LOWER AND UPPER BOUNDS FOR THE JENSEN'S GAP OF CONVEX FUNCTIONS IN TERMS OF VARIANCE

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Abstract. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. In this paper we establish some lower and upper bounds for the Jensen's gap

$$\int_\Omega (\Phi \circ f)\, d\mu - \Phi\left(\int_\Omega f d\mu\right)$$

in terms of the variance

$$\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2,$$

for some classes of convex functions Φ. Applications for exponential, logarithm and power functions are also given.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. Consider the *Lebesgue space*

$$L(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)|\, d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t)\, d\mu(t)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, the author obtained in [6] and [9] the following result:

Theorem 1. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, $\mathring{I}$ the interior of I. If $f : \Omega \to [m, M]$ is so that $\Phi \circ f$, f, $\Phi' \circ f$, $(\Phi' \circ f) f \in L(\Omega, \mu)$, then we have the*

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inequality:

$$0 \le \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \le \int_\Omega (\Phi' \circ f) f d\mu - \int_\Omega \Phi' \circ f d\mu \int_\Omega f d\mu \tag{1.1}$$
$$\le \frac{1}{2}\left[\Phi'_-(M) - \Phi'_+(m)\right] \int_\Omega \left| f - \int_\Omega f d\mu \right| d\mu$$
$$\le \frac{1}{2}\left[\Phi'_-(M) - \Phi'_+(m)\right] \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]^{\frac{1}{2}}$$
$$\le \frac{1}{4}\left[\Phi'_-(M) - \Phi'_+(m)\right](M - m).$$

Remark 1. *We notice that the inequality between the first and the second term in (1.1) in the discrete case was proved in 1994 by Dragomir & Ionescu, see* [12].

For a real function $g : [m, M] \to \mathbb{R}$ and two distinct points α, $\beta \in [m, M]$ we recall that the *divided difference* of g in these points is defined by

$$[\alpha, \beta; g] := \frac{g(\beta) - g(\alpha)}{\beta - \alpha}.$$

Upper and lower bounds for the Jensen's gap were also obtained in [10]:

Theorem 2. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and m, $M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$. If $f : \Omega \to [m, M]$, is μ-measurable and such that f, $\Phi \circ f \in L(\Omega, \mu)$, then by assuming that $\int_\Omega f d\mu \neq m$, M, we have*

$$\left| \int_\Omega \left| \Phi(f) - \Phi\left(\int_\Omega f d\mu\right) \right| \operatorname{sgn}\left(f - \int_\Omega f d\mu\right) d\mu \right| \tag{1.2}$$
$$\le \int_\Omega (\Phi \circ f) d\mu - \Phi\left(\int_\Omega f d\mu\right)$$
$$\le \frac{1}{2}\left(\left[\int_\Omega f d\mu, M; \Phi\right] - \left[m, \int_\Omega f d\mu; \Phi\right]\right) \int_\Omega \left| f - \int_\Omega f d\mu \right| d\mu$$
$$\le \frac{1}{2}\left(\left[\int_\Omega f d\mu, M; \Phi\right] - \left[m, \int_\Omega f d\mu; \Phi\right]\right) \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]^{\frac{1}{2}}$$
$$\le \frac{1}{4}\left(\left[\int_\Omega f d\mu, M; \Phi\right] - \left[m, \int_\Omega f d\mu; \Phi\right]\right)(M - m).$$

The constant $\frac{1}{2}$ in the second inequality from (2.5) is best possible.

For other recent reverses of Jensen inequality and applications to divergence measures see [8], [9], [10] and the survey paper [11]. More related results may be found in [1]-[4], [7], [11] and [11]-[14].

Motivated by the above results, in this paper we establish some lower and upper bounds for the Jensen's gap,

$$\int_\Omega (\Phi \circ f) d\mu - \Phi\left(\int_\Omega f d\mu\right)$$

in terms of the variance

$$\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2,$$

for some classes of convex functions Φ. Applications for exponential, logarithm and power functions are also given.

2. Main Results

The following theorem is well known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Lemma 1. *Let $I\subset\mathbb{R}$ be a closed interval, $a\in I$ and let n be a positive integer. If $g:I\longrightarrow\mathbb{C}$ is such that the n-derivative $g^{(n)}$ is absolutely continuous on I, then for each $x\in I$*

$$g\left(x\right)=T_{n}\left(g;a,x\right)+R_{n}\left(g;a,x\right), \tag{2.1}$$

where $T_{n}\left(g;c,y\right)$ is Taylor's polynomial, i.e.,

$$T_{n}\left(g;a,x\right):=\sum_{k=0}^{n}\frac{\left(x-a\right)^{k}}{k!}g^{(k)}\left(a\right). \tag{2.2}$$

Note that $g^{(0)}:=g$ and $0!:=1$ and the remainder is given by

$$R_{n}\left(g;a,x\right):=\frac{1}{n!}\int_{a}^{x}\left(x-t\right)^{n}g^{(n+1)}\left(t\right)dt. \tag{2.3}$$

A simple proof of this theorem can be achieved by mathematical induction using the integration by parts formula in the Lebesgue integral.

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $t=\left(1-s\right)c+sd$, $s\in[0,1]$ that

$$\int_{c}^{d}h\left(t\right)dt=\left(d-c\right)\int_{0}^{1}h\left(\left(1-s\right)c+sd\right)ds.$$

Therefore,

$$\begin{aligned}&\int_{a}^{x}g^{(n+1)}\left(t\right)\left(x-t\right)^{n}dt\\&=\left(x-a\right)\int_{0}^{1}g^{(n+1)}\left(\left(1-s\right)a+sx\right)\left(x-\left(1-s\right)a-sx\right)^{n}ds\\&=\left(x-a\right)^{n+1}\int_{0}^{1}g^{(n+1)}\left(\left(1-s\right)a+sx\right)\left(1-s\right)^{n}ds.\end{aligned}$$

The identity (2.1) can then be written as

$$\begin{aligned}g\left(x\right)&=\sum_{k=0}^{n}\frac{1}{k!}g^{(k)}\left(a\right)\left(x-a\right)^{k}\\&\quad+\frac{1}{n!}\left(x-a\right)^{n+1}\int_{0}^{1}g^{(n+1)}\left(\left(1-s\right)a+sx\right)\left(1-s\right)^{n}ds\end{aligned} \tag{2.4}$$

for all x, $a\in I$.

We have the following result concerning lower and upper bounds for the Jensen's gap:

Theorem 3. *Let $\Phi:[m,M]\subset\mathbb{R}\rightarrow\mathbb{R}$ be a twice differentiable convex function on (m,M) and $f:\Omega\rightarrow[m,M]$ so that f, f^{2}, $\Phi\circ f$, $\Phi''\circ f\in L\left(\Omega,\mu\right)$. Then we have the inequality:*

$$
\begin{aligned}
(2.5)\qquad 0 &\leq \operatorname*{essinf}_{t\in\Omega}\left(\int_0^1 \Phi''\left((1-s)\int_\Omega fd\mu + sf(t)\right)(1-s)\,ds\right) \\
&\times\left[\int_\Omega f^2 d\mu - \left(\int_\Omega fd\mu\right)^2\right] \\
&\leq \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega fd\mu\right) \\
&\leq \operatorname*{essup}_{t\in\Omega}\left(\int_0^1 \Phi''\left((1-s)\int_\Omega fd\mu + sf(t)\right)(1-s)\,ds\right) \\
&\times\left[\int_\Omega f^2 d\mu - \left(\int_\Omega fd\mu\right)^2\right],
\end{aligned}
$$

provided that $\operatorname{essup}_{t\in\Omega}(\cdot)$ *is finite.*

Proof. We have from (2.4) for $n=2$ that

$$
\Phi(x) = \Phi(c) + \Phi'(c)(x-c) + (x-c)^2\int_0^1 \Phi''((1-s)c+sx)(1-s)\,ds
$$

for all $x,\ c\in[m,M]$, where Φ is such that Φ' is absolutely continuos on $[m,M]$.

This implies that

$$
\begin{aligned}
(2.6)\qquad \Phi(f(t)) &= \Phi\left(\int_\Omega fd\mu\right) + \Phi'\left(\int_\Omega fd\mu\right)\left(f(t)-\int_\Omega fd\mu\right) \\
&+\left(f(t)-\int_\Omega fd\mu\right)^2\int_0^1 \Phi''\left((1-s)\int_\Omega fd\mu + sf(t)\right)(1-s)\,ds
\end{aligned}
$$

for all $t\in\Omega$.

If we take the integral in (2.6), then we get

$$
\begin{aligned}
(2.7)\qquad &\int_\Omega \Phi(f(t))\,d\mu(t) \\
&= \Phi\left(\int_\Omega fd\mu\right)\int_\Omega d\mu(t) + \Phi'\left(\int_\Omega fd\mu\right)\int_\Omega\left(f(t)-\int_\Omega fd\mu\right)d\mu(t) \\
&+ \int_\Omega\left(f(t)-\int_\Omega fd\mu\right)^2 \\
&\times\left(\int_0^1 \Phi''\left((1-s)\int_\Omega fd\mu + sf(t)\right)(1-s)\,ds\right)d\mu(t) \\
&= \Phi\left(\int_\Omega fd\mu\right) + \int_\Omega\left(f(t)-\int_\Omega fd\mu\right)^2 \\
&\times\left(\int_0^1 \Phi''\left((1-s)\int_\Omega fd\mu + sf(t)\right)(1-s)\,ds\right)d\mu(t),
\end{aligned}
$$

since

$$
\int_\Omega\left(f(t)-\int_\Omega fd\mu\right)d\mu(t) = \int_\Omega f(t)\,d\mu(t) - \int_\Omega fd\mu\int_\Omega d\mu(t) = 0.
$$

Therefore we have the equality of interest

$$\begin{aligned}(2.8)\qquad & \int_\Omega \Phi\left(f\left(t\right)\right)d\mu\left(t\right)-\Phi\left(\int_\Omega fd\mu\right) \\ & =\int_\Omega \left(f\left(t\right)-\int_\Omega fd\mu\right)^2 \\ & \times\left(\int_0^1 \Phi''\left(\left(1-s\right)\int_\Omega fd\mu+sf\left(t\right)\right)\left(1-s\right)ds\right)d\mu\left(t\right).\end{aligned}$$

Now, observe that

$$\begin{aligned}(2.9)\qquad & \operatorname*{essinf}_{t\in\Omega}\left(\int_0^1 \Phi''\left(\left(1-s\right)\int_\Omega fd\mu+sf\left(t\right)\right)\left(1-s\right)ds\right) \\ & \times\int_\Omega \left(f\left(t\right)-\int_\Omega fd\mu\right)^2 d\mu\left(t\right) \\ & \leq\int_\Omega \left(f\left(t\right)-\int_\Omega fd\mu\right)^2 \\ & \times\left(\int_0^1 \Phi''\left(\left(1-s\right)\int_\Omega fd\mu+sf\left(t\right)\right)\left(1-s\right)ds\right)d\mu\left(t\right) \\ & \leq\operatorname*{essup}_{t\in\Omega}\left(\int_0^1 \Phi''\left(\left(1-s\right)\int_\Omega fd\mu+sf\left(t\right)\right)\left(1-s\right)ds\right) \\ & \times\int_\Omega \left(f\left(t\right)-\int_\Omega fd\mu\right)^2 d\mu\left(t\right)\end{aligned}$$

and since

$$\begin{aligned} & \int_\Omega \left(f\left(t\right)-\int_\Omega fd\mu\right)^2 d\mu\left(t\right) \\ & =\int_\Omega \left[f^2\left(t\right)-2f\left(t\right)\int_\Omega fd\mu+\left(\int_\Omega fd\mu\right)^2\right]d\mu\left(t\right) \\ & =\int_\Omega f^2\left(t\right)d\mu\left(t\right)-2\int_\Omega f\left(t\right)d\mu\left(t\right)\int_\Omega fd\mu+\left(\int_\Omega fd\mu\right)^2\int_\Omega d\mu\left(t\right) \\ & =\int_\Omega f^2 d\mu-2\left(\int_\Omega fd\mu\right)^2+\left(\int_\Omega fd\mu\right)^2=\int_\Omega f^2 d\mu-\left(\int_\Omega fd\mu\right)^2,\end{aligned}$$

hence by (2.9) we get the desired result (2.5). □

Corollary 1. *With the assumptions of Theorem 3 and if there exist* $0<\gamma<\Gamma<\infty$ *such that*

$$(2.10)\qquad \gamma\leq\Phi''\left(t\right)\leq\Gamma \textit{ for almost every } t\in\left[m,M\right]$$

then

$$\begin{aligned}(2.11)\qquad 0 & \leq\frac{1}{2}\gamma\left[\int_\Omega f^2 d\mu-\left(\int_\Omega fd\mu\right)^2\right]\leq\int_\Omega \Phi\circ fd\mu-\Phi\left(\int_\Omega fd\mu\right) \\ & \leq\frac{1}{2}\Gamma\left[\int_\Omega f^2 d\mu-\left(\int_\Omega fd\mu\right)^2\right].\end{aligned}$$

Proof. Observe that by (2.10) we have

$$\frac{1}{2}\gamma=\gamma\int_0^1(1-s)\,ds\le\int_0^1\Phi''\left((1-s)\int_\Omega fd\mu+sf(t)\right)(1-s)\,ds$$
$$\le\Gamma\int_0^1(1-s)\,ds=\frac{1}{2}\Gamma$$

and by (2.5) we get (2.11). □

Corollary 2. *With the assumptions of Theorem 3 and if $\Phi''(\cdot)$ is monotonic nondecreasing on (m,M), then*

$$0\le\frac{1}{\int_\Omega fd\mu-m}\left\{\Phi'\left(\int_\Omega fd\mu\right)-\frac{\Phi\left(\int_\Omega fd\mu\right)-\Phi(m)}{\int_\Omega fd\mu-m}\right\}\times\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]$$
$$\le\int_\Omega\Phi\circ fd\mu-\Phi\left(\int_\Omega fd\mu\right)$$
$$\le\frac{1}{M-\int_\Omega fd\mu}\left[\frac{\Phi(M)-\Phi\left(\int_\Omega fd\mu\right)}{M-\int_\Omega fd\mu}-\Phi'\left(\int_\Omega fd\mu\right)\right]\times\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]. \tag{2.12}$$

If $\Phi''(\cdot)$ is monotonic nonincreasing on (m,M), then

$$0\le\frac{1}{M-\int_\Omega fd\mu}\left[\frac{\Phi(M)-\Phi\left(\int_\Omega fd\mu\right)}{M-\int_\Omega fd\mu}-\Phi'\left(\int_\Omega fd\mu\right)\right]\times\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]$$
$$\le\int_\Omega\Phi\circ fd\mu-\Phi\left(\int_\Omega fd\mu\right)$$
$$\le\frac{1}{\int_\Omega fd\mu-m}\left\{\Phi'\left(\int_\Omega fd\mu\right)-\frac{\Phi\left(\int_\Omega fd\mu\right)-\Phi(m)}{\int_\Omega fd\mu-m}\right\}\times\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]. \tag{2.13}$$

Proof. Since $\Phi''(\cdot)$ is monotonic nondecreasing on (m,M), then

$$\Phi''\left((1-s)\int_\Omega fd\mu+sm\right)\le\Phi''\left((1-s)\int_\Omega fd\mu+sf(t)\right)$$
$$\le\Phi''\left((1-s)\int_\Omega fd\mu+sM\right)$$

for all $s\in(0,1)$ and μ-a.e. $t\in\Omega$.

This implies that

$$\begin{aligned}\int_0^1 \Phi''\left((1-s)\int_\Omega f d\mu + sm\right)(1-s)\,ds \\ \leq \int_0^1 \Phi''\left((1-s)\int_\Omega f d\mu + sf(t)\right)(1-s)\,ds \\ \leq \int_0^1 \Phi''\left((1-s)\int_\Omega f d\mu + sM\right)(1-s)\,ds\end{aligned} \tag{2.14}$$

for μ-a.e. $t\in\Omega$.

Assume that $\int_\Omega f d\mu \in (m,M)$. Observe that, integrating by parts, we have

$$\begin{aligned}&\int_0^1 \Phi''\left((1-s)\int_\Omega f d\mu + sm\right)(1-s)\,ds \\ &= \frac{1}{m-\int_\Omega f d\mu}\int_0^1 (1-s)\,d\left(\Phi'\left((1-s)\int_\Omega f d\mu + sm\right)\right) \\ &= \frac{1}{m-\int_\Omega f d\mu}\left\{(1-s)\,\Phi'\left((1-s)\int_\Omega f d\mu + sm\right)\Big|_0^1\right. \\ &\left.+\int_0^1 \Phi'\left((1-s)\int_\Omega f d\mu + sm\right)ds\right\} \\ &= \frac{1}{m-\int_\Omega f d\mu}\left\{-\Phi'\left(\int_\Omega f d\mu\right)+\int_0^1 \Phi'\left((1-s)\int_\Omega f d\mu + sm\right)ds\right\} \\ &= \frac{1}{\int_\Omega f d\mu - m}\left\{\Phi'\left(\int_\Omega f d\mu\right)-\int_0^1 \Phi'\left((1-s)\int_\Omega f d\mu + sm\right)ds\right\} \\ &= \frac{1}{\int_\Omega f d\mu - m}\left\{\Phi'\left(\int_\Omega f d\mu\right)-\frac{\Phi\left(\int_\Omega f d\mu\right)-\Phi(m)}{\int_\Omega f d\mu - m}\right\}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\int_0^1 \Phi''\left((1-s)\int_\Omega f d\mu + sM\right)(1-s)\,ds\ \} \\ &= \frac{1}{M-\int_\Omega f d\mu}\int_0^1 (1-s)\,d\left(\Phi'\left((1-s)\int_\Omega f d\mu + sM\right)\right) \\ &= \frac{1}{M-\int_\Omega f d\mu}\left[\Phi'\left((1-s)\int_\Omega f d\mu + sM\right)(1-s)\Big|_0^1\right. \\ &\left.+\int_0^1 \Phi'\left((1-s)\int_\Omega f d\mu + sM\right)ds\right] \\ &= \frac{1}{M-\int_\Omega f d\mu}\left[\int_0^1 \Phi'\left((1-s)\int_\Omega f d\mu + sM\right)ds - \Phi'\left(\int_\Omega f d\mu\right)\right] \\ &= \frac{1}{M-\int_\Omega f d\mu}\left[\frac{\Phi(M)-\Phi\left(\int_\Omega f d\mu\right)}{M-\int_\Omega f d\mu}-\Phi'\left(\int_\Omega f d\mu\right)\right].\end{aligned}$$

By utilising (2.14) we then get

$$\frac{1}{\int_\Omega f d\mu - m}\left\{\Phi'\left(\int_\Omega f d\mu\right) - \frac{\Phi\left(\int_\Omega f d\mu\right) - \Phi(m)}{\int_\Omega f d\mu - m}\right\} \le \operatorname*{essinf}_{t\in\Omega}\left(\int_0^1 \Phi''\left((1-s)\int_\Omega f d\mu + s f(t)\right)(1-s)\,ds\right) \tag{2.15}$$

and

$$\operatorname*{essup}_{t\in\Omega}\left(\int_0^1 \Phi''\left((1-s)\int_\Omega f d\mu + s f(t)\right)(1-s)\,ds\right) \le \frac{1}{M - \int_\Omega f d\mu}\left[\frac{\Phi(M) - \Phi\left(\int_\Omega f d\mu\right)}{M - \int_\Omega f d\mu} - \Phi'\left(\int_\Omega f d\mu\right)\right]. \tag{2.16}$$

If we use (2.15), (2.16) and Theorem 3, then we get (2.12).

The case of $\Phi''(\cdot)$ is monotonic nonincreasing on (m, M) follows in a similar way. □

Corollary 3. *With the assumptions of Theorem 3 and if $\Phi''(\cdot)$ is convex on (m, M), then*

$$\begin{aligned} 0 &\le \frac{1}{2}\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right] \operatorname*{essinf}_{t\in\Omega}\left[\Phi''\left(\frac{2\int_\Omega f d\mu + f(t)}{3}\right)\right] \\ &\le \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\ &\le \frac{1}{2}\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]\left[\frac{2\Phi''\left(\int_\Omega f d\mu\right) + \operatorname{essup}_{t\in\Omega}\Phi''(f(t))}{3}\right]. \end{aligned} \tag{2.17}$$

If $\Phi''(\cdot)$ is concave on (m, M), then

$$\begin{aligned} 0 &\le \frac{1}{2}\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]\left[\frac{2\Phi''\left(\int_\Omega f d\mu\right) + \operatorname{essinf}_{t\in\Omega}\Phi''(f(t))}{3}\right] \\ &\le \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\ &\le \frac{1}{2}\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right] \operatorname*{essup}_{t\in\Omega}\left[\Phi''\left(\frac{2\int_\Omega f d\mu + f(t)}{3}\right)\right]. \end{aligned} \tag{2.18}$$

Proof. By the convexity of Φ'' and Jensen's integral inequality, we have

$$\begin{aligned}
&\int_0^1 \Phi''\left((1-s)\int_\Omega f d\mu + sf(t)\right)(1-s)\,ds \\
&\geq \left(\int_0^1 (1-s)\,ds\right)\Phi''\left(\frac{\int_0^1 (1-s)\left[(1-s)\int_\Omega f d\mu + sf(t)\right]ds}{\int_0^1 (1-s)\,ds}\right) \\
&= \frac{1}{2}\Phi''\left(\frac{\int_0^1 (1-s)\left[(1-s)\int_\Omega f d\mu + sf(t)\right]ds}{\frac{1}{2}}\right) \\
&= \frac{1}{2}\Phi''\left(\frac{\int_\Omega f d\mu \int_0^1 (1-s)^2\,ds + f(t)\int_0^1 s(1-s)\,ds}{\frac{1}{2}}\right) \\
&= \frac{1}{2}\Phi''\left(\frac{\frac{1}{3}\int_\Omega f d\mu + \frac{1}{6}f(t)}{\frac{1}{2}}\right) = \frac{1}{2}\Phi''\left(\frac{2}{3}\int_\Omega f d\mu + \frac{1}{3}f(t)\right)
\end{aligned}$$

for μ-a.e. $t \in \Omega$. $t \in \Omega$.

This implies that

$$\begin{aligned}
&\operatorname*{essinf}_{t\in\Omega}\left(\int_0^1 \Phi''\left((1-s)\int_\Omega f d\mu + sf(t)\right)(1-s)\,ds\right) \\
&\geq \frac{1}{2}\operatorname*{essinf}_{t\in\Omega}\left[\Phi''\left(\frac{2}{3}\int_\Omega f d\mu + \frac{1}{3}f(t)\right)\right],
\end{aligned}$$

which proves the first inequality in (2.17).

By the convexity of Φ'' we also have

$$\begin{aligned}
&\int_0^1 \Phi''\left((1-s)\int_\Omega f d\mu + sf(t)\right)(1-s)\,ds \\
&\leq \int_0^1 \left[(1-s)\Phi''\left(\int_\Omega f d\mu\right) + s\Phi''(f(t))\right](1-s)\,ds \\
&= \Phi''\left(\int_\Omega f d\mu\right)\int_0^1 (1-s)^2\,ds + \Phi''(f(t))\int_0^1 s(1-s)\,ds \\
&= \frac{1}{3}\Phi''\left(\int_\Omega f d\mu\right) + \frac{1}{6}\Phi''(f(t)) = \frac{1}{6}\left[2\Phi''\left(\int_\Omega f d\mu\right) + \Phi''(f(t))\right],
\end{aligned}$$

for μ-a.e. $t \in \Omega$.

This implies that

$$\begin{aligned}
&\operatorname*{essup}_{t\in\Omega}\left(\int_0^1 \Phi''\left((1-s)\int_\Omega f d\mu + sf(t)\right)(1-s)\,ds\right) \\
&\leq \operatorname*{essup}_{t\in\Omega}\frac{1}{6}\left[2\Phi''\left(\int_\Omega f d\mu\right) + \Phi''(f(t))\right] \\
&= \frac{1}{6}\left[2\Phi''\left(\int_\Omega f d\mu\right) + \operatorname*{essup}_{t\in\Omega}\Phi''(f(t))\right]
\end{aligned}$$

and by Theorem 3, we get the second part of (2.17).

In the case when $\Phi''(\cdot)$ is concave on (m, M), the proof goes in a similar way and we omit the details. □

We recall that a function $f : I \to \mathbb{R}$ is called *quasiconvex* on the interval I if

$$f((1-s)x+sy) \le \max\{f(x), f(y)\} \tag{2.19}$$

for all $x, y \in I$ and $s \in [0,1]$.

Corollary 4. *With the assumptions of Theorem 3 and if $\Phi''(\cdot)$ is quasiconvex on (m, M), then*

$$\begin{aligned} &\int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\ &\le \frac{1}{2}\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right] \max\left\{\Phi''\left(\int_\Omega f d\mu\right), \operatorname*{essup}_{t\in\Omega}\left[\Phi''(f(t))\right]\right\}. \end{aligned} \tag{2.20}$$

Proof. Since $\Phi''(\cdot)$ is quasiconvex, hence

$$\begin{aligned} &\int_0^1 \Phi''\left((1-s)\int_\Omega f d\mu + sf(t)\right)(1-s)\, ds \\ &\le \max\left\{\Phi''\left(\int_\Omega f d\mu\right), \Phi''(f(t))\right\}\int_0^1 (1-s)\, ds \\ &= \frac{1}{2}\max\left\{\Phi''\left(\int_\Omega f d\mu\right), \Phi''(f(t))\right\} \end{aligned}$$

for μ-a.e. $t \in \Omega$.

Taking the $\operatorname{essup}_{t\in\Omega}$ in this inequality, we get

$$\begin{aligned} &\operatorname*{essup}_{t\in\Omega}\int_0^1 \Phi''\left((1-s)\int_\Omega f d\mu + sf(t)\right)(1-s)\, ds \\ &\le \frac{1}{2}\operatorname*{essup}_{t\in\Omega}\left(\max\left\{\Phi''\left(\int_\Omega f d\mu\right), \Phi''(f(t))\right\}\right) \\ &= \frac{1}{2}\max\left\{\Phi''\left(\int_\Omega f d\mu\right), \operatorname*{essup}_{t\in\Omega}\Phi''(f(t))\right\}. \end{aligned}$$

By (2.5) we then obtain the desired result (2.20). □

3. Exponential Integral Inequalities

We consider the convex function $\Phi_\alpha : \mathbb{R} \to (0,\infty)$ defined by $\Phi_\alpha(t) = \exp(\alpha x)$. We have $\Phi''_\alpha(t) = \alpha^2 \exp(\alpha x)$ which shows that Φ''_α is convex and monotonic decreasing for $\alpha < 0$ and increasing for $\alpha > 0$.

We then have for $t \in [m, M]$ that

$$\begin{aligned} &E_1(\alpha, [m, M]) \\ &:= \alpha^2 \begin{cases} \exp(\alpha M), & \alpha < 0 \\ \exp(\alpha m), & \alpha > 0 \end{cases} \le \Phi''_\alpha(t) \le \alpha^2 \begin{cases} \exp(\alpha m), & \alpha < 0 \\ \exp(\alpha M), & \alpha > 0 \end{cases} \\ &:= E_2(\alpha, [m, M]). \end{aligned}$$

If $f:\Omega\rightarrow[m,M]$ is so that $f,\ f^{2},\ \exp\circ f\in L(\Omega,\mu)$, then by (2.11) we get

$$0\leq\frac{1}{2}E_{1}(\alpha,[m,M])\left[\int_{\Omega}f^{2}d\mu-\left(\int_{\Omega}fd\mu\right)^{2}\right]\leq\int_{\Omega}\exp(\alpha f)\,d\mu-\exp\left(\alpha\int_{\Omega}fd\mu\right)\leq\frac{1}{2}E_{2}(\alpha,[m,M])\left[\int_{\Omega}f^{2}d\mu-\left(\int_{\Omega}fd\mu\right)^{2}\right]. \tag{3.1}$$

If $\alpha>0$, then Φ_{α}'' is increasing and by (2.12) we get

$$\begin{aligned}0&\leq\frac{1}{\int_{\Omega}fd\mu-m}\left\{\alpha\exp\left(\alpha\int_{\Omega}fd\mu\right)-\frac{\exp\left(\alpha\int_{\Omega}fd\mu\right)-\exp(\alpha m)}{\int_{\Omega}fd\mu-m}\right\}\\&\times\left[\int_{\Omega}f^{2}d\mu-\left(\int_{\Omega}fd\mu\right)^{2}\right]\\&\leq\int_{\Omega}\exp(\alpha f)\,d\mu-\exp\left(\alpha\int_{\Omega}fd\mu\right)\\&\leq\frac{1}{M-\int_{\Omega}fd\mu}\left[\frac{\exp(\alpha M)-\exp\left(\alpha\int_{\Omega}fd\mu\right)}{M-\int_{\Omega}fd\mu}-\alpha\exp\left(\alpha\int_{\Omega}fd\mu\right)\right]\\&\times\left[\int_{\Omega}f^{2}d\mu-\left(\int_{\Omega}fd\mu\right)^{2}\right].\end{aligned} \tag{3.2}$$

If $\alpha<0$, then Φ_{α}'' is decreasing and by (2.13) we get

$$\begin{aligned}0&\leq\frac{1}{M-\int_{\Omega}fd\mu}\left[\frac{\exp(\alpha M)-\exp\left(\alpha\int_{\Omega}fd\mu\right)}{M-\int_{\Omega}fd\mu}-\alpha\exp\left(\alpha\int_{\Omega}fd\mu\right)\right]\\&\times\left[\int_{\Omega}f^{2}d\mu-\left(\int_{\Omega}fd\mu\right)^{2}\right]\end{aligned} \tag{3.3}$$

$$\begin{aligned}&\leq\int_{\Omega}\exp(\alpha f)\,d\mu-\exp\left(\alpha\int_{\Omega}fd\mu\right)\\&\leq\frac{1}{\int_{\Omega}fd\mu-m}\left\{\alpha\exp\left(\alpha\int_{\Omega}fd\mu\right)-\frac{\exp\left(\alpha\int_{\Omega}fd\mu\right)-\exp(\alpha m)}{\int_{\Omega}fd\mu-m}\right\}\\&\times\left[\int_{\Omega}f^{2}d\mu-\left(\int_{\Omega}fd\mu\right)^{2}\right].\end{aligned}$$

Since the function Φ_{α}'' is convex, then by (2.17) we have for all $\alpha \neq 0$ that

$$0 \leq \frac{1}{2}\alpha^2 \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right] \operatorname*{essinf}_{t\in\Omega}\left[\exp\left(\alpha \frac{2\int_\Omega f d\mu + f(t)}{3}\right)\right] \tag{3.4}$$
$$\leq \int_\Omega \exp(\alpha f)\, d\mu - \exp\left(\alpha \int_\Omega f d\mu\right)$$
$$\leq \frac{1}{2}\alpha^2 \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]$$
$$\times \left[\frac{2\exp\left(\alpha \int_\Omega f d\mu\right) + \operatorname{essup}_{t\in\Omega} \exp(\alpha f(t))}{3}\right].$$

If $\alpha > 0$, then by (3.4) we get

$$0 \leq \frac{1}{2}\alpha^2 \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right] \exp\left(\alpha \frac{2\int_\Omega f d\mu + m}{3}\right) \tag{3.5}$$
$$\leq \int_\Omega \exp(\alpha f)\, d\mu - \exp\left(\alpha \int_\Omega f d\mu\right)$$
$$\leq \frac{1}{2}\alpha^2 \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right] \left[\frac{2\exp\left(\alpha \int_\Omega f d\mu\right) + \exp(\alpha M)}{3}\right],$$

while for $\alpha < 0$ we get

$$0 \leq \frac{1}{2}\alpha^2 \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right] \exp\left(\alpha \frac{2\int_\Omega f d\mu + M}{3}\right) \tag{3.6}$$
$$\leq \int_\Omega \exp(\alpha f)\, d\mu - \exp\left(\alpha \int_\Omega f d\mu\right)$$
$$\leq \frac{1}{2}\alpha^2 \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right] \left[\frac{2\exp\left(\alpha \int_\Omega f d\mu\right) + \exp(\alpha m)}{3}\right].$$

4. Logarithmic Integral Inequalities

We consider the convex function $\Phi : (0, \infty) \to \mathbb{R}$, $\Phi(x) = -\ln x$. Then $\Phi''(x) = \frac{1}{x^2}$, which is convex and decreasing on $(0, \infty)$. If $x \in [m, M] \subset (0, \infty)$, then we also have

$$\frac{1}{M^2} \leq \Phi''(x) \leq \frac{1}{m^2}.$$

If $f : \Omega \to [m, M]$ is so that f, f^2, $\ln \circ f \in L(\Omega, \mu)$, then by (2.11) we get

$$0 \leq \frac{1}{2M^2}\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right] \leq \ln\left(\int_\Omega f d\mu\right) - \int_\Omega \ln(f)\, d\mu \tag{4.1}$$
$$\leq \frac{1}{2m^2}\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right].$$

Since Φ'' is decreasing, hence by (2.13) we get

$$0 \le \frac{1}{M - \int_\Omega f d\mu} \left[\left(\int_\Omega f d\mu \right)^{-1} - \frac{\ln(M) - \ln\left(\int_\Omega f d\mu\right)}{M - \int_\Omega f d\mu} \right] \times \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu \right)^2 \right] \le \ln\left(\int_\Omega f d\mu \right) - \int_\Omega \ln(f) \, d\mu \le \frac{1}{\int_\Omega f d\mu - m} \left\{ \frac{\ln\left(\int_\Omega f d\mu\right) - \ln(m)}{\int_\Omega f d\mu - m} - \left(\int_\Omega f d\mu \right)^{-1} \right\} \times \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu \right)^2 \right]. \tag{4.2}$$

Since Φ'' is convex, then by (2.17) we have

$$0 \le \frac{1}{2} \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu \right)^2 \right] \left(\frac{2 \int_\Omega f d\mu + M}{3} \right)^{-2} \le \ln\left(\int_\Omega f d\mu \right) - \int_\Omega \ln(f) \, d\mu \le \frac{1}{2} \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu \right)^2 \right] \left[\frac{2 \left(\int_\Omega f d\mu\right)^{-2} + m^{-2}}{3} \right]. \tag{4.3}$$

We consider the convex function $\Phi : (0, \infty) \to \mathbb{R}$, $\Phi(x) = x \ln x$. Then $\Phi''(x) = \frac{1}{x}$, which is convex and decreasing on $(0, \infty)$. If $x \in [m, M] \subset (0, \infty)$ then we also have

$$\frac{1}{M} \le \Phi''(x) \le \frac{1}{m}.$$

If $f : \Omega \to [m, M]$ is so that f, f^2, $f \ln \circ f \in L(\Omega, \mu)$, then by (2.11) we get

$$0 \le \frac{1}{2M} \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu \right)^2 \right] \le \int_\Omega f \ln(f) \, d\mu - \left(\int_\Omega f d\mu \right) \ln\left(\int_\Omega f d\mu \right) \le \frac{1}{2m} \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu \right)^2 \right]. \tag{4.4}$$

Since Φ'' is decreasing, hence by (2.13) we get

$$
\begin{aligned}
(4.5)\quad 0 &\leq \frac{1}{M-\int_\Omega f d\mu}\left[\frac{M\ln(M)-\left(\int_\Omega f d\mu\right)\ln\left(\int_\Omega f d\mu\right)}{M-\int_\Omega f d\mu}-\ln\left(\int_\Omega f d\mu\right)-1\right] \\
&\times\left[\int_\Omega f^2 d\mu-\left(\int_\Omega f d\mu\right)^2\right] \\
&\leq \int_\Omega f\ln(f)\,d\mu-\left(\int_\Omega f d\mu\right)\ln\left(\int_\Omega f d\mu\right) \\
&\leq \frac{1}{\int_\Omega f d\mu-m}\left\{\ln\left(\int_\Omega f d\mu\right)+1-\frac{\left(\int_\Omega f d\mu\right)\ln\left(\int_\Omega f d\mu\right)-m\ln(m)}{\int_\Omega f d\mu-m}\right\} \\
&\times\left[\int_\Omega f^2 d\mu-\left(\int_\Omega f d\mu\right)^2\right].
\end{aligned}
$$

Since Φ'' is convex, then by (2.17) we have

$$
\begin{aligned}
(4.6)\quad 0 &\leq \frac{1}{2}\left[\int_\Omega f^2 d\mu-\left(\int_\Omega f d\mu\right)^2\right]\left(\frac{2\int_\Omega f d\mu+M}{3}\right)^{-1} \\
&\leq \int_\Omega f\ln(f)\,d\mu-\left(\int_\Omega f d\mu\right)\ln\left(\int_\Omega f d\mu\right) \\
&\leq \frac{1}{2}\left[\int_\Omega f^2 d\mu-\left(\int_\Omega f d\mu\right)^2\right]\left[\frac{2\left(\int_\Omega f d\mu\right)^{-1}+M^{-1}}{3}\right].
\end{aligned}
$$

5. Power Inequalities

The power function $\Phi_p:(0,\infty)\to(0,\infty)$, $\Phi_p(x)=x^p$ is convex for $p\in(-\infty,0)\cup[1,\infty)$. We have $\Phi_p''(x)=p(p-1)x^{p-2}$, $x\in(0,\infty)$. If $p\in(-\infty,0)\cup[1,2)$ then Φ_p'' is decreasing and if $p\in[2,\infty)$ then Φ_p'' is increasing.

For $p\in(-\infty,0)\cup[1,\infty)$ and $[m,M]\subset(0,\infty)$ we define the bounds

$$
(5.1)\qquad B_1(p,[m,M]):=p(p-1)\begin{cases} M^{p-2} \text{ if } (-\infty,0)\cup[1,2), \\ \\ m^{p-2} \text{ if } p\in[2,\infty)\end{cases}
$$

and

$$
(5.2)\qquad B_2(p,[m,M]):=p(p-1)\begin{cases} m^{p-2} \text{ if } (-\infty,0)\cup[1,2), \\ \\ M^{p-2} \text{ if } p\in[2,\infty).\end{cases}
$$

Then we have

$$
B_1(p,[m,M])\leq\Phi_p''(x)\leq B_2(p,[m,M]) \text{ for } x\in[m,M].
$$

If $f:\Omega\to[m,M]$ is so that $f,\ f^2,\ f^p\in L(\Omega,\mu)$, then by (2.11) we get

$$0\leq\frac{1}{2}B_1(p,[m,M])\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]\tag{5.3}$$
$$\leq\int_\Omega f^pd\mu-\left(\int_\Omega fd\mu\right)^p$$
$$\leq\frac{1}{2}B_2(p,[m,M])\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right].$$

If $p\in(-\infty,0)\cup[1,2)$ then Φ_p'' is decreasing and by (2.13) we have

$$0\leq\frac{1}{M-\int_\Omega fd\mu}\left[\frac{M^p-\left(\int_\Omega fd\mu\right)^p}{M-\int_\Omega fd\mu}-p\left(\int_\Omega fd\mu\right)^{p-1}\right]\tag{5.4}$$
$$\times\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]$$
$$\leq\int_\Omega f^pd\mu-\left(\int_\Omega fd\mu\right)^p$$
$$\leq\frac{1}{\int_\Omega fd\mu-m}\left\{p\left(\int_\Omega fd\mu\right)^{p-1}-\frac{\left(\int_\Omega fd\mu\right)^p-m^p}{\int_\Omega fd\mu-m}\right\}$$
$$\times\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right].$$

If $p\in[2,\infty)$ then Φ_p'' is increasing and by (2.12) we get

$$0\leq\frac{1}{\int_\Omega fd\mu-m}\left\{p\left(\int_\Omega fd\mu\right)^{p-1}-\frac{\left(\int_\Omega fd\mu\right)^p-m^p}{\int_\Omega fd\mu-m}\right\}\tag{5.5}$$
$$\times\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]$$
$$\leq\int_\Omega f^pd\mu-\left(\int_\Omega fd\mu\right)^p$$
$$\leq\frac{1}{M-\int_\Omega fd\mu}\left[\frac{M^p-\left(\int_\Omega fd\mu\right)^p}{M-\int_\Omega fd\mu}-p\left(\int_\Omega fd\mu\right)^{p-1}\right]$$
$$\times\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right].$$

If $p\in(-\infty,0)\cup[1,2]\cup[3,\infty)$, then Φ_p'' is convex and by (2.17) we get

$$0\le\frac{1}{2}p\left(p-1\right)\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]\times\operatorname*{essinf}_{t\in\Omega}\left[\left(\frac{2\int_\Omega fd\mu+f\left(t\right)}{3}\right)^{p-2}\right]\le\int_\Omega f^pd\mu-\left(\int_\Omega fd\mu\right)^p\le\frac{1}{2}p\left(p-1\right)\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]\times\left[\frac{2\left(\int_\Omega fd\mu\right)^{p-2}+\operatorname{essup}_{t\in\Omega}\left(f\left(t\right)\right)^{p-2}}{3}\right].\tag{5.6}$$

Therefore, if $p\in(-\infty,0)\cup[1,2]$, then

$$0\le\frac{1}{2}p\left(p-1\right)\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]\left[\left(\frac{2\int_\Omega fd\mu+M}{3}\right)^{p-2}\right]\le\int_\Omega f^pd\mu-\left(\int_\Omega fd\mu\right)^p\le\frac{1}{2}p\left(p-1\right)\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]\left[\frac{2\left(\int_\Omega fd\mu\right)^{p-2}+m^{p-2}}{3}\right].\tag{5.7}$$

If $p\in[3,\infty)$, then

$$0\le\frac{1}{2}p\left(p-1\right)\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]\left[\left(\frac{2\int_\Omega fd\mu+m}{3}\right)^{p-2}\right]\le\int_\Omega f^pd\mu-\left(\int_\Omega fd\mu\right)^p\le\frac{1}{2}p\left(p-1\right)\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]\left[\frac{2\left(\int_\Omega fd\mu\right)^{p-2}+M^{p-2}}{3}\right].\tag{5.8}$$

If $p\in(2,3)$, then Φ_p'' is concave and by (2.17) we get

$$0\le\frac{1}{2}p\left(p-1\right)\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]\left[\frac{2\left(\int_\Omega fd\mu\right)^{p-2}+m^{p-2}}{3}\right]\le\int_\Omega f^pd\mu-\left(\int_\Omega fd\mu\right)^p\le\frac{1}{2}p\left(p-1\right)\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]\left(\frac{2\int_\Omega fd\mu+M}{3}\right)^{p-2}.\tag{5.9}$$

6. Discrete Case

The discrete case is useful for applications and we state some examples here.

In this section the function $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ is a twice differentiable convex function on (m,M) and $x_k\in[m,M]$, $p_k\ge0$ for $k\in\{1,...,n\}$ with $\sum_{k=1}^n p_k=1$.

By the inequality (2.5) we have

$$0 \leq \min_{k\in\{1,\ldots,n\}} \left(\int_0^1 \Phi'' \left((1-s) \sum_{i=1}^n p_i x_i + s x_k \right) (1-s)\, ds \right) \times \left[\sum_{i=1}^n p_i x_i^2 - \left(\sum_{i=1}^n p_i x_i \right)^2 \right] \leq \sum_{i=1}^n p_i \Phi(x_i) - \Phi\left(\sum_{i=1}^n p_i x_i \right) \leq \max_{k\in\{1,\ldots,n\}} \left(\int_0^1 \Phi'' \left((1-s) \sum_{i=1}^n p_i x_i + s x_k \right) (1-s)\, ds \right) \times \left[\sum_{i=1}^n p_i x_i^2 - \left(\sum_{i=1}^n p_i x_i \right)^2 \right]. \tag{6.1}$$

If there exist $0 < \gamma < \Gamma < \infty$ such that

$$\gamma \leq \Phi''(t) \leq \Gamma \text{ for almost every } t \in [m, M],$$

then, see also [5],

$$0 \leq \frac{1}{2}\gamma \left[\sum_{i=1}^n p_i x_i^2 - \left(\sum_{i=1}^n p_i x_i \right)^2 \right] \leq \sum_{i=1}^n p_i \Phi(x_i) - \Phi\left(\sum_{i=1}^n p_i x_i \right) \leq \frac{1}{2}\Gamma \left[\sum_{i=1}^n p_i x_i^2 - \left(\sum_{i=1}^n p_i x_i \right)^2 \right]. \tag{6.2}$$

If $\Phi''(\cdot)$ is monotonic nondecreasing on (m, M), then

$$0 \leq \frac{1}{\sum_{i=1}^n p_i x_i - m} \left\{ \Phi'\left(\sum_{i=1}^n p_i x_i \right) - \frac{\Phi\left(\sum_{i=1}^n p_i x_i\right) - \Phi(m)}{\sum_{i=1}^n p_i x_i - m} \right\} \times \left[\sum_{i=1}^n p_i x_i^2 - \left(\sum_{i=1}^n p_i x_i \right)^2 \right] \leq \sum_{i=1}^n p_i \Phi(x_i) - \Phi\left(\sum_{i=1}^n p_i x_i \right) \leq \frac{1}{M - \sum_{i=1}^n p_i x_i} \left[\frac{\Phi(M) - \Phi\left(\sum_{i=1}^n p_i x_i\right)}{M - \sum_{i=1}^n p_i x_i} - \Phi'\left(\sum_{i=1}^n p_i x_i \right) \right] \times \left[\sum_{i=1}^n p_i x_i^2 - \left(\sum_{i=1}^n p_i x_i \right)^2 \right]. \tag{6.3}$$

If $\Phi''(\cdot)$ is monotonic nonincreasing on (m, M), then

$$0 \leq \frac{1}{M - \sum_{i=1}^{n} p_i x_i} \left[\frac{\Phi(M) - \Phi\left(\sum_{i=1}^{n} p_i x_i\right)}{M - \sum_{i=1}^{n} p_i x_i} - \Phi'\left(\sum_{i=1}^{n} p_i x_i\right) \right] \times \left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2 \right] \leq \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{\sum_{i=1}^{n} p_i x_i - m} \left\{ \Phi'\left(\sum_{i=1}^{n} p_i x_i\right) - \frac{\Phi\left(\sum_{i=1}^{n} p_i x_i\right) - \Phi(m)}{\sum_{i=1}^{n} p_i x_i - m} \right\} \times \left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2 \right]. \tag{6.4}$$

If $\Phi''(\cdot)$ is convex on (m, M), then

$$0 \leq \frac{1}{2} \left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2 \right] \min_{k \in \{1,\dots,n\}} \left[\Phi''\left(\frac{2\sum_{i=1}^{n} p_i x_i + x_k}{3} \right) \right] \leq \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{2} \left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2 \right] \times \left[\frac{2\Phi''\left(\sum_{i=1}^{n} p_i x_i\right) + \max_{k \in \{1,\dots,n\}} \Phi''(x_k)}{3} \right]. \tag{6.5}$$

If $\Phi''(\cdot)$ is concave on (m, M), then

$$0 \leq \frac{1}{2} \left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2 \right] \left[\frac{2\Phi''\left(\sum_{i=1}^{n} p_i x_i\right) + \min_{k \in \{1,\dots,n\}} \Phi''(x_k)}{3} \right] \leq \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{2} \left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2 \right] \max_{k \in \{1,\dots,n\}} \left[\Phi''\left(\frac{2\sum_{i=1}^{n} p_i x_i + x_k}{3} \right) \right]. \tag{6.6}$$

If $\Phi''(\cdot)$ is quasiconvex on (m, M), then

$$\sum_{i=1}^{n} p_i (x_i) - \Phi\left(\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{2} \left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2 \right] \max\left\{ \Phi''\left(\sum_{i=1}^{n} p_i x_i\right), \max_{k \in \{1,\dots,n\}} \left[\Phi''(x_k)\right] \right\}. \tag{6.7}$$

If $\Phi(x) = \exp x$, then Φ'' is convex and increasing on any interval $[m, M]$. If $x_k \in [m, M]$, $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$, then by (6.2) we get

$$0 \leq \frac{1}{2}\exp(m)\left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right] \leq \sum_{i=1}^{n} p_i \exp(x_i) - \exp\left(\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{2}\exp(M)\left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right]. \tag{6.8}$$

From (6.3) we get

$$\begin{aligned} 0 &\leq \frac{1}{\sum_{i=1}^{n} p_i x_i - m}\left\{\exp\left(\sum_{i=1}^{n} p_i x_i\right) - \frac{\exp\left(\sum_{i=1}^{n} p_i x_i\right) - \exp(m)}{\sum_{i=1}^{n} p_i x_i - m}\right\} \\ &\times \left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right] \\ &\leq \sum_{i=1}^{n} p_i \exp(x_i) - \exp\left(\sum_{i=1}^{n} p_i x_i\right) \\ &\leq \frac{1}{M - \sum_{i=1}^{n} p_i x_i}\left[\frac{\exp(M) - \exp\left(\sum_{i=1}^{n} p_i x_i\right)}{M - \sum_{i=1}^{n} p_i x_i} - \exp\left(\sum_{i=1}^{n} p_i x_i\right)\right] \\ &\times \left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right], \end{aligned} \tag{6.9}$$

while from (6.5) we derive

$$\begin{aligned} 0 &\leq \frac{1}{2}\left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right]\exp\left(\frac{2\sum_{i=1}^{n} p_i x_i + m}{3}\right) \\ &\leq \sum_{i=1}^{n} p_i \exp(x_i) - \exp\left(\sum_{i=1}^{n} p_i x_i\right) \\ &\leq \frac{1}{2}\left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right]\left[\frac{2\exp\left(\sum_{i=1}^{n} p_i x_i\right) + \exp M}{3}\right]. \end{aligned} \tag{6.10}$$

Now, let $y_k \in [a,b] \subset (0,\infty)$, $k \in \{1,...,n\}$. If we take $x_k = \ln y_k$, $m = \ln a$ and $M = \ln b$, then by (6.8), (6.9) and (6.10) we obtain

$$0 \le \frac{1}{2} a \left[\sum_{i=1}^n p_i (\ln y_i)^2 - \left(\ln \left(\prod_{i=1}^n y_i^{p_i} \right) \right)^2 \right] \le \sum_{i=1}^n p_i y_i - \prod_{i=1}^n y_i^{p_i} \le \frac{1}{2} b \left[\sum_{i=1}^n p_i (\ln y_i)^2 - \left(\ln \left(\prod_{i=1}^n y_i^{p_i} \right) \right)^2 \right], \tag{6.11}$$

$$\begin{aligned} 0 &\le \frac{1}{\ln \left(\prod_{i=1}^n y_i^{p_i}\right) - \ln a} \left\{ \prod_{i=1}^n y_i^{p_i} - \frac{\prod_{i=1}^n y_i^{p_i} - a}{\ln \left(\prod_{i=1}^n y_i^{p_i}\right) - \ln a} \right\} \\ &\quad \times \left[\sum_{i=1}^n p_i (\ln y_i)^2 - \left(\ln \left(\prod_{i=1}^n y_i^{p_i} \right) \right)^2 \right] \\ &\le \sum_{i=1}^n p_i y_i - \prod_{i=1}^n y_i^{p_i} \\ &\le \frac{1}{\ln b - \ln \left(\prod_{i=1}^n y_i^{p_i}\right)} \left[\frac{b - \prod_{i=1}^n y_i^{p_i}}{\ln b - \ln \left(\prod_{i=1}^n y_i^{p_i}\right)} - \prod_{i=1}^n y_i^{p_i} \right] \\ &\quad \times \left[\sum_{i=1}^n p_i (\ln y_i)^2 - \left(\ln \left(\prod_{i=1}^n y_i^{p_i} \right) \right)^2 \right] \end{aligned} \tag{6.12}$$

and

$$\begin{aligned} 0 &\le \frac{1}{2} \left[\sum_{i=1}^n p_i (\ln y_i)^2 - \left(\ln \left(\prod_{i=1}^n y_i^{p_i} \right) \right)^2 \right] \left(\prod_{i=1}^n y_i^{p_i} \right)^{2/3} a^{1/3} \\ &\le \sum_{i=1}^n p_i y_i - \prod_{i=1}^n y_i^{p_i} \\ &\le \frac{1}{2} \left[\sum_{i=1}^n p_i (\ln y_i)^2 - \left(\ln \left(\prod_{i=1}^n y_i^{p_i} \right) \right)^2 \right] \left(\frac{2 \prod_{i=1}^n y_i^{p_i} + b}{3} \right). \end{aligned} \tag{6.13}$$

We consider the convex function $\Phi : (0,\infty) \to \mathbb{R}$, $\Phi(x) = -\ln x$. Then $\Phi''(x) = \frac{1}{x^2}$ which is convex and decreasing on $(0,\infty)$. If $x \in [m,M] \subset (0,\infty)$ then we also have

$$\frac{1}{M^2} \le \Phi''(x) \le \frac{1}{m^2}.$$

If $x_k \in [m, M]$, $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^n p_k = 1$, then by (6.2) we obtain

$$0 \leq \frac{1}{2M^2} \left[\sum_{i=1}^n p_i x_i^2 - \left(\sum_{i=1}^n p_i x_i \right)^2 \right] \leq \ln \left(\sum_{i=1}^n p_i x_i \right) - \ln \left(\prod_{i=1}^n x_i^{p_i} \right) \leq \frac{1}{2m^2} \left[\sum_{i=1}^n p_i x_i^2 - \left(\sum_{i=1}^n p_i x_i \right)^2 \right], \tag{6.14}$$

which is equivalent to

$$1 \leq \exp \left\{ \frac{1}{2M^2} \left[\sum_{i=1}^n p_i x_i^2 - \left(\sum_{i=1}^n p_i x_i \right)^2 \right] \right\} \leq \frac{\sum_{i=1}^n p_i x_i}{\prod_{i=1}^n x_i^{p_i}} \leq \exp \left\{ \frac{1}{2m^2} \left[\sum_{i=1}^n p_i x_i^2 - \left(\sum_{i=1}^n p_i x_i \right)^2 \right] \right\}. \tag{6.15}$$

From (6.4) we obtain

$$\begin{aligned} 0 &\leq \frac{1}{M - \sum_{i=1}^n p_i x_i} \left[\left(\sum_{i=1}^n p_i x_i \right)^{-1} - \frac{\ln (M) - \ln \left(\sum_{i=1}^n p_i x_i \right)}{M - \sum_{i=1}^n p_i x_i} \right] \\ &\quad \times \left[\sum_{i=1}^n p_i x_i^2 - \left(\sum_{i=1}^n p_i x_i \right)^2 \right] \\ &\leq \ln \left(\sum_{i=1}^n p_i x_i \right) - \ln \left(\prod_{i=1}^n x_i^{p_i} \right) \\ &\leq \frac{1}{\sum_{i=1}^n p_i x_i - m} \left\{ \frac{\ln \left(\sum_{i=1}^n p_i x_i \right) - \ln (m)}{\sum_{i=1}^n p_i x_i - m} - \left(\sum_{i=1}^n p_i x_i \right)^{-1} \right\} \\ &\quad \times \left[\sum_{i=1}^n p_i x_i^2 - \left(\sum_{i=1}^n p_i x_i \right)^2 \right], \end{aligned} \tag{6.16}$$

while from (6.5) we get

$$\begin{aligned} 0 &\leq \frac{1}{2} \left[\sum_{i=1}^n p_i x_i^2 - \left(\sum_{i=1}^n p_i x_i \right)^2 \right] \left(\frac{2 \sum_{i=1}^n p_i x_i + M}{3} \right)^{-2} \\ &\leq \ln \left(\sum_{i=1}^n p_i x_i \right) - \ln \left(\prod_{i=1}^n x_i^{p_i} \right) \\ &\leq \frac{1}{2} \left[\sum_{i=1}^n p_i x_i^2 - \left(\sum_{i=1}^n p_i x_i \right)^2 \right] \left[\frac{2 \left(\sum_{i=1}^n p_i x_i \right)^{-2} + m^{-2}}{3} \right]. \end{aligned} \tag{6.17}$$

The power function $\Phi_p : (0, \infty) \to (0, \infty)$, $\Phi_p(x) = x^p$ is convex for $p \in (-\infty, 0) \cup [1, \infty)$. We have $\Phi_p''(x) = p(p-1)x^{p-2}$, $x \in (0, \infty)$. If $p \in (-\infty, 0) \cup [1, 2)$ then Φ_p'' is decreasing and if $p \in [2, \infty)$ then Φ_p'' is increasing. Assume that $x_k \in [m, M]$, $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$.

By the discrete version of (5.3) we have for $p \in (-\infty, 0) \cup [1, \infty)$ that

$$0 \leq \frac{1}{2} B_1(p, [m, M]) \left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i \right)^2 \right] \leq \sum_{i=1}^{n} p_i x_i^p - \left(\sum_{i=1}^{n} p_i x_i \right)^p \leq \frac{1}{2} B_2(p, [m, M]) \left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i \right)^2 \right], \tag{6.18}$$

where $B_1(p, [m, M])$ and $B_2(p, [m, M])$ are defined by (5.1) and (5.2).

If $p \in (-\infty, 0) \cup [1, 2)$, then

$$\begin{aligned} 0 &\leq \frac{1}{M - \sum_{i=1}^{n} p_i x_i} \left[\frac{M^p - \left(\sum_{i=1}^{n} p_i x_i\right)^p}{M - \sum_{i=1}^{n} p_i x_i} - p \left(\sum_{i=1}^{n} p_i x_i \right)^{p-1} \right] \\ &\times \left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i \right)^2 \right] \\ &\leq \sum_{i=1}^{n} p_i x_i^p - \left(\sum_{i=1}^{n} p_i x_i \right)^p \\ &\leq \frac{1}{\sum_{i=1}^{n} p_i x_i - m} \left\{ p \left(\sum_{i=1}^{n} p_i x_i \right)^{p-1} - \frac{\left(\sum_{i=1}^{n} p_i x_i\right)^p - m^p}{\sum_{i=1}^{n} p_i x_i - m} \right\} \\ &\times \left[\sum_{i=1}^{n} p_i x_i^2 - \left(\int_{\Omega} f d\mu \right)^2 \right]. \end{aligned} \tag{6.19}$$

If $p \in [2, \infty)$, then

$$\begin{aligned} 0 &\leq \frac{1}{\sum_{i=1}^{n} p_i x_i - m} \left\{ p \left(\sum_{i=1}^{n} p_i x_i \right)^{p-1} - \frac{\left(\sum_{i=1}^{n} p_i x_i\right)^p - m^p}{\sum_{i=1}^{n} p_i x_i - m} \right\} \\ &\times \left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i \right)^2 \right] \\ &\leq \sum_{i=1}^{n} p_i x_i^p - \left(\sum_{i=1}^{n} p_i x_i \right)^p \end{aligned} \tag{6.20}$$

$$\leq \frac{1}{M-\sum_{i=1}^{n}p_ix_i}\left[\frac{M^p-\left(\sum_{i=1}^{n}p_ix_i\right)^p}{M-\sum_{i=1}^{n}p_ix_i}-p\left(\sum_{i=1}^{n}p_ix_i\right)^{p-1}\right]$$
$$\times\left[\sum_{i=1}^{n}p_ix_i^2-\left(\sum_{i=1}^{n}p_ix_i\right)^2\right].$$

If $p\in(-\infty,0)\cup[1,2]$, then

$$0\leq\frac{1}{2}p\,(p-1)\left[\sum_{i=1}^{n}p_ix_i^2-\left(\sum_{i=1}^{n}p_ix_i\right)^2\right]\left(\frac{2\sum_{i=1}^{n}p_ix_i+M}{3}\right)^{p-2} \tag{6.21}$$
$$\leq\sum_{i=1}^{n}p_ix_i^p-\left(\sum_{i=1}^{n}p_ix_i\right)^p$$
$$\leq\frac{1}{2}p\,(p-1)\left[\sum_{i=1}^{n}p_ix_i^2-\left(\sum_{i=1}^{n}p_ix_i\right)^2\right]\left[\frac{2\left(\sum_{i=1}^{n}p_ix_i\right)^{p-2}+m^{p-2}}{3}\right].$$

If $p\in[3,\infty)$, then

$$0\leq\frac{1}{2}p\,(p-1)\left[\sum_{i=1}^{n}p_ix_i^2-\left(\sum_{i=1}^{n}p_ix_i\right)^2\right]\left(\frac{2\sum_{i=1}^{n}p_ix_i+m}{3}\right)^{p-2} \tag{6.22}$$
$$\leq\sum_{i=1}^{n}p_ix_i^p-\left(\sum_{i=1}^{n}p_ix_i\right)^p$$
$$\leq\frac{1}{2}p\,(p-1)\left[\sum_{i=1}^{n}p_ix_i^2-\left(\sum_{i=1}^{n}p_ix_i\right)^2\right]\left[\frac{2\left(\sum_{i=1}^{n}p_ix_i\right)^{p-2}+M^{p-2}}{3}\right].$$

If $p\in(2,3)$, then

$$0\leq\frac{1}{2}p\,(p-1)\left[\sum_{i=1}^{n}p_ix_i^2-\left(\sum_{i=1}^{n}p_ix_i\right)^2\right]\left[\frac{2\left(\sum_{i=1}^{n}p_ix_i\right)^{p-2}+m^{p-2}}{3}\right] \tag{6.23}$$
$$\leq\sum_{i=1}^{n}p_ix_i^p-\left(\sum_{i=1}^{n}p_ix_i\right)^p$$
$$\leq\frac{1}{2}p\,(p-1)\left[\sum_{i=1}^{n}p_ix_i^2-\left(\sum_{i=1}^{n}p_ix_i\right)^2\right]\left(\frac{2\sum_{i=1}^{n}p_ix_i+M}{3}\right)^{p-2}.$$

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