq \beta_{p,q}(|\overline{x}|, |\overline{y}|, W), \tag{5.4}$$

where p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In inequality (5.4) we can choose

$$B_{p,q}(k, K, l, L) = \max\left\{ g_{\frac{1}{p}}\left(\left[\left(\frac{K}{k}\right)^p \left(\frac{L}{l}\right)^q \right]^{-1} \right), g_{\frac{1}{p}}\left(\left(\frac{K}{k}\right)^p \left(\frac{L}{l}\right)^q \right) \right\}$$

or

$$B_{p,q}(k, K, l, L) = S\left(\left(\frac{K}{k}\right)^p \left(\frac{L}{l}\right)^q \right)$$

or

$$B_{p,q}(k, K, l, L) = U^T\left(\left(\frac{K}{k}\right)^p \left(\frac{L}{l}\right)^q \right),$$

where p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $T = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

If there exists the constants m_1, M_1, m_2, M_2 such that

$$0 < m_1 \leq x_j \leq M_1 < \infty, \; 0 < m_2 \leq y_j \leq M_2 < \infty$$

for all $j \in \{1, ..., m\}$, then

$$\beta_{p,q}(\overline{x}, \overline{y}, W) \leq M_{p,q}(m_1, M_1, m_2, M_2) \sum_{j=1}^{m} w_j x_j y_j, \tag{5.5}$$

where p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In inequality (5.5) we can choose

$$\begin{aligned} &M_{p,q}(m_1, M_1, m_2, M_2) \\ &= \max\left\{ g_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q \right]^{-1} \right), g_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q \right) \right\} \end{aligned}$$

or

$$M_{p,q}(m_1, M_1, m_2, M_2) = S\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q \right)$$

or

$$M_{p,q}(m_1, M_1, m_2, M_2) = U^T\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q \right),$$

where p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $T = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

References

[1] S. Abramovich, Jensen, Hölder, Minkowski, Jensen-Steffensen and Slater-Pečarić inequalities derived through N-quasiconvexity. *Math. Inequal. Appl.* **19** (2016), no. 4, 1203–1226.

[2] A. A. El-Deeb, H. A. Elsennary and W.-S. Cheung, Some reverse Hölder inequalities with Specht's ratio on time scales. *J. Nonlinear Sci. Appl.* **11** (2018), no. 4, 444–455.

[3] S. S. Dragomir, Refining CBS integral inequality for partitions of weights, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 111, 15 pp. [http://rgmia.org/papers/v18/v18a111.pdf].

[4] S. S. Dragomir, Some multiplicative reverses of Callebaut and Hölder inequalities for isotonic functionals, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 180, 10 pp. [Online http://rgmia.org/papers/v18/v18a180.pdf].

[5] İ. İşcan, New Refinements for integral and sum forms of Hölder inequality, *Journal of Inequalities and Applications* (2019) **2019**:304. Preprint arXiv:1901.05841.

[6] E. G. Kwon and J. E. Bae, On a refined Hölder's inequality. *J. Math. Inequal.* **10** (2016), no. 1, 261–268.

[7] X.-F. Ma, New refinements of Hölder's inequality. *Aust. J. Math. Anal. Appl.* **14** (2017), no. 1, Art. 6, 8 pp

[8] I. Pinelis, On the Hölder and Cauchy-Schwarz inequalities. *Amer. Math. Monthly* **122** (2015), no. 6, 593–595.

[9] M. Raïssouli and R. Zine, On some Hölder-type inequalities with applications. *J. Inequal. Appl.* 2015, 2015:93, 13 pp.

[10] K. Razminia, Hölder's inequality revisited. *Math. Gaz.* **103** (2019), no. 558, 512–514.

[11] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.* **74** (1960), pp. 91-98.

[12] G. Taztepe, The sharpening Hölder inequality via abstract convexity. *Turkish J. Math.* **40** (2016), no. 2, 438–444.

[13] F. Yan and Q. Gao, Extensions and demonstrations of Hölder's inequality. *J. Inequal. Appl.* **2019**, Paper No. 97, 12 pp.

[14] C.-J. Zhao and W.- S. Cheung, Hölder's reverse inequality and its applications. *Publ. Inst. Math.* (Beograd) (N.S.) **99** (113) (2016), 211–216.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa