REFINING HÖLDER INTEGRAL INEQUALITY FOR DIVISIONS OF MEASURABLE SPACE

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Abstract. In this paper we establish a refinement and some reverses for Hölder inequality for the general Lebesgue integral on divisions of measurable space. Applications for discrete inequalities and weighted means of positive numbers are also given.

1. Introduction

The Hölder inequality plays an important role in different branches of Modern Mathematics including Hilbert Spaces Theory, Probability & Statistics, Classical Real and Complex Analysis, Numerical Analysis, Qualitative Theory of Differential Equations and their applications.

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a σ-algebra of subsets of Ω denoted by Σ and μ a countably additive and positive measure on Σ with values in $\mathbb{R} \cup \{\infty\}$. Denote by $L_w^r(\Omega, \mu)$, $r \geq 1$ the Banach space of all $\mathbb{C}$-valued functions f defined on Ω that are r-w-integrable on Ω, i.e., $\int_\Omega w(x) |f(x)|^r d\mu(x) < \infty$, where $w : \Omega \to [0, \infty)$ is a given μ-measurable function on Ω. We write for simplicity $\int_\Omega w |f|^r d\mu$ instead of $\int_\Omega w(x) |f(x)|^r d\mu(x)$.

The following inequality is well known in the literature as the *integral Hölder inequality*:

$$\left| \int_\Omega wfgd\mu \right| \leq \left(\int_\Omega w |f|^p d\mu \right)^{1/p} \left(\int_\Omega w |g|^q d\mu \right)^{1/q} \tag{1.1}$$

provided that $f \in L_w^p(\Omega, \mu)$, $g \in L_w^q(\Omega, \mu)$ with $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

For some recent results related to Hölder inequality, see [1]-[2] and [6]-[14].

We say that the family of measurable sets $F_n(\Omega) = \{\Omega_i\}_{i \in \{1,\ldots,n\}}$ is a *n-division* for Ω if $\Omega = \bigcup_{i=1}^n \Omega_i$, $\Omega_i \cap \Omega_j = \emptyset$ for any $i, j \in \{1, ..., n\}$ with $i \neq j$ and $\mu(\Omega_i) > 0$ for any $i \in \{1, ..., n\}$. In this situation, if $f \in L_w(\Omega, \mu)$ then $f \in L_w(\Omega_i, \mu)$ for any $i \in \{1, ..., n\}$ and $\int_\Omega fwd\mu = \sum_{i=1}^n \int_{\Omega_i} fwd\mu$. Also, $\int_\Omega wd\mu = \sum_{i=1}^n \int_{\Omega_i} wd\mu$ with $\int_{\Omega_i} wd\mu > 0$ for any $i \in \{1, ..., n\}$.

For a given $n \geq 2$ we denote by $\mathfrak{D}_n(\Omega)$ the set of all n-divisions of Ω and consider the functional $\gamma_{p,q}(|f|, |g|, \cdot) : \mathfrak{D}_n(\Omega) \to \mathbb{R}$ defined by

$$\gamma_{p,q}(|f|, |g|, F_n(\Omega)) := \sum_{i=1}^n \left(\int_{\Omega_i} w |f|^p d\mu \right)^{1/p} \left(\int_{\Omega_i} w |g|^q d\mu \right)^{1/q}, \tag{1.2}$$

where $f \in L_w^p(\Omega, \mu)$, $g \in L_w^q(\Omega, \mu)$ with $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 94A17.

Key words and phrases. Hölder-Inequality, Measurable spaces, Lebesgue integral, Weighted means.

RGMIA Res. Rep. Coll. 23 (2020), Art. 2, 8 pp. Received 12/01/20

In this paper we establish some inequalities concerning the functional $\gamma_{p,q}(|f|,|g|,\cdot)$ that provide refinements and reverses for the Hölder integral inequality (1.1). Applications for discrete inequalities and weighted means of positive numbers are also given.

2. The Main Results

We state the following refinements of Hölder inequality:

Theorem 1. *For $f \in L_w^p(\Omega,\mu)$, $g \in L_w^q(\Omega,\mu)$ with p, $q > 1$ and $\frac{1}{p}+\frac{1}{q}=1$ and $F_n(\Omega) \in \mathfrak{D}_n(\Omega)$ we have*

$$\left|\int_\Omega wfgd\mu\right| \le \gamma_{p,q}(|f|,|g|,F_n(\Omega)) \le \left(\int_\Omega w|f|^p d\mu\right)^{1/p}\left(\int_\Omega w|g|^q d\mu\right)^{1/q}. \tag{2.1}$$

Proof. Let $F_n(\Omega)=\{\Omega_i\}_{i\in\{1,\dots,n\}}$ be a n-division for Ω.

We have by Hölder integral inequality that

$$\begin{aligned}\left|\int_\Omega wfgd\mu\right| &\le \int_\Omega w|fg|\,d\mu = \sum_{i=1}^n \int_{\Omega_i} w|fg|\,d\mu \\ &\le \sum_{i=1}^n \left(\int_{\Omega_i} w|f|^p d\mu\right)^{1/p}\left(\int_{\Omega_i} w|g|^q d\mu\right)^{1/q} = \gamma_{p,q}(|f|,|g|,\overline{w}),\end{aligned}$$

which proves the first inequality in (2.1).

By the Hölder discrete inequality we also have

$$\begin{aligned}\gamma_{p,q}(|f|,|g|,\overline{w}) &= \sum_{i=1}^n \left(\int_{\Omega_i} w|f|^p d\mu\right)^{1/p}\left(\int_{\Omega_i} w|g|^q d\mu\right)^{1/q} \\ &\le \left[\sum_{i=1}^n\left(\left(\int_{\Omega_i} w|f|^p d\mu\right)^{1/p}\right)^p\right]^{1/p}\left[\sum_{i=1}^n\left(\left(\int_{\Omega_i} w|g|^q d\mu\right)^{1/q}\right)^q\right]^{1/q} \\ &= \left[\sum_{i=1}^n \int_{\Omega_i} w|f|^p d\mu\right]^{1/p}\left[\sum_{i=1}^n \int_{\Omega_i} w|g|^q d\mu\right]^{1/q} \\ &= \left[\int_\Omega w|f|^p d\mu\right]^{1/p}\left[\int_\Omega w|g|^q d\mu\right]^{1/q},\end{aligned}$$

which proves the second inequality in (2.1). □

Remark 1. *For $p=q=2$ we recapture the result from* [3] *related to the Cauchy-Bunyakovsky-Schwarz inequality.*

If $\Omega = [a,b] \subset \mathbb{R}$ and if we take $\Omega_1 = [a,y]$ and $\Omega_1 = [y,b]$, $y \in (a,b)$, then we get the refinement of Hölder's univariate integral inequality

$$\left|\int_a^b w(x) f(x) g(x)\,dx\right| \leq \left(\int_a^y w(x)|f(x)|^p dx\right)^{1/p} \left(\int_a^y w(x)|g(x)|^q dx\right)^{1/q} + \left(\int_y^b w(x)|f(x)|^p dx\right)^{1/p} \left(\int_y^b w(x)|g(x)|^q dx\right)^{1/q} \leq \left(\int_a^b w(x)|f(x)|^p dx\right)^{1/p} \left(\int_a^b w(x)|g(x)|^q dx\right)^{1/q}, \tag{2.2}$$

for all $y \in (a,b)$.

If there exists the constants $m_1,\ M_1,\ m_2,\ M_2$ such that

$$0 < m_1 \leq a_i \leq M_1 < \infty,\ 0 < m_2 \leq b_i \leq M_2 < \infty \text{ for any } i \in \{1,...,n\} \tag{2.3}$$

and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have the following reverse of Hölder's inequality [4]

$$(1 \leq)\, \frac{\left(\sum_{i=1}^n p_i a_i^p\right)^{1/p}\left(\sum_{i=1}^n p_i b_i^q\right)^{1/q}}{\sum_{i=1}^n p_i a_i b_i} \leq \max\left\{g_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right), g_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)\right\}, \tag{2.4}$$

where $g_{\frac{1}{p}}$ is defined by

$$g_{\frac{1}{p}}(x) = \frac{1}{q}x^{-\frac{1}{p}} + \frac{1}{p}x^{\frac{1}{q}},\ x>0. \tag{2.5}$$

We recall that *Specht's ratio* is defined by [11]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty), \\ 1 \text{ if } h=1. \end{cases} \tag{2.6}$$

It is well known that $\lim_{h\to1} S(h) = 1,\ S(h) = S\left(\frac{1}{h}\right) > 1$ for $h>0,\ h\neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In terms of the *Specht's ratio* we have the following reverse of Hölder's inequality [4]

$$(1 \leq)\, \frac{\left(\sum_{i=1}^n p_i a_i^p\right)^{1/p}\left(\sum_{i=1}^n p_i b_i^q\right)^{1/q}}{\sum_{i=1}^n p_i a_i b_i} \leq S\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right). \tag{2.7}$$

We consider the *Kantorovich's constant* defined by

$$U(h) := \frac{(h+1)^2}{4h},\ h>0. \tag{2.8}$$

The function U is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $U(h) \geq 1$ for any $h>0$ and $U(h) = U\left(\frac{1}{h}\right)$ for any $h>0$.

In [4] we also obtained the reverse of Hölder's inequality

$$(1\leq)\ \frac{\left(\sum_{i=1}^{n}p_ia_i^p\right)^{1/p}\left(\sum_{i=1}^{n}p_ib_i^q\right)^{1/q}}{\sum_{i=1}^{n}p_ia_ib_i}\leq U^T\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right), \tag{2.9}$$

where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

We assume that, in general, we have the following reverse of Hölder's inequality

$$(1\leq)\ \frac{\left(\sum_{i=1}^{n}p_ia_i^p\right)^{1/p}\left(\sum_{i=1}^{n}p_ib_i^q\right)^{1/q}}{\sum_{i=1}^{n}p_ia_ib_i}\leq B_{p,q}\left(m_1,M_1,m_2,M_2\right), \tag{2.10}$$

provided that $a_i,\ b_i,\ i\in\{1,...,n\}$ satisfy the condition (2.3), where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

The following result holds:

Theorem 2. *Let $f\in L_w^p(\Omega,\mu)$, $g\in L_w^q(\Omega,\mu)$ with $p,\ q>1$ and $\frac{1}{p}+\frac{1}{q}=1$ and $F_n(\Omega)\in\mathfrak{D}_n(\Omega)$ be such that there exists $k,\ K,\ l,\ L>0$ with the property*

$$0<k\leq\left(\int_{\Omega_i}w\left|f\right|^p d\mu\right)^{1/p}\leq K<\infty \tag{2.11}$$

and

$$0<l\leq\left(\int_{\Omega_i}w\left|g\right|^q d\mu\right)^{1/q}\leq L<\infty \tag{2.12}$$

for each $i\in\{1,...,n\}$. Then

$$\frac{\left(\int_{\Omega}w\left|f\right|^p d\mu\right)^{1/p}\left(\int_{\Omega}w\left|g\right|^q d\mu\right)^{1/q}}{B_{p,q}\left(k,K,l,L\right)}\leq\gamma_{p,q}\left(\left|f\right|,\left|g\right|,\overline{w}\right). \tag{2.13}$$

Proof. Now, if we take $a_i=\left(\int_{\Omega_i}w\left|f\right|^p d\mu\right)^{1/p}$, $b_i=\left(\int_{\Omega_i}w\left|g\right|^q d\mu\right)^{1/q}$, $p_i=1$, $i\in\{1,...,n\}$, $m_1=k$, $M_1=K$, $m_2=l$ and $M_2=L$, then by (2.10) we get

$$\frac{\left(\sum_{i=1}^{n}\left(\left(\int_{\Omega_i}w\left|f\right|^p d\mu\right)^{1/p}\right)^p\right)^{1/p}\left(\sum_{i=1}^{n}\left(\left(\int_{\Omega_i}w\left|g\right|^q d\mu\right)^{1/q}\right)^q\right)^{1/q}}{\sum_{i=1}^{n}\left(\int_{\Omega_i}w\left|f\right|^p d\mu\right)^{1/p}\left(\int_{\Omega_i}w\left|g\right|^q d\mu\right)^{1/q}}$$
$$\leq B_{p,q}\left(m_1,M_1,m_2,M_2\right),$$

namely

$$\frac{\left(\sum_{i=1}^{n}\int_{\Omega_i}w\left|f\right|^p d\mu\right)^{1/p}\left(\sum_{i=1}^{n}\int_{\Omega_i}w\left|g\right|^q d\mu\right)^{1/q}}{\sum_{i=1}^{n}\left(\int_{\Omega_i}w\left|f\right|^p d\mu\right)^{1/p}\left(\int_{\Omega_i}w\left|g\right|^q d\mu\right)^{1/q}}$$
$$\leq B_{p,q}\left(m_1,M_1,m_2,M_2\right),$$

which is equivalent to

$$\frac{\left(\int_{\Omega}w\left|f\right|^p d\mu\right)^{1/p}\left(\int_{\Omega}w\left|g\right|^q d\mu\right)^{1/q}}{\sum_{i=1}^{n}\left(\int_{\Omega_i}w\left|f\right|^p d\mu\right)^{1/p}\left(\int_{\Omega_i}w\left|g\right|^q d\mu\right)^{1/q}}$$
$$\leq B_{p,q}\left(m_1,M_1,m_2,M_2\right),$$

and the inequality (2.13) is obtained. $\square$

Corollary 1. *With the assumptions of Theorem 2 we have the inequality (2.13) with*

$$B_{p,q}(k,K,l,L)=\max\left\{g_{\frac{1}{p}}\left(\left[\left(\frac{K}{k}\right)^p\left(\frac{L}{l}\right)^q\right]^{-1}\right),g_{\frac{1}{p}}\left(\left(\frac{K}{k}\right)^p\left(\frac{L}{l}\right)^q\right)\right\}$$

or

$$B_{p,q}(k,K,l,L)=S\left(\left(\frac{K}{k}\right)^p\left(\frac{L}{l}\right)^q\right)$$

or

$$B_{p,q}(k,K,l,L)=U^T\left(\left(\frac{K}{k}\right)^p\left(\frac{L}{l}\right)^q\right),$$

where $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

Let f, g be μ-measurable functions with the property that there exists the constants m_1, M_1, m_2, M_2 such that

$$0<m_1\le f\le M_1<\infty,\ 0<m_2\le g\le M_2<\infty\ \mu\text{-a.e. on }\Omega. \tag{2.14}$$

Let $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have the following reverse of Hölder's integral inequality [4]

$$(1\le)\frac{\left(\int_\Omega wf^pd\mu\right)^{1/p}\left(\int_\Omega wg^qd\mu\right)^{1/q}}{\int_\Omega wfgd\mu} \le\max\left\{g_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right),g_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)\right\}, \tag{2.15}$$

where $g_{\frac{1}{p}}$ is defined by (2.5).

We also we have [4]

$$(1\le)\frac{\left(\int_\Omega wf^pd\mu\right)^{1/p}\left(\int_\Omega wg^qd\mu\right)^{1/q}}{\int_\Omega wfgd\mu}\le S\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right) \tag{2.16}$$

and

$$(1\le)\frac{\left(\int_\Omega wf^pd\mu\right)^{1/p}\left(\int_\Omega wg^qd\mu\right)^{1/q}}{\int_\Omega wfgd\mu}\le U^T\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right), \tag{2.17}$$

where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

We assume that, in general, we have the following reverse of Hölder's inequality

$$(1\le)\frac{\left(\int_\Omega wf^pd\mu\right)^{1/p}\left(\int_\Omega wg^qd\mu\right)^{1/q}}{\int_\Omega wfgd\mu}\le M_{p,q}(m_1,M_1,m_2,M_2) \tag{2.18}$$

provided that f, g be μ-measurable functions with the property that there exists the constants m_1, M_1, m_2, M_2 such that (2.14) is valid.

Theorem 3. *Let* $f\in L_w^p(\Omega,\mu)$, $g\in L_w^q(\Omega,\mu)$ *with* $p,\ q>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$ *and* $F_n(\Omega)\in\mathfrak{D}_n(\Omega)$ *be such that there exists the constants* m_1, M_1, m_2, M_2 *such that (2.14) is valid. Then*

$$\gamma_{p,q}(f,g,\overline{w})\le M_{p,q}(m_1,M_1,m_2,M_2)\int_\Omega wfgd\mu. \tag{2.19}$$

Proof. By the inequality (2.18) we get

$$\left(\int_{\Omega_i} wf^p d\mu\right)^{1/p}\left(\int_{\Omega_i} wg^q d\mu\right)^{1/q} \le M_{p,q}\left(m_1, M_1, m_2, M_2\right)\int_{\Omega_i} wfgd\mu$$

for all $i \in \{1, ..., n\}$.

If we sum over i from 1 to n, then we get

$$\sum_{i=1}^{n}\left(\int_{\Omega_i} wf^p d\mu\right)^{1/p}\left(\int_{\Omega_i} wg^q d\mu\right)^{1/q}$$
$$\le M_{p,q}\left(m_1, M_1, m_2, M_2\right)\sum_{i=1}^{n}\int_{\Omega_i} wfgd\mu = M_{p,q}\left(m_1, M_1, m_2, M_2\right)\int_{\Omega} wfgd\mu$$

and the inequality (2.19) is proved. □

Corollary 2. *With the assumptions of Theorem 3 we have the inequality (2.19) with*

$$M_{p,q}\left(m_1, M_1, m_2, M_2\right)$$
$$= \max\left\{g_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right), g_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)\right\}$$

or

$$M_{p,q}\left(m_1, M_1, m_2, M_2\right) = S\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)$$

or

$$M_{p,q}\left(m_1, M_1, m_2, M_2\right) = U^T\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right),$$

where $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, $T = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

3. Discrete Inequalities

For a nonempty finite family of indices J and positive weights $w_j,\ j \in J$ we denote $W_J := \sum_{j\in J} w_j$. Assume that, for $n \ge 2$, the family J of indices containing more than n elements and $F_n(J) = \{J_i\}_{i\in\{1,...,n\}}$ is a n-division for J, namely $J = \bigcup_{i=1}^{n} J_i$ and $J_i \cap J_j = \emptyset$ for any $i,\ j \in \{1, ..., n\}$ with $i \ne j$.

For a given $n \ge 2$ we denote by $\mathfrak{D}_n(J)$ the set of all n-divisions of J and consider the functional $\gamma_{p,q}(x, y, \cdot) : \mathfrak{D}_n(J) \to \mathbb{R}$ with $p,\ q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ defined by

$$\gamma_{p,q}\left(x, y, F_n(J)\right) := \sum_{i=1}^{n}\left(\sum_{j\in J_i} w_j\left|x_j\right|^p\right)^{1/p}\left(\sum_{j\in J_i} w_j\left|y_j\right|^q\right)^{1/q}, \tag{3.1}$$

where $x = \{x_j\}_{j\in J}$ and $y = \{y_j\}_{j\in J} \subset \mathbb{C}$.

From (2.1) we have

$$\left|\sum_{j\in J} w_j x_j y_j\right| \le \gamma_{p,q}\left(x, y, F_n(J)\right) \le \left(\sum_{j\in J} w_j\left|x_j\right|^p\right)^{1/p}\left(\sum_{j\in J} w_j\left|y_j\right|^q\right)^{1/q}. \tag{3.2}$$

If $F_n(J) = \{J_i\}_{i\in\{1,\dots,n\}}$ is a n-division for J and there exists k, K, l, L so that

$$0 < k \leq \left(\sum_{j\in J_i} w_j |x_j|^p\right)^{1/p} \leq K < \infty \tag{3.3}$$

and

$$0 < l \leq \left(\sum_{j\in J_i} w_j |y_j|^q\right)^{1/q} \leq L < \infty. \tag{3.4}$$

By (2.13) we have

$$\frac{\left(\sum_{j\in J} w_j |x_j|^p\right)^{1/p}\left(\sum_{j\in J} w_j |y_j|^q\right)^{1/q}}{B_{p,q}(k,K,l,L)} \leq \gamma_{p,q}(x,y,F_n(J)). \tag{3.5}$$

With the assumptions (3.3) and (3.4) we have the inequality (3.5) with

$$B_{p,q}(k,K,l,L) = \max\left\{g_{\frac{1}{p}}\left(\left[\left(\frac{K}{k}\right)^p\left(\frac{L}{l}\right)^q\right]^{-1}\right), g_{\frac{1}{p}}\left(\left(\frac{K}{k}\right)^p\left(\frac{L}{l}\right)^q\right)\right\}$$

or

$$B_{p,q}(k,K,l,L) = S\left(\left(\frac{K}{k}\right)^p\left(\frac{L}{l}\right)^q\right)$$

or

$$B_{p,q}(k,K,l,L) = U^T\left(\left(\frac{K}{k}\right)^p\left(\frac{L}{l}\right)^q\right),$$

where p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $T = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

Assume that there exists the constants m_1, M_1, m_2, M_2 such that

$$0 < m_1 \leq x_j \leq M_1 < \infty,\ 0 < m_2 \leq y_j \leq M_2 < \infty \tag{3.6}$$

for $j \in J$.

Then (2.19) we have

$$\gamma_{p,q}(x,y,F_n(J)) \leq M_{p,q}(m_1,M_1,m_2,M_2)\sum_{j\in J} w_j x_j y_j. \tag{3.7}$$

We observe that the inequality (3.7) also holds if

$$M_{p,q}(m_1,M_1,m_2,M_2) = \max\left\{g_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right), g_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)\right\}$$

or

$$M_{p,q}(m_1,M_1,m_2,M_2) = S\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)$$

or

$$M_{p,q}(m_1,M_1,m_2,M_2) = U^T\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right),$$

where p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $T = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$ and provided that the condition (3.6) is satisfied.

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