LOWER AND UPPER BOUNDS FOR THE JENSEN'S GAP OF CONVEX FUNCTIONS IN TERMS OF INF/SUP-SQUARE-MEAN-DEVIATION

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Abstract. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. In this paper we establish some lower and upper bounds for the Jensen's gap

$$\int_\Omega (\Phi \circ f)\, d\mu - \Phi\left(\int_\Omega f d\mu\right)$$

in terms of the *inf-square-mean-deviation*

$$\operatorname*{essinf}_{t\in\Omega}\left(f(t) - \int_\Omega f d\mu\right)^2$$

and *sup-square-mean-deviation*

$$\operatorname*{essup}_{t\in\Omega}\left(f(t) - \int_\Omega f d\mu\right)^2,$$

for some classes of convex functions Φ. Applications for exponential, logarithm and power functions are also given.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. Consider the *Lebesgue space*

$$L(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)|\, d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t)\, d\mu(t)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, the author obtained in [6] and [9] the following result:

Theorem 1. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, $\mathring{I}$ the interior of I. If $f : \Omega \to [m, M]$ is so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L(\Omega, \mu)$, then we have the*

1991 *Mathematics Subject Classification.* Primary 26D15, 26D20..

Key words and phrases. Jensen's inequality, Measurable functions, Lebesgue integral, Discrete inequalities.

RGMIA Res. Rep. Coll. 23 (2020), Art. 4, 28 pp. Received 12/01/20

inequality:

$$0\le\int_\Omega\Phi\circ fd\mu-\Phi\left(\int_\Omega fd\mu\right)\le\int_\Omega(\Phi'\circ f)fd\mu-\int_\Omega\Phi'\circ fd\mu\int_\Omega fd\mu \tag{1.1}$$
$$\le\frac{1}{2}\left[\Phi'_-(M)-\Phi'_+(m)\right]\int_\Omega\left|f-\int_\Omega fd\mu\right|d\mu$$
$$\le\frac{1}{2}\left[\Phi'_-(M)-\Phi'_+(m)\right]\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]^{\frac{1}{2}}$$
$$\le\frac{1}{4}\left[\Phi'_-(M)-\Phi'_+(m)\right](M-m).$$

Remark 1. *We notice that the inequality between the first and the second term in (1.1) in the discrete case was proved in 1994 by Dragomir & Ionescu, see* [12].

For a real function $g:[m,M]\to\mathbb{R}$ and two distinct points $\alpha,\ \beta\in[m,M]$ we recall that the *divided difference* of g in these points is defined by

$$[\alpha,\beta;g]:=\frac{g(\beta)-g(\alpha)}{\beta-\alpha}.$$

Upper and lower bounds for the Jensen's gap were also obtained in [10]:

Theorem 2. *Let $\Phi:I\to\mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m,\ M\in\mathbb{R},\ m<M$ with $[m,M]\subset\mathring{I}$. If $f:\Omega\to[m,M]$, is μ-measurable and such that $f,\ \Phi\circ f\in L(\Omega,\mu)$, then by assuming that $\int_\Omega fd\mu\neq m$, M, we have*

$$\left|\int_\Omega\left|\Phi(f)-\Phi\left(\int_\Omega fd\mu\right)\right|\operatorname{sgn}\left(f-\int_\Omega fd\mu\right)d\mu\right| \tag{1.2}$$
$$\le\int_\Omega(\Phi\circ f)d\mu-\Phi\left(\int_\Omega fd\mu\right)$$
$$\le\frac{1}{2}\left(\left[\int_\Omega fd\mu,M;\Phi\right]-\left[m,\int_\Omega fd\mu;\Phi\right]\right)\int_\Omega\left|f-\int_\Omega fd\mu\right|d\mu$$
$$\le\frac{1}{2}\left(\left[\int_\Omega fd\mu,M;\Phi\right]-\left[m,\int_\Omega fd\mu;\Phi\right]\right)\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]^{\frac{1}{2}}$$
$$\le\frac{1}{4}\left(\left[\int_\Omega fd\mu,M;\Phi\right]-\left[m,\int_\Omega fd\mu;\Phi\right]\right)(M-m).$$

The constant $\frac{1}{2}$ in the second inequality from (2.5) is best possible.

For other recent reverses of Jensen inequality and applications to divergence measures see [8], [9], [10] and the survey paper [11]. More related results may be found in [1]-[4], [7], [11] and [11]-[14].

Motivated by the above results, in this paper we establish some lower and upper bounds for the Jensen's gap of convex function Φ,

$$\int_\Omega(\Phi\circ f)d\mu-\Phi\left(\int_\Omega fd\mu\right)$$

in terms of the quantities

$$\operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_\Omega fd\mu\right)^2,\text{ we call the \textit{inf-square-mean-deviation}}$$

and

$$\operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2, \text{ we call the } \textit{sup-square-mean-deviation,}$$

respectively. Applications for exponential, logarithm and power functions are also given.

2. Main Results

The following theorem is well known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Lemma 1. *Let $I \subset \mathbb{R}$ be a closed interval, $a \in I$ and let n be a positive integer. If $g : I \longrightarrow \mathbb{C}$ is such that the n-derivative $g^{(n)}$ is absolutely continuous on I, then for each $x \in I$*

$$g(x) = T_n(g; a, x) + R_n(g; a, x), \tag{2.1}$$

where $T_n(g; c, y)$ is Taylor's polynomial, i.e.,

$$T_n(g; a, x) := \sum_{k=0}^{n} \frac{(x-a)^k}{k!} g^{(k)}(a). \tag{2.2}$$

Note that $g^{(0)} := g$ and $0! := 1$ and the remainder is given by

$$R_n(g; a, x) := \frac{1}{n!} \int_a^x (x-t)^n g^{(n+1)}(t)\, dt. \tag{2.3}$$

A simple proof of this theorem can be achieved by mathematical induction using the integration by parts formula in the Lebesgue integral.

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $t = (1-s)c + sd$, $s \in [0, 1]$ that

$$\int_c^d h(t)\, dt = (d-c) \int_0^1 h((1-s)c + sd)\, ds.$$

Therefore,

$$\begin{aligned}
&\int_a^x g^{(n+1)}(t)(x-t)^n\, dt \\
&= (x-a) \int_0^1 g^{(n+1)}((1-s)a + sx)(x - (1-s)a - sx)^n\, ds \\
&= (x-a)^{n+1} \int_0^1 g^{(n+1)}((1-s)a + sx)(1-s)^n\, ds.
\end{aligned}$$

The identity (2.1) can then be written as

$$\begin{aligned}
g(x) = &\sum_{k=0}^{n} \frac{1}{k!} g^{(k)}(a)(x-a)^k \\
&+ \frac{1}{n!}(x-a)^{n+1} \int_0^1 g^{(n+1)}((1-s)a + sx)(1-s)^n\, ds
\end{aligned} \tag{2.4}$$

for all x, $a \in I$.

We have:

Theorem 3. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that f, f^2, $\Phi \circ f$, $\Phi'' \circ f \in L(\Omega, \mu)$. Then we have the inequalities:*

$$
\begin{aligned}
(2.5)\qquad & 0 \leq \operatorname*{essinf}_{t\in\Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \\
& \times \int_\Omega \left(\int_0^1 \Phi'' \left((1-s) \int_\Omega f d\mu + s f(t) \right) (1-s)\, ds \right) d\mu(t) \\
& \leq \int_\Omega \Phi \circ f d\mu - \Phi \left(\int_\Omega f d\mu \right) \\
& \leq \operatorname*{essup}_{t\in\Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \\
& \times \int_\Omega \left(\int_0^1 \Phi'' \left((1-s) \int_\Omega f d\mu + s f(t) \right) (1-s)\, ds \right) d\mu(t).
\end{aligned}
$$

Proof. We have from (2.4) for $n = 2$ that

$$
\Phi(x) = \Phi(c) + \Phi'(c)(x-c) + (x-c)^2 \int_0^1 \Phi''((1-s)c + sx)(1-s)\, ds
$$

for all x, $c \in [m, M]$, where Φ is such that Φ' is absolutely continuos on $[m, M]$.

This implies that

$$
\begin{aligned}
(2.6)\qquad \Phi(f(t)) = {} & \Phi \left(\int_\Omega f d\mu \right) + \Phi' \left(\int_\Omega f d\mu \right) \left(f(t) - \int_\Omega f d\mu \right) \\
& + \left(f(t) - \int_\Omega f d\mu \right)^2 \int_0^1 \Phi'' \left((1-s) \int_\Omega f d\mu + s f(t) \right) (1-s)\, ds
\end{aligned}
$$

for all $t \in \Omega$.

If we take the integral in (2.6), then we get

$$
\begin{aligned}
(2.7)\qquad & \int_\Omega \Phi(f(t))\, d\mu(t) \\
& = \Phi \left(\int_\Omega f d\mu \right) \int_\Omega d\mu(t) + \Phi' \left(\int_\Omega f d\mu \right) \int_\Omega \left(f(t) - \int_\Omega f d\mu \right) d\mu(t) \\
& + \int_\Omega \left(f(t) - \int_\Omega f d\mu \right)^2 \\
& \times \left(\int_0^1 \Phi'' \left((1-s) \int_\Omega f d\mu + s f(t) \right) (1-s)\, ds \right) d\mu(t) \\
& = \Phi \left(\int_\Omega f d\mu \right) + \int_\Omega \left(f(t) - \int_\Omega f d\mu \right)^2 \\
& \times \left(\int_0^1 \Phi'' \left((1-s) \int_\Omega f d\mu + s f(t) \right) (1-s)\, ds \right) d\mu(t),
\end{aligned}
$$

since

$$
\int_\Omega \left(f(t) - \int_\Omega f d\mu \right) d\mu(t) = \int_\Omega f(t)\, d\mu(t) - \int_\Omega f d\mu \int_\Omega d\mu(t) = 0.
$$

Therefore we have the equality of interest

$$\begin{aligned}\int_\Omega \Phi\left(f\left(t\right)\right)d\mu\left(t\right)-\Phi\left(\int_\Omega fd\mu\right) \\ = \int_\Omega \left(f\left(t\right)-\int_\Omega fd\mu\right)^2 \\ \times \left(\int_0^1 \Phi''\left(\left(1-s\right)\int_\Omega fd\mu+sf\left(t\right)\right)\left(1-s\right)ds\right)d\mu\left(t\right).\end{aligned} \tag{2.8}$$

Observe that, by the properties of infimum and supremum, we have

$$\begin{aligned}&\operatorname*{essinf}_{t\in\Omega}\left(f\left(t\right)-\int_\Omega fd\mu\right)^2 \\ &\times \int_\Omega \left(\int_0^1 \Phi''\left(\left(1-s\right)\int_\Omega fd\mu+sf\left(t\right)\right)\left(1-s\right)ds\right)d\mu\left(t\right) \\ &\le \frac{1}{2}\int_\Omega \left(f\left(t\right)-\int_\Omega fd\mu\right)^2 \\ &\times \left(\int_0^1 \Phi''\left(\left(1-s\right)\int_\Omega fd\mu+sf\left(t\right)\right)\left(1-s\right)ds\right)d\mu\left(t\right) \\ &\le \operatorname*{essup}_{t\in\Omega}\left(f\left(t\right)-\int_\Omega fd\mu\right)^2 \\ &\times \int_\Omega \left(\int_0^1 \Phi''\left(\left(1-s\right)\int_\Omega fd\mu+sf\left(t\right)\right)\left(1-s\right)ds\right)d\mu\left(t\right),\end{aligned}$$

then by using the identity (2.8) we get the desired result (2.5). □

Corollary 1. *With the assumptions of Theorem 3 and if there exist* $0<\gamma<\Gamma<\infty$ *such that*

$$\gamma \le \Phi''\left(t\right) \le \Gamma \text{ for almost every } t\in[m,M] \tag{2.9}$$

then

$$\begin{aligned}0 &\le \frac{1}{2}\gamma \operatorname*{essinf}_{t\in\Omega}\left(f\left(t\right)-\int_\Omega fd\mu\right)^2 \le \int_\Omega \Phi\circ fd\mu-\Phi\left(\int_\Omega fd\mu\right) \\ &\le \frac{1}{2}\Gamma \operatorname*{essup}_{t\in\Omega}\left(f\left(t\right)-\int_\Omega fd\mu\right)^2.\end{aligned} \tag{2.10}$$

Proof. Observe that by (2.9) we have

$$\begin{aligned}\frac{1}{2}\gamma &= \gamma\int_0^1 \left(1-s\right)ds \le \int_0^1 \Phi''\left(\left(1-s\right)\int_\Omega fd\mu+sf\left(t\right)\right)\left(1-s\right)ds \\ &\le \Gamma\int_0^1 \left(1-s\right)ds=\frac{1}{2}\Gamma.\end{aligned}$$

This implies that

$$\begin{aligned}\frac{1}{2}\gamma\int_\Omega d\mu\left(t\right) &\le \int_\Omega \left(\int_0^1 \Phi''\left(\left(1-s\right)\int_\Omega fd\mu+sf\left(t\right)\right)\left(1-s\right)ds\right)d\mu\left(t\right) \\ &\le \frac{1}{2}\Gamma\int_\Omega d\mu\left(t\right),\end{aligned}$$

namely

$$\frac{1}{2}\gamma \leq \int_{\Omega}\left(\int_{0}^{1}\Phi''\left((1-s)\int_{\Omega}fd\mu+sf(t)\right)(1-s)\,ds\right)d\mu(t)$$

and

$$\int_{\Omega}\left(\int_{0}^{1}\Phi''\left((1-s)\int_{\Omega}fd\mu+sf(t)\right)(1-s)\,ds\right)d\mu(t)\leq\frac{1}{2}\Gamma.$$

By employing (2.5) we deduce (2.10). □

In the case of monotonic second derivatives, we have:

Corollary 2. *With the assumptions of Theorem 3 and if $\Phi''(\cdot)$ is monotonic nondecreasing on (m,M), then*

$$\begin{aligned}(2.11)\quad 0 &\leq \operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_{\Omega}fd\mu\right)^{2}\\ &\quad\times\frac{1}{\int_{\Omega}fd\mu-m}\left\{\Phi'\left(\int_{\Omega}fd\mu\right)-\frac{\Phi\left(\int_{\Omega}fd\mu\right)-\Phi(m)}{\int_{\Omega}fd\mu-m}\right\}\\ &\leq\int_{\Omega}\Phi\circ fd\mu-\Phi\left(\int_{\Omega}fd\mu\right)\\ &\leq\operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_{\Omega}fd\mu\right)^{2}\\ &\quad\times\frac{1}{M-\int_{\Omega}fd\mu}\left[\frac{\Phi(M)-\Phi\left(\int_{\Omega}fd\mu\right)}{M-\int_{\Omega}fd\mu}-\Phi'\left(\int_{\Omega}fd\mu\right)\right].\end{aligned}$$

If $\Phi''(\cdot)$ is monotonic nonincreasing on (m,M), then

$$\begin{aligned}(2.12)\quad 0 &\leq \operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_{\Omega}fd\mu\right)^{2}\\ &\quad\times\frac{1}{M-\int_{\Omega}fd\mu}\left[\frac{\Phi(M)-\Phi\left(\int_{\Omega}fd\mu\right)}{M-\int_{\Omega}fd\mu}-\Phi'\left(\int_{\Omega}fd\mu\right)\right]\\ &\leq\int_{\Omega}\Phi\circ fd\mu-\Phi\left(\int_{\Omega}fd\mu\right)\\ &\leq\operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_{\Omega}fd\mu\right)^{2}\\ &\quad\times\frac{1}{\int_{\Omega}fd\mu-m}\left\{\Phi'\left(\int_{\Omega}fd\mu\right)-\frac{\Phi\left(\int_{\Omega}fd\mu\right)-\Phi(m)}{\int_{\Omega}fd\mu-m}\right\}.\end{aligned}$$

Proof. Since $\Phi''(\cdot)$ is monotonic nondecreasing on (m,M), then

$$\begin{aligned}\Phi''\left((1-s)\int_{\Omega}fd\mu+sm\right)&\leq\Phi''\left((1-s)\int_{\Omega}fd\mu+sf(t)\right)\\ &\leq\Phi''\left((1-s)\int_{\Omega}fd\mu+sM\right)\end{aligned}$$

for all $s\in(0,1)$ and $t\in\Omega$.

This implies that

$$\begin{aligned}\int_0^1 \Phi''\left((1-s)\int_\Omega fd\mu + sm\right)(1-s)\,ds \\ \leq \int_0^1 \Phi''\left((1-s)\int_\Omega fd\mu + sf(t)\right)(1-s)\,ds \\ \leq \int_0^1 \Phi''\left((1-s)\int_\Omega fd\mu + sM\right)(1-s)\,ds\end{aligned} \tag{2.13}$$

for all $t \in \Omega$.

Assume that $\int_\Omega fd\mu \in (m, M)$. Observe that, integrating by parts, we have

$$\begin{aligned}&\int_0^1 \Phi''\left((1-s)\int_\Omega fd\mu + sm\right)(1-s)\,ds \\ &= \frac{1}{m - \int_\Omega fd\mu}\int_0^1 (1-s)\,d\left(\Phi'\left((1-s)\int_\Omega fd\mu + sm\right)\right) \\ &= \frac{1}{m - \int_\Omega fd\mu}\left\{(1-s)\,\Phi'\left((1-s)\int_\Omega fd\mu + sm\right)\Big|_0^1\right. \\ &\left.+\int_0^1 \Phi'\left((1-s)\int_\Omega fd\mu + sm\right)ds\right\} \\ &= \frac{1}{m - \int_\Omega fd\mu}\left\{-\Phi'\left(\int_\Omega fd\mu\right) + \int_0^1 \Phi'\left((1-s)\int_\Omega fd\mu + sm\right)ds\right\} \\ &= \frac{1}{\int_\Omega fd\mu - m}\left\{\Phi'\left(\int_\Omega fd\mu\right) - \int_0^1 \Phi'\left((1-s)\int_\Omega fd\mu + sm\right)ds\right\} \\ &= \frac{1}{\int_\Omega fd\mu - m}\left\{\Phi'\left(\int_\Omega fd\mu\right) - \frac{\Phi\left(\int_\Omega fd\mu\right) - \Phi(m)}{\int_\Omega fd\mu - m}\right\}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\int_0^1 \Phi''\left((1-s)\int_\Omega fd\mu + sM\right)(1-s)\,ds \\ &= \frac{1}{M - \int_\Omega fd\mu}\int_0^1 (1-s)\,d\left(\Phi'\left((1-s)\int_\Omega fd\mu + sM\right)\right) \\ &= \frac{1}{M - \int_\Omega fd\mu}\left[\Phi'\left((1-s)\int_\Omega fd\mu + sM\right)(1-s)\Big|_0^1\right. \\ &\left.+\int_0^1 \Phi'\left((1-s)\int_\Omega fd\mu + sM\right)ds\right] \\ &= \frac{1}{M - \int_\Omega fd\mu}\left[\int_0^1 \Phi'\left((1-s)\int_\Omega fd\mu + sM\right)ds - \Phi'\left(\int_\Omega fd\mu\right)\right] \\ &= \frac{1}{M - \int_\Omega fd\mu}\left[\frac{\Phi(M) - \Phi\left(\int_\Omega fd\mu\right)}{M - \int_\Omega fd\mu} - \Phi'\left(\int_\Omega fd\mu\right)\right].\end{aligned}$$

By utilising (2.13) we then get

$$\frac{1}{\int_\Omega f d\mu - m}\left\{\Phi'\left(\int_\Omega f d\mu\right) - \frac{\Phi\left(\int_\Omega f d\mu\right) - \Phi(m)}{\int_\Omega f d\mu - m}\right\} \le \operatorname*{essinf}_{t\in\Omega}\left(\int_0^1 \Phi''\left((1-s)\int_\Omega f d\mu + s f(t)\right)(1-s)\,ds\right) \tag{2.14}$$

and

$$\operatorname*{essup}_{t\in\Omega}\left(\int_0^1 \Phi''\left((1-s)\int_\Omega f d\mu + s f(t)\right)(1-s)\,ds\right) \le \frac{1}{M-\int_\Omega f d\mu}\left[\frac{\Phi(M) - \Phi\left(\int_\Omega f d\mu\right)}{M - \int_\Omega f d\mu} - \Phi'\left(\int_\Omega f d\mu\right)\right]. \tag{2.15}$$

If we use (2.14), (2.15) and Theorem 3, then we get (2.12). □

From a complementary view point, we also have:

Corollary 3. *With the assumptions of Theorem 3 and if $\Phi''(\cdot)$ is monotonic nondecreasing on (m, M), then*

$$\begin{aligned} 0 &\le \operatorname*{essinf}_{t\in\Omega}\left(f(t) - \int_\Omega f d\mu\right)^2 \\ &\quad\times \int_\Omega \frac{1}{f(t)-m}\left\{\frac{\Phi(f(t)) - \Phi(m)}{f(t)-m} - \Phi'(m)\right\} d\mu(t) \\ &\le \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\ &\le \operatorname*{essup}_{t\in\Omega}\left(f(t) - \int_\Omega f d\mu\right)^2 \\ &\quad\times \int_\Omega \frac{1}{M-f(t)}\left[\Phi'(M) - \frac{\Phi(M) - \Phi(f(t))}{M-f(t)}\right] d\mu(t). \end{aligned} \tag{2.16}$$

If $\Phi''(\cdot)$ is monotonic nonincreasing on (m, M), then

$$\begin{aligned} 0 &\le \operatorname*{essinf}_{t\in\Omega}\left(f(t) - \int_\Omega f d\mu\right)^2 \\ &\quad\times \int_\Omega \frac{1}{M-f(t)}\left[\Phi'(M) - \frac{\Phi(M) - \Phi(f(t))}{M-f(t)}\right] d\mu(t) \\ &\le \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\ &\le \operatorname*{essup}_{t\in\Omega}\left(f(t) - \int_\Omega f d\mu\right)^2 \\ &\quad\times \int_\Omega \frac{1}{f(t)-m}\left\{\frac{\Phi(f(t)) - \Phi(m)}{f(t)-m} - \Phi'(m)\right\} d\mu(t). \end{aligned} \tag{2.17}$$

Proof. Since $\Phi''(\cdot)$ is monotonic nondecreasing on (m, M) and $m \leq \int_\Omega f d\mu \leq M$, then

$$\Phi''((1-s)m + sf(t)) \leq \Phi''\left((1-s)\int_\Omega f d\mu + sf(t)\right) \leq \Phi''((1-s)M + sf(t))$$

for all $s \in (0,1)$ and $t \in \Omega$.

This implies that

$$\int_0^1 \Phi''((1-s)m + sf(t))(1-s)\,ds \leq \int_0^1 \Phi''\left((1-s)\int_\Omega f d\mu + sf(t)\right)(1-s)\,ds \leq \int_0^1 \Phi''((1-s)M + sf(t))(1-s)\,ds \tag{2.18}$$

for all $t \in \Omega$.

As above, we have

$$\int_0^1 \Phi''((1-s)m + sf(t))(1-s)\,ds = \frac{1}{f(t)-m}\left\{\frac{\Phi(f(t)) - \Phi(m)}{f(t)-m} - \Phi'(m)\right\} \tag{2.19}$$

and

$$\int_0^1 \Phi''((1-s)M + sf(t))(1-s)\,ds = \frac{1}{M-f(t)}\left[\Phi'(M) - \frac{\Phi(M) - \Phi(f(t))}{M-f(t)}\right]. \tag{2.20}$$

By integrating (2.18) and using the equalities (2.19) and (2.20), we get

$$\int_\Omega \frac{1}{f(t)-m}\left\{\frac{\Phi(f(t)) - \Phi(m)}{f(t)-m} - \Phi'(m)\right\} d\mu(t) \leq \int_\Omega \left(\int_0^1 \Phi''\left((1-s)\int_\Omega f d\mu + sf(t)\right)(1-s)\,ds\right) d\mu(t) \leq \int_\Omega \frac{1}{M-f(t)}\left[\Phi'(M) - \frac{\Phi(M) - \Phi(f(t))}{M-f(t)}\right] d\mu(t). \tag{2.21}$$

On utilising the inequality (2.5) and (2.21), we get the desired result (2.16). □

The case of second derivative being convex/concave is as follows:

Corollary 4. *With the assumptions of Theorem 3 and if $\Phi''(\cdot)$ is convex on (m,M), then*

$$0 \le \frac{1}{2}\operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2 \Phi''\left(\int_\Omega f d\mu\right) \tag{2.22}$$
$$\le \frac{1}{2}\operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2 \int_\Omega \Phi''\left(\frac{2}{3}\int_\Omega f d\mu+\frac{1}{3}f(t)\right)d\mu(t)$$
$$\le \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right)$$
$$\le \frac{1}{2}\operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2 \left[\frac{2\Phi''\left(\int_\Omega f d\mu\right)+\int_\Omega \Phi''\circ f d\mu}{3}\right]$$
$$\le \frac{1}{2}\operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2 \int_\Omega \Phi''\circ f d\mu.$$

If $\Phi''(\cdot)$ is concave on (m,M), then

$$0 \le \frac{1}{2}\operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2 \int_\Omega \Phi''\circ f d\mu \tag{2.23}$$
$$\le \frac{1}{2}\operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2 \left[\frac{2\Phi''\left(\int_\Omega f d\mu\right)+\int_\Omega \Phi''\circ f d\mu}{3}\right]$$
$$\le \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right)$$
$$\le \frac{1}{2}\operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2 \int_\Omega \Phi''\left(\frac{2}{3}\int_\Omega f d\mu+\frac{1}{3}f(t)\right)d\mu(t)$$
$$\le \frac{1}{2}\operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2 \Phi''\left(\int_\Omega f d\mu\right).$$

Proof. If $\Phi''(\cdot)$ is convex on (m,M), then by Jensen's inequality we have

$$\int_0^1 \Phi''\left((1-s)\int_\Omega f d\mu + sf(t)\right)(1-s)\,ds$$
$$\ge \left(\int_0^1 (1-s)\,ds\right)\Phi''\left(\frac{\int_0^1 (1-s)\left[(1-s)\int_\Omega f d\mu + sf(t)\right]ds}{\int_0^1 (1-s)\,ds}\right)$$
$$= \frac{1}{2}\Phi''\left(\frac{\int_0^1 (1-s)\left[(1-s)\int_\Omega f d\mu + sf(t)\right]ds}{\frac{1}{2}}\right)$$
$$= \frac{1}{2}\Phi''\left(\frac{\int_\Omega f d\mu \int_0^1 (1-s)^2 ds + f(t)\int_0^1 s(1-s)\,ds}{\frac{1}{2}}\right)$$
$$= \frac{1}{2}\Phi''\left(\frac{\frac{1}{3}\int_\Omega f d\mu + \frac{1}{6}f(t)}{\frac{1}{2}}\right) = \frac{1}{2}\Phi''\left(\frac{2}{3}\int_\Omega f d\mu + \frac{1}{3}f(t)\right)$$

for all $t\in\Omega$.

This, implies that

$$\begin{aligned}
&\int_{\Omega}\left(\int_{0}^{1}\Phi''\left((1-s)\int_{\Omega}fd\mu+sf(t)\right)(1-s)\,ds\right)d\mu(t)\\
&\geq\frac{1}{2}\int_{\Omega}\Phi''\left(\frac{2}{3}\int_{\Omega}fd\mu+\frac{1}{3}f(t)\right)d\mu(t)\\
&\geq\frac{1}{2}\Phi''\left(\int_{\Omega}\frac{2}{3}\left[\int_{\Omega}fd\mu+\frac{1}{3}f(t)\right]d\mu(t)\right)=\frac{1}{2}\Phi''\left(\int_{\Omega}fd\mu\right),
\end{aligned}$$

where for the second inequality we used Jensen's inequality again.

This together with (2.5) proves the first two inequalities in (2.22).

By the convexity of Φ'' we also have

$$\begin{aligned}
&\int_{0}^{1}\Phi''\left((1-s)\int_{\Omega}fd\mu+sf(t)\right)(1-s)\,ds\\
&\leq\int_{0}^{1}\left[(1-s)\,\Phi''\left(\int_{\Omega}fd\mu\right)+s\Phi''(f(t))\right](1-s)\,ds\\
&=\Phi''\left(\int_{\Omega}fd\mu\right)\int_{0}^{1}(1-s)^{2}\,ds+\Phi''(f(t))\int_{0}^{1}s(1-s)\,ds\\
&=\frac{1}{3}\Phi''\left(\int_{\Omega}fd\mu\right)+\frac{1}{6}\Phi''(f(t))=\frac{1}{6}\left[2\Phi''\left(\int_{\Omega}fd\mu\right)+\Phi''(f(t))\right].
\end{aligned}$$

This, implies that

$$\begin{aligned}
&\int_{\Omega}\left(\int_{0}^{1}\Phi''\left((1-s)\int_{\Omega}fd\mu+sf(t)\right)(1-s)\,ds\right)d\mu(t)\\
&\leq\frac{1}{6}\int_{\Omega}\left[2\Phi''\left(\int_{\Omega}fd\mu\right)+\Phi''(f(t))\right]d\mu(t)\\
&=\frac{1}{6}\left[2\Phi''\left(\int_{\Omega}fd\mu\right)+\int_{\Omega}\Phi''(f(t))\,d\mu(t)\right]\\
&\leq\frac{1}{6}\left[2\int_{\Omega}\Phi''(f(t))\,d\mu(t)+\int_{\Omega}\Phi''(f(t))\,d\mu(t)\right]=\frac{1}{2}\int_{\Omega}\Phi''\circ fd\mu.
\end{aligned}$$

This together with (2.5) proves the last two inequalities in (2.22). □

We recall that a function $f:I\to\mathbb{R}$ is called *quasiconvex* on the interval I if

$$f((1-s)\,x+sy)\leq\max\{f(x),f(y)\}$$

for all $x,\,y\in I$ and $s\in[0,1]$.

Corollary 5. *With the assumptions of Theorem 3 and if $\Phi''(\cdot)$ is quasiconvex on (m,M), then*

$$\begin{aligned}
&\int_{\Omega}\Phi\circ fd\mu-\Phi\left(\int_{\Omega}fd\mu\right)\\
&\leq\frac{1}{4}\operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_{\Omega}fd\mu\right)^{2}\\
&\times\left[\Phi''\left(\int_{\Omega}fd\mu\right)+\int_{\Omega}\Phi''\circ fd\mu+\int_{\Omega}\left|\Phi''(f(t))-\Phi''\left(\int_{\Omega}fd\mu\right)\right|d\mu(t)\right].
\end{aligned}\tag{2.24}$$

Proof. Since $\Phi''(\cdot)$ is quasiconvex, hence

$$\begin{aligned}
&\int_0^1 \Phi''\left((1-s)\int_\Omega f d\mu + sf(t)\right)(1-s)\,ds \\
&\leq \max\left\{\Phi''\left(\int_\Omega f d\mu\right), \Phi''(f(t))\right\}\int_0^1 (1-s)\,ds \\
&= \frac{1}{2}\max\left\{\Phi''\left(\int_\Omega f d\mu\right), \Phi''(f(t))\right\} \\
&= \frac{1}{4}\left[\Phi''\left(\int_\Omega f d\mu\right) + \Phi''(f(t)) + \left|\Phi''(f(t)) - \Phi''\left(\int_\Omega f d\mu\right)\right|\right]
\end{aligned}$$

for $t \in \Omega$.

Taking the integral $\int_\Omega$ in this inequality, we get

$$\begin{aligned}
&\int_\Omega \left(\int_0^1 \Phi''\left((1-s)\int_\Omega f d\mu + sf(t)\right)(1-s)\,ds\right) d\mu(t) \\
&\leq \frac{1}{4}\int_\Omega \left[\Phi''\left(\int_\Omega f d\mu\right) + \Phi''(f(t)) + \left|\Phi''(f(t)) - \Phi''\left(\int_\Omega f d\mu\right)\right|\right] d\mu(t) \\
&= \frac{1}{4}\left[\Phi''\left(\int_\Omega f d\mu\right) + \int_\Omega \Phi''(f(t))\,d\mu(t)\right] \\
&+ \frac{1}{4}\int_\Omega \left|\Phi''(f(t)) - \Phi''\left(\int_\Omega f d\mu\right)\right| d\mu(t),
\end{aligned}$$

which together with (2.5), produce the desired result (2.24). □

3. Exponential Integral Inequalities

We consider the convex function $\Phi_\alpha : \mathbb{R} \to (0,\infty)$ defined by $\Phi_\alpha(t) = \exp(\alpha x)$. We have $\Phi''_\alpha(t) = \alpha^2 \exp(\alpha x)$ which shows that Φ''_α is convex and monotonic decreasing for $\alpha < 0$ and increasing for $\alpha > 0$.

We then have for $t \in [m, M]$ that

$$\begin{aligned}
&E_1(\alpha, [m, M]) \\
&:= \alpha^2 \begin{cases} \exp(\alpha M), & \alpha < 0 \\ \\ \exp(\alpha m), & \alpha > 0 \end{cases} \leq \Phi''_\alpha(t) \leq \alpha^2 \begin{cases} \exp(\alpha m), & \alpha < 0 \\ \\ \exp(\alpha M), & \alpha > 0 \end{cases} \\
&:= E_2(\alpha, [m, M]).
\end{aligned}$$

If $f : \Omega \to [m, M]$ is so that f, f^2, $\exp \circ f \in L(\Omega, \mu)$, then by (2.10) we get

$$\begin{aligned}
0 &\leq \frac{1}{2}E_1(\alpha, [m, M]) \operatorname*{essinf}_{t\in\Omega}\left(f(t) - \int_\Omega f d\mu\right)^2 \\
&\leq \int_\Omega \exp(\alpha f)\,d\mu - \exp\left(\alpha\int_\Omega f d\mu\right) \\
&\leq \frac{1}{2}E_2(\alpha, [m, M]) \operatorname*{essup}_{t\in\Omega}\left(f(t) - \int_\Omega f d\mu\right)^2. \qquad (3.1)
\end{aligned}$$

If $\alpha > 0$, then Φ''_α is increasing and by (2.16) we get

$$\begin{aligned}
(3.2)\qquad 0 &\le \frac{1}{\int_\Omega f d\mu - m}\left\{\alpha\exp\left(\alpha\int_\Omega f d\mu\right) - \frac{\exp\left(\alpha\int_\Omega f d\mu\right) - \exp(\alpha m)}{\int_\Omega f d\mu - m}\right\}\\
&\times\left[\operatorname*{essinf}_{t\in\Omega}\left(f(t) - \int_\Omega f d\mu\right)^2\right]\\
&\le \int_\Omega \exp(\alpha f)\, d\mu - \exp\left(\alpha\int_\Omega f d\mu\right)\\
&\le \frac{1}{M - \int_\Omega f d\mu}\left[\frac{\exp(\alpha M) - \exp\left(\alpha\int_\Omega f d\mu\right)}{M - \int_\Omega f d\mu} - \alpha\exp\left(\alpha\int_\Omega f d\mu\right)\right]\\
&\times\left[\operatorname*{essup}_{t\in\Omega}\left(f(t) - \int_\Omega f d\mu\right)^2\right].
\end{aligned}$$

If $\alpha < 0$, then Φ''_α is decreasing and by (2.17) we get

$$\begin{aligned}
(3.3)\qquad 0 &\le \frac{1}{M - \int_\Omega f d\mu}\left[\frac{\exp(\alpha M) - \exp\left(\alpha\int_\Omega f d\mu\right)}{M - \int_\Omega f d\mu} - \alpha\exp\left(\alpha\int_\Omega f d\mu\right)\right]\\
&\times\left[\operatorname*{essinf}_{t\in\Omega}\left(f(t) - \int_\Omega f d\mu\right)^2\right]
\end{aligned}$$

$$\begin{aligned}
&\le \int_\Omega \exp(\alpha f)\, d\mu - \exp\left(\alpha\int_\Omega f d\mu\right)\\
&\le \frac{1}{\int_\Omega f d\mu - m}\left\{\alpha\exp\left(\alpha\int_\Omega f d\mu\right) - \frac{\exp\left(\alpha\int_\Omega f d\mu\right) - \exp(\alpha m)}{\int_\Omega f d\mu - m}\right\}\\
&\times\left[\operatorname*{essup}_{t\in\Omega}\left(f(t) - \int_\Omega f d\mu\right)^2\right].
\end{aligned}$$

Also, if $\alpha > 0$ then by (2.16) we have

$$\begin{aligned}
(3.4)\qquad 0 &\le \operatorname*{essinf}_{t\in\Omega}\left(f(t) - \int_\Omega f d\mu\right)^2\\
&\times\int_\Omega \frac{1}{f(t) - m}\left\{\frac{\exp(\alpha f(t)) - \exp(\alpha m)}{f(t) - m} - \alpha\exp(\alpha m)\right\} d\mu(t)\\
&\le \int_\Omega \exp(\alpha f)\, d\mu - \exp\left(\alpha\int_\Omega f d\mu\right)\\
&\le \operatorname*{essup}_{t\in\Omega}\left(f(t) - \int_\Omega f d\mu\right)^2\\
&\times\int_\Omega \frac{1}{M - f(t)}\left[\alpha\exp(\alpha M) - \frac{\exp(\alpha M) - \exp(\alpha f(t))}{M - f(t)}\right] d\mu(t),
\end{aligned}$$

while for $\alpha < 0$ we have, by (2.17) that

$$\begin{aligned} (3.5) \qquad & 0 \le \operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_\Omega fd\mu\right)^2 \\ & \times \int_\Omega \frac{1}{M-f(t)}\left[\alpha\exp(\alpha M)-\frac{\exp(\alpha M)-\exp(\alpha f(t))}{M-f(t)}\right]d\mu(t) \\ & \le \int_\Omega \exp(\alpha f)\,d\mu-\exp\left(\alpha\int_\Omega fd\mu\right) \\ & \le \operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_\Omega fd\mu\right)^2 \\ & \times \int_\Omega \frac{1}{f(t)-m}\left\{\frac{\exp(\alpha f(t))-\exp(\alpha m)}{f(t)-m}-\alpha\exp(\alpha m)\right\}d\mu(t). \end{aligned}$$

Since the function Φ_α'' is convex, then by (2.22) we have for all $\alpha \neq 0$ that

$$\begin{aligned} (3.6) \qquad & 0 \le \alpha^2 \operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_\Omega fd\mu\right)^2 \exp\left(\alpha\int_\Omega fd\mu\right) \\ & \le \alpha^2 \operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_\Omega fd\mu\right)^2 \\ & \times \int_\Omega \exp\left(\frac{2}{3}\alpha\int_\Omega fd\mu+\frac{1}{3}\alpha f(t)\right)d\mu(t) \\ & \le \int_\Omega \exp(\alpha f)\,d\mu-\exp\left(\alpha\int_\Omega fd\mu\right) \\ & \le \alpha^2 \operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_\Omega fd\mu\right)^2 \\ & \times \left[\frac{2\exp\left(\alpha\int_\Omega fd\mu\right)+\int_\Omega \exp(\alpha f)\,d\mu}{3}\right] \\ & \le \alpha^2 \operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_\Omega fd\mu\right)^2 \int_\Omega \exp(\alpha f)\,d\mu. \end{aligned}$$

This inequality provided the following simple and nice result

$$\begin{aligned} (3.7) \qquad & 0 \le \alpha^2 \operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_\Omega fd\mu\right)^2 \le 1-\frac{\exp\left(\alpha\int_\Omega fd\mu\right)}{\int_\Omega \exp(\alpha f)\,d\mu} \\ & \le \alpha^2 \operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_\Omega fd\mu\right)^2. \end{aligned}$$

4. Logarithmic Integral Inequalities

We consider the convex function $\Phi : (0,\infty) \to \mathbb{R}$, $\Phi(x) = -\ln x$. Then $\Phi''(x) = \frac{1}{x^2}$, which is convex and decreasing on $(0,\infty)$. If $x \in [m, M] \subset (0,\infty)$, then we also have

$$\frac{1}{M^2} \le \Phi''(x) \le \frac{1}{m^2}.$$

If $f:\Omega\rightarrow[m,M]$ is so that $f,\ f^{2},\ \ln\circ f\in L(\Omega,\mu)$, then by (2.10) we get

$$0\leq\frac{1}{M^{2}}\left[\operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_{\Omega}fd\mu\right)^{2}\right]\leq\ln\left(\int_{\Omega}fd\mu\right)-\int_{\Omega}\ln(f)\,d\mu \tag{4.1}$$
$$\leq\frac{1}{m^{2}}\left[\operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_{\Omega}fd\mu\right)^{2}\right].$$

Since Φ'' is decreasing, hence by (2.12) we get

$$0\leq\frac{1}{M-\int_{\Omega}fd\mu}\left[\left(\int_{\Omega}fd\mu\right)^{-1}-\frac{\ln(M)-\ln\left(\int_{\Omega}fd\mu\right)}{M-\int_{\Omega}fd\mu}\right] \tag{4.2}$$
$$\times\left[\operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_{\Omega}fd\mu\right)^{2}\right]$$
$$\leq\ln\left(\int_{\Omega}fd\mu\right)-\int_{\Omega}\ln(f)\,d\mu$$
$$\leq\frac{1}{\int_{\Omega}fd\mu-m}\left\{\frac{\ln\left(\int_{\Omega}fd\mu\right)-\ln(m)}{\int_{\Omega}fd\mu-m}-\left(\int_{\Omega}fd\mu\right)^{-1}\right\}$$
$$\times\left[\operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_{\Omega}fd\mu\right)^{2}\right].$$

Also, by (2.17) we get

$$0\leq\operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_{\Omega}fd\mu\right)^{2} \tag{4.3}$$
$$\times\int_{\Omega}\frac{1}{M-f(t)}\left[\frac{\ln(M)-\ln(f(t))}{M-f(t)}-\frac{1}{M}\right]d\mu(t)$$
$$\leq\ln\left(\int_{\Omega}fd\mu\right)-\int_{\Omega}\ln(f)\,d\mu$$
$$\leq\operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_{\Omega}fd\mu\right)^{2}$$
$$\times\int_{\Omega}\frac{1}{f(t)-m}\left\{\frac{1}{m}-\frac{\ln(f(t))-\ln(m)}{f(t)-m}\right\}d\mu(t).$$

Since Φ'' is convex, then by (2.22) we have

$$0 \le \frac{1}{2} \operatorname*{essinf}_{t\in\Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \left(\int_\Omega f d\mu \right)^{-2} \tag{4.4}$$
$$\le \frac{1}{2} \operatorname*{essinf}_{t\in\Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \int_\Omega \left(\frac{2}{3} \int_\Omega f d\mu + \frac{1}{3} f(t) \right)^{-2} d\mu(t)$$
$$\le \ln \left(\int_\Omega f d\mu \right) - \int_\Omega \ln(f) d\mu$$
$$\le \frac{1}{2} \operatorname*{essup}_{t\in\Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \left[\frac{2 \left(\int_\Omega f d\mu \right)^{-2} + \int_\Omega f^{-2} d\mu}{3} \right]$$
$$\le \frac{1}{2} \operatorname*{essup}_{t\in\Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \int_\Omega f^{-2} d\mu.$$

We consider the convex function $\Phi : (0, \infty) \to \mathbb{R}$, $\Phi(x) = x \ln x$. Then $\Phi''(x) = \frac{1}{x}$, which is convex and decreasing on $(0, \infty)$. If $x \in [m, M] \subset (0, \infty)$ then we also have

$$\frac{1}{M} \le \Phi''(x) \le \frac{1}{m}.$$

If $f : \Omega \to [m, M]$ is so that f, f^2, $f \ln \circ f \in L(\Omega, \mu)$, then by (2.10) we get

$$0 \le \frac{1}{2M} \left[\operatorname*{essinf}_{t\in\Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \right] \tag{4.5}$$
$$\le \int_\Omega f \ln(f) d\mu - \left(\int_\Omega f d\mu \right) \ln \left(\int_\Omega f d\mu \right)$$
$$\le \frac{1}{2m} \left[\operatorname*{essup}_{t\in\Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \right].$$

Since Φ'' is decreasing, hence by (2.12) we get

$$0 \le \frac{1}{M - \int_\Omega f d\mu} \left[\frac{M \ln(M) - \left(\int_\Omega f d\mu \right) \ln \left(\int_\Omega f d\mu \right)}{M - \int_\Omega f d\mu} - \ln \left(\int_\Omega f d\mu \right) - 1 \right] \tag{4.6}$$
$$\times \left[\operatorname*{essinf}_{t\in\Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \right]$$
$$\le \int_\Omega f \ln(f) d\mu - \left(\int_\Omega f d\mu \right) \ln \left(\int_\Omega f d\mu \right)$$
$$\le \frac{1}{\int_\Omega f d\mu - m} \left\{ \ln \left(\int_\Omega f d\mu \right) + 1 - \frac{\left(\int_\Omega f d\mu \right) \ln \left(\int_\Omega f d\mu \right) - m \ln(m)}{\int_\Omega f d\mu - m} \right\}$$
$$\times \left[\operatorname*{essup}_{t\in\Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \right].$$

Since Φ'' is convex, then by (2.22) we have

$$
\begin{aligned}
(4.7)\qquad 0 &\leq \frac{1}{2}\operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2\left(\int_\Omega f d\mu\right)^{-1} \\
&\leq \frac{1}{2}\operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2\int_\Omega\left(\frac{2}{3}\int_\Omega f d\mu+\frac{1}{3}f(t)\right)^{-1}d\mu(t) \\
&\leq \int_\Omega f\ln(f)\,d\mu-\left(\int_\Omega f d\mu\right)\ln\left(\int_\Omega f d\mu\right) \\
&\leq \frac{1}{2}\operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2\left[\frac{2\left(\int_\Omega f d\mu\right)^{-1}+\int_\Omega f^{-1}d\mu}{3}\right] \\
&\leq \frac{1}{2}\operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2\int_\Omega f^{-1}d\mu.
\end{aligned}
$$

5. Power Inequalities

The power function $\Phi_p:(0,\infty)\to(0,\infty)$, $\Phi_p(x)=x^p$ is convex for $p\in(-\infty,0)\cup[1,\infty)$. We have $\Phi_p''(x)=p(p-1)x^{p-2}$, $x\in(0,\infty)$. If $p\in(-\infty,0)\cup[1,2)$ then Φ_p'' is decreasing and if $p\in[2,\infty)$ then Φ_p'' is increasing.

For $p\in(-\infty,0)\cup[1,\infty)$ and $[m,M]\subset(0,\infty)$ we define the bounds

$$
(5.1)\qquad B_1(p,[m,M]) := p(p-1)\begin{cases} M^{p-2} \text{ if } (-\infty,0)\cup[1,2), \\ m^{p-2} \text{ if } p\in[2,\infty) \end{cases}
$$

and

$$
(5.2)\qquad B_2(p,[m,M]) := p(p-1)\begin{cases} m^{p-2} \text{ if } (-\infty,0)\cup[1,2), \\ M^{p-2} \text{ if } p\in[2,\infty). \end{cases}
$$

Then we have

$$
B_1(p,[m,M]) \leq \Phi_p''(x) \leq B_2(p,[m,M]) \text{ for } x\in[m,M].
$$

If $f:\Omega\to[m,M]$ is so that $f,\ f^2,\ f^p\in L(\Omega,\mu)$, then by (2.10) we get

$$
\begin{aligned}
(5.3)\qquad 0 &\leq \frac{1}{2}B_1(p,[m,M])\left[\operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2\right] \\
&\leq \int_\Omega f^p d\mu-\left(\int_\Omega f d\mu\right)^p \\
&\leq \frac{1}{2}B_2(p,[m,M])\left[\operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2\right].
\end{aligned}
$$

If $p \in (-\infty, 0) \cup [1, 2)$ then Φ_p'' is decreasing and by (2.12) we have

$$\begin{aligned} (5.4) \qquad 0 &\leq \frac{1}{M - \int_\Omega f d\mu} \left[\frac{M^p - \left(\int_\Omega f d\mu\right)^p}{M - \int_\Omega f d\mu} - p \left(\int_\Omega f d\mu \right)^{p-1} \right] \\ &\times \left[\operatorname*{essinf}_{t \in \Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \right] \\ &\leq \int_\Omega f^p d\mu - \left(\int_\Omega f d\mu \right)^p \\ &\leq \frac{1}{\int_\Omega f d\mu - m} \left\{ p \left(\int_\Omega f d\mu \right)^{p-1} - \frac{\left(\int_\Omega f d\mu\right)^p - m^p}{\int_\Omega f d\mu - m} \right\} \\ &\times \left[\operatorname*{essup}_{t \in \Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \right]. \end{aligned}$$

If $p \in [2, \infty)$ then Φ_p'' is increasing and by (2.16) we get

$$\begin{aligned} (5.5) \qquad 0 &\leq \frac{1}{\int_\Omega f d\mu - m} \left\{ p \left(\int_\Omega f d\mu \right)^{p-1} - \frac{\left(\int_\Omega f d\mu\right)^p - m^p}{\int_\Omega f d\mu - m} \right\} \\ &\times \left[\operatorname*{essinf}_{t \in \Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \right] \\ &\leq \int_\Omega f^p d\mu - \left(\int_\Omega f d\mu \right)^p \\ &\leq \frac{1}{M - \int_\Omega f d\mu} \left[\frac{M^p - \left(\int_\Omega f d\mu\right)^p}{M - \int_\Omega f d\mu} - p \left(\int_\Omega f d\mu \right)^{p-1} \right] \\ &\times \left[\operatorname*{essup}_{t \in \Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \right]. \end{aligned}$$

If $p \in (-\infty, 0) \cup [1, 2] \cup [3, \infty)$, then Φ_p'' is convex and by (2.22) we get

$$\begin{aligned} (5.6) \quad 0 &\leq \frac{1}{2} p (p-1) \operatorname*{essinf}_{t \in \Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \left(\int_\Omega f d\mu \right)^{p-2} \\ &\leq \frac{1}{2} p (p-1) \operatorname*{essinf}_{t \in \Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \int_\Omega \left(\frac{2}{3} \int_\Omega f d\mu + \frac{1}{3} f(t) \right)^{p-2} d\mu(t) \\ &\leq \int_\Omega f^p d\mu - \left(\int_\Omega f d\mu \right)^p \\ &\leq \frac{1}{2} p (p-1) \operatorname*{essup}_{t \in \Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \left[\frac{2 \left(\int_\Omega f d\mu\right)^{p-2} + \int_\Omega f^{p-2} d\mu}{3} \right] \\ &\leq \frac{1}{2} p (p-1) \operatorname*{essup}_{t \in \Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \int_\Omega f^{p-2} d\mu. \end{aligned}$$

If $p \in (2,3)$, then Φ_p'' is concave and by (2.23) we get

$$0 \leq \frac{1}{2} p (p-1) \operatorname*{essinf}_{t \in \Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \int_\Omega f^{p-2} d\mu \tag{5.7}$$
$$\leq \frac{1}{2} p (p-1) \operatorname*{essinf}_{t \in \Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \left[\frac{2 \left(\int_\Omega f d\mu \right)^{p-2} + \int_\Omega f^{p-2} d\mu}{3} \right]$$
$$\leq \int_\Omega f^p d\mu - \left(\int_\Omega f d\mu \right)^p$$
$$\leq \frac{1}{2} p (p-1) \operatorname*{essup}_{t \in \Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \int_\Omega \left(\frac{2}{3} \int_\Omega f d\mu + \frac{1}{3} f(t) \right)^{p-2} d\mu(t)$$
$$\leq \frac{1}{2} p (p-1) \operatorname*{essup}_{t \in \Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \left(\int_\Omega f d\mu \right)^{p-2}.$$

6. Discrete Case

The discrete case is useful for applications and we will sate the corresponding inequalities here.

In this section the function $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ is a twice differentiable convex function on (m, M) and $x_k \in [m, M]$, $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^n p_k = 1$.

By the inequality (2.5) we have

$$0 \leq \min_{k \in \{1,\ldots,n\}} \left(x_k - \sum_{i=1}^n p_i x_i \right)^2 \tag{6.1}$$
$$\times \sum_{k=1}^n p_k \left(\int_0^1 \Phi'' \left((1-s) \sum_{i=1}^n p_i x_i + s x_k \right) (1-s) ds \right)$$
$$\leq \sum_{i=1}^n p_i \Phi(x_i) - \Phi\left(\sum_{i=1}^n p_i x_i \right)$$
$$\leq \max_{k \in \{1,\ldots,n\}} \left(x_k - \sum_{i=1}^n p_i x_i \right)^2$$
$$\times \sum_{k=1}^n p_k \left(\int_0^1 \Phi'' \left((1-s) \sum_{i=1}^n p_i x_i + s x_k \right) (1-s) ds \right).$$

If there exist $0 < \gamma < \Gamma < \infty$ such that

$$\gamma \leq \Phi''(t) \leq \Gamma \text{ for almost every } t \in [m, M]$$

then,

$$0 \leq \frac{1}{2} \gamma \min_{k \in \{1,\ldots,n\}} \left(x_k - \sum_{i=1}^n p_i x_i \right)^2 \leq \sum_{i=1}^n p_i \Phi(x_i) - \Phi\left(\sum_{i=1}^n p_i x_i \right) \tag{6.2}$$
$$\leq \frac{1}{2} \Gamma \max_{k \in \{1,\ldots,n\}} \left(x_k - \sum_{i=1}^n p_i x_i \right)^2.$$

If $\Phi''(\cdot)$ is monotonic nondecreasing on (m, M), then by (2.11)

$$0 \leq \frac{1}{\sum_{i=1}^{n} p_i x_i - m} \left\{ \Phi'\left(\sum_{i=1}^{n} p_i x_i \right) - \frac{\Phi\left(\sum_{i=1}^{n} p_i x_i\right) - \Phi(m)}{\sum_{i=1}^{n} p_i x_i - m} \right\} \times \min_{k \in \{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \leq \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} p_i x_i \right) \leq \frac{1}{M - \sum_{i=1}^{n} p_i x_i} \left[\frac{\Phi(M) - \Phi\left(\sum_{i=1}^{n} p_i x_i\right)}{M - \sum_{i=1}^{n} p_i x_i} - \Phi'\left(\sum_{i=1}^{n} p_i x_i \right) \right] \times \max_{k \in \{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 . \tag{6.3}$$

If $\Phi''(\cdot)$ is monotonic nonincreasing on (m, M), then by (2.12)

$$0 \leq \frac{1}{M - \sum_{i=1}^{n} p_i x_i} \left[\frac{\Phi(M) - \Phi\left(\sum_{i=1}^{n} p_i x_i\right)}{M - \sum_{i=1}^{n} p_i x_i} - \Phi'\left(\sum_{i=1}^{n} p_i x_i \right) \right] \times \min_{k \in \{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \leq \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} p_i x_i \right) \leq \frac{1}{\sum_{i=1}^{n} p_i x_i - m} \left\{ \Phi'\left(\sum_{i=1}^{n} p_i x_i \right) - \frac{\Phi\left(\sum_{i=1}^{n} p_i x_i\right) - \Phi(m)}{\sum_{i=1}^{n} p_i x_i - m} \right\} \times \max_{k \in \{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 . \tag{6.4}$$

Also, if $\Phi''(\cdot)$ is monotonic nondecreasing on (m, M), then by (2.16)

$$0 \leq \min_{k \in \{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \sum_{i=1}^{n} \frac{p_i}{x_i - m} \left\{ \frac{\Phi(x_i) - \Phi(m)}{x_i - m} - \Phi'(m) \right\} \leq \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} p_i x_i \right) \leq \max_{k \in \{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \sum_{i=1}^{n} \frac{p_i}{M - x_i} \left[\Phi'(M) - \frac{\Phi(M) - \Phi(x_i)}{M - x_i} \right] . \tag{6.5}$$

If $\Phi''(\cdot)$ is monotonic nonincreasing on (m,M), then by (2.17)

$$
\begin{aligned}
(6.6)\quad 0 &\le \min_{k\in\{1,\dots,n\}} \left(x_k-\sum_{i=1}^n p_i x_i\right)^2 \sum_{i=1}^n \frac{p_i}{M-x_i}\left[\Phi'(M)-\frac{\Phi(M)-\Phi(x_i)}{M-x_i}\right] \\
&\le \sum_{i=1}^n p_i\Phi(x_i)-\Phi\left(\sum_{i=1}^n p_i x_i\right) \\
&\le \max_{k\in\{1,\dots,n\}} \left(x_k-\sum_{i=1}^n p_i x_i\right)^2 \sum_{i=1}^n \frac{p_i}{x_i-m}\left\{\frac{\Phi(x_i)-\Phi(m)}{x_i-m}-\Phi'(m)\right\}.
\end{aligned}
$$

If $\Phi''(\cdot)$ is convex on (m,M), then by (2.22) we have

$$
\begin{aligned}
(6.7)\quad 0 &\le \frac{1}{2}\min_{k\in\{1,\dots,n\}} \left(x_k-\sum_{i=1}^n p_i x_i\right)^2 \Phi''\left(\sum_{i=1}^n p_i x_i\right) \\
&\le \frac{1}{2}\min_{k\in\{1,\dots,n\}} \left(x_k-\sum_{i=1}^n p_i x_i\right)^2 \sum_{k=1}^n p_k \Phi''\left(\frac{2}{3}\sum_{i=1}^n p_i x_i+\frac{1}{3}x_k\right) \\
&\le \sum_{i=1}^n p_i\Phi(x_i)-\Phi\left(\sum_{i=1}^n p_i x_i\right) \\
&\le \frac{1}{2}\max_{k\in\{1,\dots,n\}} \left(x_k-\sum_{i=1}^n p_i x_i\right)^2 \left[\frac{2\Phi''\left(\sum_{i=1}^n p_i x_i\right)+\sum_{i=1}^n p_i\Phi''(x_i)}{3}\right] \\
&\le \frac{1}{2}\max_{k\in\{1,\dots,n\}} \left(x_k-\sum_{i=1}^n p_i x_i\right)^2 \sum_{i=1}^n p_i\Phi''(x_i).
\end{aligned}
$$

If $\Phi''(\cdot)$ is concave on (m,M), then by (2.23)

$$
\begin{aligned}
(6.8)\quad 0 &\le \frac{1}{2}\min_{k\in\{1,\dots,n\}} \left(x_k-\sum_{i=1}^n p_i x_i\right)^2 \sum_{i=1}^n p_i\Phi''(x_i) \\
&\le \frac{1}{2}\min_{k\in\{1,\dots,n\}} \left(x_k-\sum_{i=1}^n p_i x_i\right)^2 \left[\frac{2\Phi''\left(\sum_{i=1}^n p_i x_i\right)+\sum_{i=1}^n p_i\Phi''(x_i)}{3}\right] \\
&\le \sum_{i=1}^n p_i\Phi(x_i)-\Phi\left(\sum_{i=1}^n p_i x_i\right) \\
&\le \frac{1}{2}\max_{k\in\{1,\dots,n\}} \left(x_k-\sum_{i=1}^n p_i x_i\right)^2 \sum_{k=1}^n p_k \Phi''\left(\frac{2}{3}\sum_{i=1}^n p_i x_i+\frac{1}{3}x_k\right) \\
&\le \frac{1}{2}\max_{k\in\{1,\dots,n\}} \left(x_k-\sum_{i=1}^n p_i x_i\right)^2 \Phi''\left(\sum_{i=1}^n p_i x_i\right).
\end{aligned}
$$

If $\Phi''(\cdot)$ is quasiconvex on (m, M), then by (2.24)

$$\begin{aligned}
&\sum_{i=1}^{n} p_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} p_i x_i\right) \\
&\leq \frac{1}{4} \max_{k\in\{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i\right)^2 \\
&\times \left[\Phi''\left(\sum_{i=1}^{n} p_i x_i\right) + \sum_{i=1}^{n} p_i \Phi''(x_i) + \sum_{k=1}^{n} p_k \left|\Phi''(x_k) - \Phi''\left(\sum_{i=1}^{n} p_i x_i\right)\right|\right].
\end{aligned} \tag{6.9}$$

If $x_k \in [m, M] \subset \mathbb{R}$, $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$, then

$$\begin{aligned}
0 &\leq \frac{1}{2} E_1(\alpha, [m, M]) \min_{k\in\{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i\right)^2 \\
&\leq \sum_{i=1}^{n} p_i \exp(\alpha x_i) - \exp\left(\alpha \sum_{i=1}^{n} p_i x_i\right) \\
&\leq \frac{1}{2} E_2(\alpha, [m, M]) \max_{k\in\{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i\right)^2.
\end{aligned} \tag{6.10}$$

If $\alpha > 0$, then

$$\begin{aligned}
0 &\leq \frac{1}{\sum_{i=1}^{n} p_i x_i - m} \min_{k\in\{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i\right)^2 \\
&\times \left\{\alpha \exp\left(\alpha \sum_{i=1}^{n} p_i x_i\right) - \frac{\exp\left(\alpha \sum_{i=1}^{n} p_i x_i\right) - \exp(\alpha m)}{\sum_{i=1}^{n} p_i x_i - m}\right\} \\
&\leq \sum_{i=1}^{n} p_i \exp(\alpha x_i) - \exp\left(\alpha \sum_{i=1}^{n} p_i x_i\right) \\
&\leq \frac{1}{M - \sum_{i=1}^{n} p_i x_i} \max_{k\in\{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i\right)^2 \\
&\left[\frac{\exp(\alpha M) - \exp\left(\alpha \sum_{i=1}^{n} p_i x_i\right)}{M - \sum_{i=1}^{n} p_i x_i} - \alpha \exp\left(\alpha \sum_{i=1}^{n} p_i x_i\right)\right].
\end{aligned} \tag{6.11}$$

If $\alpha < 0$, then

$$\begin{aligned}
0 &\leq \frac{1}{M - \sum_{i=1}^{n} p_i x_i} \min_{k\in\{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i\right)^2 \\
&\times \left[\frac{\exp(\alpha M) - \exp\left(\alpha \sum_{i=1}^{n} p_i x_i\right)}{M - \sum_{i=1}^{n} p_i x_i} - \alpha \exp\left(\alpha \sum_{i=1}^{n} p_i x_i\right)\right]
\end{aligned} \tag{6.12}$$

$$\leq \sum_{i=1}^{n} p_i \exp(\alpha x_i) - \exp\left(\alpha \sum_{i=1}^{n} p_i x_i\right)$$
$$\leq \frac{1}{\sum_{i=1}^{n} p_i x_i - m} \max_{k\in\{1,\ldots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i\right)^2$$
$$\times \left\{\alpha \exp\left(\alpha \sum_{i=1}^{n} p_i x_i\right) - \frac{\exp\left(\alpha \sum_{i=1}^{n} p_i x_i\right) - \exp(\alpha m)}{\sum_{i=1}^{n} p_i x_i - m}\right\}$$

Also, if $\alpha > 0$ then

$$0 \leq \min_{k\in\{1,\ldots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i\right)^2 \tag{6.13}$$
$$\times \sum_{i=1}^{n} \frac{p_i}{x_i - m} \left\{\frac{\exp(\alpha x_i) - \exp(\alpha m)}{x_i - m} - \alpha \exp(\alpha m)\right\}$$
$$\leq \sum_{i=1}^{n} p_i \exp(\alpha x_i) - \exp\left(\alpha \sum_{i=1}^{n} p_i x_i\right)$$
$$\leq \max_{k\in\{1,\ldots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i\right)^2$$
$$\times \sum_{i=1}^{n} \frac{p_i}{M - x_i} \left[\alpha \exp(\alpha M) - \frac{\exp(\alpha M) - \exp(\alpha x_i)}{M - x_i}\right],$$

while for $\alpha < 0$ we have,

$$0 \leq \min_{k\in\{1,\ldots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i\right)^2 \tag{6.14}$$
$$\times \sum_{i=1}^{n} \frac{p_i}{M - x_i} \left[\alpha \exp(\alpha M) - \frac{\exp(\alpha M) - \exp(\alpha x_i)}{M - x_i}\right]$$
$$\leq \sum_{i=1}^{n} p_i \exp(\alpha x_i) - \exp\left(\alpha \sum_{i=1}^{n} p_i x_i\right)$$
$$\leq \max_{k\in\{1,\ldots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i\right)^2$$
$$\times \sum_{i=1}^{n} \frac{p_i}{x_i - m} \left\{\frac{\exp(\alpha x_i) - \exp(\alpha m)}{x_i - m} - \alpha \exp(\alpha m)\right\}.$$

Since the function Φ_{α}'' is convex, then for all $\alpha \neq 0$

$$\begin{aligned}
(6.15)\qquad 0 &\leq \alpha^2 \min_{k\in\{1,...,n\}} \left(x_k - \sum_{i=1}^n p_i x_i\right)^2 \exp\left(\alpha \sum_{i=1}^n p_i x_i\right) \\
&\leq \alpha^2 \min_{k\in\{1,...,n\}} \left(x_k - \sum_{i=1}^n p_i x_i\right)^2 \\
&\times \sum_{i=1}^n p_k \exp\left(\frac{2}{3}\alpha \sum_{i=1}^n p_i x_i + \frac{1}{3}\alpha x_k\right) \\
&\leq \sum_{i=1}^n p_i \exp(\alpha x_i) - \exp\left(\alpha \sum_{i=1}^n p_i x_i\right) \\
&\leq \alpha^2 \max_{k\in\{1,...,n\}} \left(x_k - \sum_{i=1}^n p_i x_i\right)^2 \\
&\times \left[\frac{2\exp\left(\alpha \sum_{i=1}^n p_i x_i\right) + \sum_{i=1}^n p_i \exp(\alpha x_i)}{3}\right] \\
&\leq \alpha^2 \max_{k\in\{1,...,n\}} \left(x_k - \sum_{i=1}^n p_i x_i\right)^2 \sum_{i=1}^n p_i \exp(\alpha x_i).
\end{aligned}$$

This inequality provides the following simple and nice result

$$\begin{aligned}
(6.16)\qquad 0 &\leq \alpha^2 \min_{k\in\{1,...,n\}} \left(x_k - \sum_{i=1}^n p_i x_i\right)^2 \leq 1 - \frac{\exp\left(\alpha \sum_{i=1}^n p_i x_i\right)}{\sum_{i=1}^n p_i \exp(\alpha x_i)} \\
&\leq \alpha^2 \max_{k\in\{1,...,n\}} \left(x_k - \sum_{i=1}^n p_i x_i\right)^2.
\end{aligned}$$

If in (6.16) we take $\alpha = 1$, $x_k := \ln y_k$, with $y_k > 0$ for $k \in \{1, ..., n\}$, then we get

$$\begin{aligned}
(6.17)\qquad 0 &\leq \min_{k\in\{1,...,n\}} \left(\ln\left(\frac{y_k}{\prod_{i=1}^n y_i^{p_i}}\right)\right)^2 \leq 1 - \frac{\prod_{i=1}^n y_i^{p_i}}{\sum_{i=1}^n p_i y_i} \\
&\leq \max_{k\in\{1,...,n\}} \left(\ln\left(\frac{y_k}{\prod_{i=1}^n y_i^{p_i}}\right)\right)^2.
\end{aligned}$$

If $x_k \in [m, M] \subset (0, \infty)$, $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^n p_k = 1$, then

$$\begin{aligned}
0 &\leq \frac{1}{M^2} \min_{k\in\{1,...,n\}} \left(x_k - \sum_{i=1}^n p_i x_i\right)^2 \leq \ln\left(\sum_{i=1}^n p_i x_i\right) - \ln\left(\prod_{i=1}^n x_i^{p_i}\right) \\
&\leq \frac{1}{m^2} \max_{k\in\{1,...,n\}} \left(x_k - \sum_{i=1}^n p_i x_i\right)^2,
\end{aligned}$$

namely

$$0 \leq \exp\left[\frac{1}{M^2}\min_{k\in\{1,\dots,n\}}\left(x_k - \sum_{i=1}^n p_i x_i\right)^2\right] \leq \frac{\sum_{i=1}^n p_i x_i}{\prod_{i=1}^n x_i^{p_i}} \leq \exp\left[\frac{1}{m^2}\max_{k\in\{1,\dots,n\}}\left(x_k - \sum_{i=1}^n p_i x_i\right)^2\right]. \tag{6.18}$$

Also,

$$\begin{aligned} 0 &\leq \frac{1}{M - \sum_{i=1}^n p_i x_i}\left[\left(\sum_{i=1}^n p_i x_i\right)^{-1} - \frac{\ln(M) - \ln\left(\sum_{i=1}^n p_i x_i\right)}{M - \sum_{i=1}^n p_i x_i}\right] \\ &\times \min_{k\in\{1,\dots,n\}}\left(x_k - \sum_{i=1}^n p_i x_i\right)^2 \\ &\leq \ln\left(\sum_{i=1}^n p_i x_i\right) - \ln\left(\prod_{i=1}^n x_i^{p_i}\right) \\ &\leq \frac{1}{\sum_{i=1}^n p_i x_i - m}\left\{\frac{\ln\left(\sum_{i=1}^n p_i x_i\right) - \ln(m)}{\sum_{i=1}^n p_i x_i - m} - \left(\sum_{i=1}^n p_i x_i\right)^{-1}\right\} \\ &\times \max_{k\in\{1,\dots,n\}}\left(x_k - \sum_{i=1}^n p_i x_i\right)^2 \end{aligned} \tag{6.19}$$

and

$$\begin{aligned} 0 &\leq \min_{k\in\{1,\dots,n\}}\left(x_k - \sum_{i=1}^n p_i x_i\right)^2 \\ &\times \sum_{i=1}^n \frac{p_i}{M - x_i}\left[\frac{\ln(M) - \ln(x_i)}{M - x_i} - \frac{1}{M}\right] \\ &\leq \ln\left(\sum_{i=1}^n p_i x_i\right) - \ln\left(\prod_{i=1}^n x_i^{p_i}\right) \\ &\leq \max_{k\in\{1,\dots,n\}}\left(x_k - \sum_{i=1}^n p_i x_i\right)^2 \\ &\times \sum_{i=1}^n \frac{p_i}{x_i - m}\left\{\frac{1}{m} - \frac{\ln(x_i) - \ln(m)}{x_i - m}\right\}. \end{aligned} \tag{6.20}$$

Moreover, we have

$$0 \leq \frac{1}{2} \min_{k\in\{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \left(\sum_{i=1}^{n} p_i x_i \right)^{-2} \tag{6.21}$$
$$\leq \frac{1}{2} \min_{k\in\{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \sum_{k=1}^{n} p_k \left(\frac{2}{3} \sum_{i=1}^{n} p_i x_i + \frac{1}{3} x_k \right)^{-2}$$
$$\leq \ln \left(\sum_{i=1}^{n} p_i x_i \right) - \ln \left(\prod_{i=1}^{n} x_i^{p_i} \right)$$
$$\leq \frac{1}{2} \max_{k\in\{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \left[\frac{2 \left(\sum_{i=1}^{n} p_i x_i \right)^{-2} + \sum_{i=1}^{n} p_i x_i^{-2}}{3} \right]$$
$$\leq \frac{1}{2} \max_{k\in\{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \sum_{i=1}^{n} p_i x_i^{-2}.$$

If $x_k \in [m, M] \subset (0, \infty)$, $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$, then we also have

$$0 \leq \frac{1}{2M} \min_{k\in\{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \tag{6.22}$$
$$\leq \sum_{i=1}^{n} p_i x_i \ln x_i - \left(\sum_{i=1}^{n} p_i x_i \right) \ln \left(\sum_{i=1}^{n} p_i x_i \right)$$
$$\leq \frac{1}{2m} \max_{k\in\{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 .$$

Also, we have

$$0 \leq \frac{1}{2} \min_{k\in\{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \left(\sum_{i=1}^{n} p_i x_i \right)^{-1} \tag{6.23}$$
$$\leq \frac{1}{2} \min_{k\in\{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \sum_{k=1}^{n} p_k \left(\frac{2}{3} \sum_{i=1}^{n} p_i x_i + \frac{1}{3} x_k \right)^{-1}$$
$$\leq \sum_{i=1}^{n} p_i x_i \ln x_i - \left(\sum_{i=1}^{n} p_i x_i \right) \ln \left(\sum_{i=1}^{n} p_i x_i \right)$$
$$\leq \frac{1}{2} \max_{k\in\{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \left[\frac{2 \left(\sum_{i=1}^{n} p_i x_i \right)^{-1} + \sum_{k=1}^{n} p_k x_k^{-1}}{3} \right]$$
$$\leq \frac{1}{2} \max_{k\in\{1,\dots,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \sum_{k=1}^{n} p_k x_k^{-1}.$$

Finally, for the power function, we only mention that, for $p \in (-\infty, 0) \cup [1, 2] \cup [3, \infty)$, $x_k \in [m, M] \subset (0, \infty)$, $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$, we have

$$
\begin{aligned}
(6.24)\quad 0 &\leq \frac{1}{2} p (p-1) \min_{k \in \{1,...,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \left(\sum_{i=1}^{n} p_i x_i \right)^{p-2} \\
&\leq \frac{1}{2} p (p-1) \min_{k \in \{1,...,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \sum_{i=1}^{n} p_k \left(\frac{2}{3} \sum_{i=1}^{n} p_i x_i + \frac{1}{3} x_k \right)^{p-2} \\
&\leq \sum_{i=1}^{n} p_i x_i^p - \left(\sum_{i=1}^{n} p_i x_i \right)^p \\
&\leq \frac{1}{2} p (p-1) \max_{k \in \{1,...,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \\
&\times \left[\frac{2 \left(\sum_{i=1}^{n} p_i x_i \right)^{p-2} + \sum_{i=1}^{n} p_i x_i^{p-2}}{3} \right] \\
&\leq \frac{1}{2} p (p-1) \max_{k \in \{1,...,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \sum_{i=1}^{n} p_i x_i^{p-2},
\end{aligned}
$$

while for $p \in (2, 3)$, we have

$$
\begin{aligned}
(6.25)\quad 0 &\leq \frac{1}{2} p (p-1) \min_{k \in \{1,...,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \sum_{i=1}^{n} p_i x_i^{p-2} \\
&\leq \frac{1}{2} p (p-1) \min_{k \in \{1,...,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \\
&\times \left[\frac{2 \left(\sum_{i=1}^{n} p_i x_i \right)^{p-2} + \sum_{i=1}^{n} p_i x_i^{p-2}}{3} \right] \\
&\leq \sum_{i=1}^{n} p_i x_i^p - \left(\sum_{i=1}^{n} p_i x_i \right)^p \\
&\leq \frac{1}{2} p (p-1) \max_{k \in \{1,...,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \sum_{i=1}^{n} p_k \left(\frac{2}{3} \sum_{i=1}^{n} p_i x_i + \frac{1}{3} x_k \right)^{p-2} \\
&\leq \frac{1}{2} p (p-1) \max_{k \in \{1,...,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \left(\sum_{i=1}^{n} p_i x_i \right)^{p-2}.
\end{aligned}
$$

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