SOME UPPER BOUNDS FOR THE PERTURBED JENSEN'S GAP OF CONVEX FUNCTIONS

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Abstract. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. In this paper we establish some upper bounds for the perturbed Jensen's gap

$$\int_\Omega \Phi \circ f d\mu - \Phi(v) - \Phi'(v)\left(\int_\Omega f d\mu - v\right)$$

for some classes of differentiable convex functions Φ defined on an interval I and $v \in I$. Applications for exponential, logarithm and power functions are also given.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. Consider the *Lebesgue space*

$$L(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)| \, d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t) \, d\mu(t)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, the author obtained in [7] and [9] the following result:

Theorem 1. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, $\mathring{I}$ the interior of I. If $f : \Omega \to [m, M]$ is so that $\Phi \circ f$, f, $\Phi' \circ f$, $(\Phi' \circ f) f \in L(\Omega, \mu)$, then we have the inequality:*

$$(1.1) \quad 0 \le \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \le \int_\Omega (\Phi' \circ f) f d\mu - \int_\Omega \Phi' \circ f d\mu \int_\Omega f d\mu$$
$$\le \frac{1}{2}\left[\Phi'_-(M) - \Phi'_+(m)\right] \int_\Omega \left| f - \int_\Omega f d\mu \right| d\mu$$
$$\le \frac{1}{2}\left[\Phi'_-(M) - \Phi'_+(m)\right] \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]^{\frac{1}{2}}$$
$$\le \frac{1}{4}\left[\Phi'_-(M) - \Phi'_+(m)\right](M - m).$$

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Remark 1. *We notice that the inequality between the first and the second term in (1.1) in the discrete case was proved in 1994 by Dragomir & Ionescu, see* [12].

For a real function $g : [m, M] \to \mathbb{R}$ and two distinct points α, $\beta \in [m, M]$ we recall that the *divided difference* of g in these points is defined by

$$[\alpha, \beta; g] := \frac{g(\beta) - g(\alpha)}{\beta - \alpha}.$$

Upper and lower bounds for the Jensen's gap were also obtained in [10]:

Theorem 2. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and m, $M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$. If $f : \Omega \to [m, M]$, is μ-measurable and such that f, $\Phi \circ f \in L(\Omega, \mu)$, then by assuming that $\int_\Omega f d\mu \neq m$, M, we have*

$$\left| \int_\Omega \left| \Phi(f) - \Phi\left(\int_\Omega f d\mu \right) \right| \operatorname{sgn}\left(f - \int_\Omega f d\mu \right) d\mu \right| \tag{1.2}$$
$$\leq \int_\Omega (\Phi \circ f) d\mu - \Phi\left(\int_\Omega f d\mu \right)$$
$$\leq \frac{1}{2}\left(\left[\int_\Omega f d\mu, M; \Phi \right] - \left[m, \int_\Omega f d\mu; \Phi \right] \right) \int_\Omega \left| f - \int_\Omega f d\mu \right| d\mu$$
$$\leq \frac{1}{2}\left(\left[\int_\Omega f d\mu, M; \Phi \right] - \left[m, \int_\Omega f d\mu; \Phi \right] \right) \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu \right)^2 \right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\left(\left[\int_\Omega f d\mu, M; \Phi \right] - \left[m, \int_\Omega f d\mu; \Phi \right] \right) (M - m).$$

The constant $\frac{1}{2}$ in the second inequality from (1.2) is best possible.

For other recent reverses of Jensen inequality and applications to divergence measures see [8], [9], [10] and the survey paper [11]. More related results may be found in [1]-[4], [11] and [11]-[13].

Motivated by the above results, in this paper we establish some upper bounds for the perturbed Jensen's gap

$$\int_\Omega \Phi \circ f d\mu - \Phi(v) - \Phi'(v) \left(\int_\Omega f d\mu - v \right)$$

for some classes of differentiable convex functions Φ defined on an interval I and $v \in I$. Applications for exponential, logarithm and power functions are also given.

2. Inequalities Related to Jensen's Result

We have:

Theorem 3. *Let $\Phi : I \to \mathbb{R}$ be a differentiable convex function on the interior of I denoted by $\mathring{I}$. Assume that $f : \Omega \to I$ is μ-measurable and such that f, $\Phi \circ f$,*

$f\cdot\Phi'\circ f\in L(\Omega,\mu)$. *Then for all* $v\in I$,

$$0\le\int_\Omega\Phi\circ fd\mu-\Phi(v)-\Phi'(v)\left(\int_\Omega fd\mu-v\right) \tag{2.1}$$
$$\le\int_\Omega[\Phi'\circ f-\Phi'(v)](f-v)\,d\mu$$
$$\le\begin{cases}\operatorname{essup}_\Omega|\Phi'\circ f-\Phi'(v)|\int_\Omega|f-v|\,d\mu,\\ \\ \left(\int_\Omega|\Phi'\circ f-\Phi'(v)|^p\,d\mu\right)^{1/p}\left(\int_\Omega|f-v|^q\,d\mu\right)^{1/q}\\ p,\ q>1\ \textit{with}\ \frac{1}{p}+\frac{1}{q}=1,\\ \\ \operatorname{essup}_\Omega|f-v|\int_\Omega|\Phi'\circ f-\Phi'(v)|\,d\mu,\end{cases}$$

provided the integrals in the last term are finite.

In particular, we have the reverse of Jensen's inequality

$$0\le\int_\Omega\Phi\circ fd\mu-\Phi\left(\int_\Omega fd\mu\right) \tag{2.2}$$
$$\le\int_\Omega\left[\Phi'\circ f-\Phi'\left(\int_\Omega fd\mu\right)\right]\left(f-\int_\Omega fd\mu\right)d\mu$$
$$=\int_\Omega f\Phi'\circ fd\mu-\int_\Omega\Phi'\circ fd\mu\int_\Omega fd\mu$$
$$\le\begin{cases}\operatorname{essup}_\Omega\left|\Phi'\circ f-\Phi'\left(\int_\Omega fd\mu\right)\right|\int_\Omega\left|f-\int_\Omega fd\mu\right|d\mu,\\ \\ \left(\int_\Omega\left|\Phi'\circ f-\Phi'\left(\int_\Omega fd\mu\right)\right|^p d\mu\right)^{1/p}\left(\int_\Omega\left|f-\int_\Omega fd\mu\right|^q d\mu\right)^{1/q}\\ p,\ q>1\ \textit{with}\ \frac{1}{p}+\frac{1}{q}=1,\\ \\ \operatorname{essup}_\Omega\left|f-\int_\Omega fd\mu\right|\int_\Omega\left|\Phi'\circ f-\Phi'\left(\int_\Omega fd\mu\right)\right|d\mu.\end{cases}$$

Proof. By the gradient inequality we have

$$\Phi'(u)(u-v)\ge\Phi(u)-\Phi(v)\ge\Phi'(v)(u-v)$$

for all u, $v\in I$. This can be written as

$$\Phi'(u)(u-v)-\Phi'(v)(u-v)\ge\Phi(u)-\Phi(v)-\Phi'(v)(u-v)\ge 0,$$

which implies that

$$[\Phi'(f(x))-\Phi'(v)](f(x)-v)\ge\Phi(f(x))-\Phi(v)-\Phi'(v)(f(x)-v)\ge 0,$$

for all $v\in I$ and $x\in\Omega$.

If we take the integral $\int_\Omega$, then we get

$$\int_\Omega[\Phi'(f(x))-\Phi'(v)](f(x)-v)\,d\mu(x)$$
$$\ge\int_\Omega\Phi(f(x))-\Phi(v)-\Phi'(v)\left(\int_\Omega f(x)\,d\mu(x)-v\right)\ge 0,$$

which proves the first inequality in (2.1).

Using Hölder's integral inequality, we have

$$\int_{\Omega}\left[\Phi'\circ f-\Phi'(v)\right](f-v)\,d\mu \leq \begin{cases} \operatorname{essup}_{\Omega}\left|\Phi'\circ f-\Phi'(v)\right|\int_{\Omega}|f-v|\,d\mu, \\ \\ \left(\int_{\Omega}\left|\Phi'\circ f-\Phi'(v)\right|^{p}d\mu\right)^{1/p}\left(\int_{\Omega}|f-v|^{q}\,d\mu\right)^{1/q} \\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \operatorname{essup}_{\Omega}|f-v|\int_{\Omega}\left|\Phi'\circ f-\Phi'(v)\right|d\mu, \end{cases}$$

which proves the last part of (2.1).

The inequality (2.2) follows by (2.1) on taking $v=\int_{\Omega}fd\mu$ and observing that

$$\begin{aligned} &\int_{\Omega}\left[\Phi'\circ f-\Phi'\left(\int_{\Omega}fd\mu\right)\right]\left(f-\int_{\Omega}fd\mu\right)d\mu \\ &=\int_{\Omega}\Phi'\circ f\left(f-\int_{\Omega}fd\mu\right)d\mu-\Phi'\left(\int_{\Omega}fd\mu\right)\int_{\Omega}\left(f-\int_{\Omega}fd\mu\right)d\mu \\ &=\int_{\Omega}f\Phi'\circ fd\mu-\int_{\Omega}\Phi'\circ fd\mu\int_{\Omega}fd\mu. \end{aligned}$$

□

Remark 2. *The discrete case is useful for applications and we will state the corresponding inequalities here. Let $\Phi:I\to\mathbb{R}$ be a differentiable convex function on $\mathring{I}$ and $x_k\in[m,M]$, $p_k\geq 0$ for $k\in\{1,...,n\}$ with $\sum_{k=1}^{n}p_k=1$. Then for all $v\in I$,*

$$\begin{aligned} 0 &\leq \sum_{k=1}^{n}p_k\Phi(x_k)-\Phi(v)-\Phi'(v)\left(\sum_{k=1}^{n}p_kx_k-v\right) \\ &\leq \sum_{k=1}^{n}p_k\left[\Phi'(x_k)-\Phi'(v)\right](x_k-v) \\ &\leq \begin{cases} \max_{k\in\{1,\dots,n\}}\left|\Phi'(x_k)-\Phi'(v)\right|\sum_{k=1}^{n}p_k|x_k-v|, \\ \\ \left(\sum_{k=1}^{n}p_k\left|\Phi'(x_k)-\Phi'(v)\right|^{p}\right)^{1/p}\left(\sum_{k=1}^{n}p_k|x_k-v|^{q}\right)^{1/q} \\ p,\ q>1 \textit{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \max_{k\in\{1,\dots,n\}}|x_k-v|\sum_{k=1}^{n}p_k\left|\Phi'(x_k)-\Phi'(v)\right|. \end{cases} \end{aligned} \tag{2.3}$$

In particular, we have the reverse of Jensen's inequality

$$\begin{aligned} 0 &\leq \sum_{k=1}^{n}p_k\Phi(x_k)-\Phi\left(\sum_{k=1}^{n}p_kx_k\right) \\ &\leq \sum_{k=1}^{n}p_kx_k\Phi'(x_k)-\sum_{k=1}^{n}p_k\Phi'(x_k)\sum_{k=1}^{n}p_kx_k \end{aligned} \tag{2.4}$$

$$\leq \begin{cases} \max_{k\in\{1,\dots,n\}} \left|\Phi'\left(x_k\right) - \Phi'\left(\sum_{j=1}^n p_j x_j\right)\right| \\ \times \sum_{k=1}^n p_k \left|x_k - \sum_{j=1}^n p_j x_j\right|, \\ \\ \left(\sum_{k=1}^n p_k \left|\Phi'\left(x_k\right) - \Phi'\left(\sum_{j=1}^n p_j x_j\right)\right|^p\right)^{1/p} \\ \times \left(\sum_{k=1}^n p_k \left|x_k - \sum_{j=1}^n p_j x_j\right|^q\right)^{1/q} \\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \max_{k\in\{1,\dots,n\}} \left|x_k - \sum_{j=1}^n p_j x_j\right| \\ \times \sum_{k=1}^n p_k \left|\Phi'\left(x_k\right) - \Phi'\left(\sum_{j=1}^n p_j x_j\right)\right|. \end{cases}$$

Corollary 1. *With the assumptions of Theorem 3 and if $\int_\Omega \Phi' \circ f d\mu \neq 0$ and the Slater point*

$$s := \frac{\int_\Omega f \cdot (\Phi' \circ f)\, d\mu}{\int_\Omega \Phi' \circ f d\mu} \in I, \tag{2.5}$$

then

$$\begin{aligned} 0 &\leq \int_\Omega \Phi \circ f d\mu - \Phi(s) - \Phi'(s)\left(\int_\Omega f d\mu - s\right) \\ &\leq \int_\Omega \left[\Phi' \circ f - \Phi'(s)\right](f-s)\, d\mu \\ &\leq \begin{cases} \operatorname{essup}_\Omega \left|\Phi' \circ f - \Phi'(v)\right| \int_\Omega |f-v|\, d\mu, \\ \\ \left(\int_\Omega \left|\Phi' \circ f - \Phi'(v)\right|^p d\mu\right)^{1/p} \left(\int_\Omega |f-v|^q d\mu\right)^{1/q} \\ p, q > 1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \operatorname{essup}_\Omega |f-v| \int_\Omega \left|\Phi' \circ f - \Phi'(v)\right| d\mu. \end{cases} \end{aligned} \tag{2.6}$$

Remark 3. *Since, for the Slater point s defined above in (2.5), we have*

$$\begin{aligned} &\int_\Omega \left[\Phi' \circ f - \Phi'(s)\right](f-s)\, d\mu \\ &= \int_\Omega f\Phi' \circ f - \Phi'(s)\int_\Omega f d\mu - s\int_\Omega \Phi' \circ f + \Phi'(s)\, s \\ &= \Phi'(s)\left(s - \int_\Omega f d\mu\right), \end{aligned}$$

hence (2.6) can also be written as

$$\begin{aligned} 0 \leq \int_\Omega \Phi \circ f d\mu - \Phi(s) + \Phi'(s)\left(s - \int_\Omega f d\mu\right) &\leq \Phi'(s)\left(s - \int_\Omega f d\mu\right) \\ \leq \begin{cases} \operatorname{essup}_\Omega \left|\Phi' \circ f - \Phi'(s)\right| \int_\Omega |f-s|\, d\mu, \\ \\ \left(\int_\Omega \left|\Phi' \circ f - \Phi'(s)\right|^p d\mu\right)^{1/p} \left(\int_\Omega |f-s|^q d\mu\right)^{1/q} \\ p,\ q > 1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \operatorname{essup}_\Omega |f-s| \int_\Omega \left|\Phi' \circ f - \Phi'(s)\right| d\mu. \end{cases} \end{aligned} \tag{2.7}$$

Corollary 2. *With the assumptions of Theorem 3 and if there exists an interval* $[m,M]\subset I$ *such that* $m\leq f\leq M$ μ*-a.e. on* Ω*, then*

$$
\begin{aligned}
(2.8)\qquad 0 &\leq \int_{\Omega}\Phi\circ f d\mu-\Phi\left(\frac{m+M}{2}\right)-\Phi'\left(\frac{m+M}{2}\right)\left(\int_{\Omega}fd\mu-\frac{m+M}{2}\right)\\
&\leq \int_{\Omega}\left[\Phi'\circ f-\Phi'\left(\frac{m+M}{2}\right)\right]\left(f-\frac{m+M}{2}\right)d\mu\\
&\leq\begin{cases}
\operatorname{essup}_{\Omega}\left|\Phi'\circ f-\Phi'\left(\frac{m+M}{2}\right)\right|\int_{\Omega}\left|f-\frac{m+M}{2}\right|d\mu,\\[2ex]
\left(\int_{\Omega}\left|\Phi'\circ f-\Phi'\left(\frac{m+M}{2}\right)\right|^{p}d\mu\right)^{1/p}\left(\int_{\Omega}\left|f-\frac{m+M}{2}\right|^{q}d\mu\right)^{1/q}\\
p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,\\[2ex]
\operatorname{essup}_{\Omega}\left|f-\frac{m+M}{2}\right|\int_{\Omega}\left|\Phi'\circ f-\Phi'\left(\frac{m+M}{2}\right)\right|d\mu.
\end{cases}
\end{aligned}
$$

Remark 4. *Since* $m\leq f\leq M$ μ*-a.e. on* Ω *is equivalent to*

$$\left|f-\frac{m+M}{2}\right|\leq\frac{1}{2}(M-m)\ \ \mu\text{-a.e.on }\Omega,$$

then from the third branch of (2.8) we obtain the following inequality of interest

$$
\begin{aligned}
(2.9)\qquad 0 &\leq \int_{\Omega}\Phi\circ f d\mu-\Phi\left(\frac{m+M}{2}\right)-\Phi'\left(\frac{m+M}{2}\right)\left(\int_{\Omega}fd\mu-\frac{m+M}{2}\right)\\
&\leq \int_{\Omega}\left[\Phi'\circ f-\Phi'\left(\frac{m+M}{2}\right)\right]\left(f-\frac{m+M}{2}\right)d\mu\\
&\leq \frac{1}{2}(M-m)\int_{\Omega}\left|\Phi'\circ f-\Phi'\left(\frac{m+M}{2}\right)\right|d\mu.
\end{aligned}
$$

Corollary 3. *With the assumptions of Corollary 2 we have*

$$
\begin{aligned}
(2.10)\qquad 0 &\leq \int_{\Omega}\Phi\circ f d\mu-\frac{2}{M-m}\int_{m}^{M}\Phi(v)\,dv\\
&\quad+\frac{1}{M-m}\left[\Phi(M)\left(M-\int_{\Omega}fd\mu\right)+\Phi(m)\left(\int_{\Omega}fd\mu-m\right)\right]\\
&\leq \frac{1}{M-m}\int_{m}^{M}\left(\int_{\Omega}\left[\Phi'\circ f-\Phi'(v)\right](f-v)\,d\mu\right)dv
\end{aligned}
$$

$$
\leq\begin{cases}
\frac{1}{M-m}\int_{m}^{M}\left(\operatorname{essup}_{\Omega}\left|\Phi'\circ f-\Phi'(v)\right|\int_{\Omega}|f-v|\,d\mu\right)dv,\\[2ex]
\frac{1}{M-m}\int_{m}^{M}\left[\left(\int_{\Omega}\left|\Phi'\circ f-\Phi'(v)\right|^{p}d\mu\right)^{1/p}\left(\int_{\Omega}|f-v|^{q}\,d\mu\right)^{1/q}\right]dv\\
p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,\\[2ex]
\frac{1}{M-m}\int_{m}^{M}\left(\operatorname{essup}_{\Omega}|f-v|\int_{\Omega}\left|\Phi'\circ f-\Phi'(v)\right|d\mu\right)dv,
\end{cases}
$$

$$\leq \begin{cases} \sup_{v\in[m,M]}\left(\operatorname{essup}_{\Omega}\left|\Phi'\circ f-\Phi'(v)\right|\right)\frac{1}{M-m}\int_m^M\left(\int_\Omega |f-v|\,d\mu\right)dv, \\ \left(\frac{1}{M-m}\int_m^M\int_\Omega \left|\Phi'\circ f-\Phi'(v)\right|^p d\mu dv\right)^{1/p}\left(\int_\Omega |f-v|^q\,d\mu dv\right)^{1/q} \\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \sup_{v\in[m,M]}\left(\operatorname{essup}_{\Omega}|f-v|\right)\frac{1}{M-m}\int_m^M\int_\Omega \left|\Phi'\circ f-\Phi'(v)\right|d\mu dv. \end{cases}$$

Proof. It follows by taking the integral mean on (2.1) and observing that, integrating by parts, we have

$$\begin{aligned} &\int_m^M \Phi(v)\,dv+\int_m^M \Phi'(v)\left(\int_\Omega f d\mu-v\right)dv \\ &=\int_m^M \Phi(v)\,dv+\Phi(v)\left(\int_\Omega f d\mu-v\right)\Big|_m^M+\int_m^M \Phi(v)\,dv \\ &=2\int_m^M \Phi(v)\,dv-\Phi(M)\left(M-\int_\Omega f d\mu\right)-\Phi(m)\left(\int_\Omega f d\mu-m\right). \end{aligned}$$

□

Corollary 4. *With the assumptions of Corollary 2 and if there is a $t\in I$, a trapezoid point, such that*

$$\Phi(t)=\frac{\Phi(m)+\Phi(M)}{2}, \tag{2.11}$$

then we have

$$\begin{aligned} 0 &\leq \int_\Omega \Phi\circ f d\mu-\frac{\Phi(m)+\Phi(M)}{2}-\Phi'(t)\left(\int_\Omega f d\mu-t\right) \\ &\leq \int_\Omega \left[\Phi'\circ f-\Phi'(t)\right](f-t)\,d\mu \\ &\leq \begin{cases} \operatorname{essup}_{\Omega}\left|\Phi'\circ f-\Phi'(t)\right|\int_\Omega |f-t|\,d\mu, \\ \left(\int_\Omega \left|\Phi'\circ f-\Phi'(t)\right|^p d\mu\right)^{1/p}\left(\int_\Omega |f-t|^q\,d\mu\right)^{1/q} \\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \operatorname{essup}_{\Omega}|f-t|\int_\Omega \left|\Phi'\circ f-\Phi'(t)\right|d\mu. \end{cases} \end{aligned} \tag{2.12}$$

When more conditions are imposed on the derivative, we have:

Corollary 5. *With the assumptions of Theorem 3 and if Φ' is Lipschitzian with the constant $L>0$ on $\mathring{I}$, namely*

$$\left|\Phi'(u)-\Phi'(v)\right|\leq L|u-v| \text{ for all } u,v\in\mathring{I},$$

then

$$0\leq \int_\Omega \Phi\circ f d\mu-\Phi(v)-\Phi'(v)\left(\int_\Omega f d\mu-v\right)\leq L\int_\Omega (f-v)^2\,d\mu. \tag{2.13}$$

In particular, we have the following upper bound in terms of variance

$$0\leq \int_\Omega \Phi\circ f d\mu-\Phi\left(\int_\Omega f d\mu\right)\leq L\left[\int_\Omega f^2 d\mu-\left(\int_\Omega f d\mu\right)^2\right].$$

If, in addition, the condition (2.5) is satisfied, then

$$0 \leq \int_\Omega \Phi \circ f d\mu - \Phi(s) - \Phi'(s)\left(\int_\Omega f d\mu - s\right) \leq L\left(\int_\Omega f^2 d\mu - 2s\int_\Omega f d\mu + s^2\right). \tag{2.14}$$

If there exists an interval $[m, M] \subset I$ *such that* $m \leq f \leq M$ μ*-a.e. on* Ω*, then*

$$0 \leq \int_\Omega \Phi \circ f d\mu - \Phi\left(\frac{m+M}{2}\right) - \Phi'\left(\frac{m+M}{2}\right)\left(\int_\Omega f d\mu - \frac{m+M}{2}\right) \leq L\int_\Omega \left(f - \frac{m+M}{2}\right)^2 d\mu \leq \frac{1}{4}L(M-m)^2. \tag{2.15}$$

We also have

$$\begin{aligned} 0 &\leq \frac{1}{2}\left\{\int_\Omega \Phi \circ f d\mu + \frac{1}{M-m}\left[\Phi(M)\left(M - \int_\Omega f d\mu\right) + \Phi(m)\left(\int_\Omega f d\mu - m\right)\right]\right\} \\ &\qquad - \frac{1}{M-m}\int_m^M \Phi(v)\, dv \\ &\leq \frac{L}{2(M-m)}\int_m^M \left(\int_\Omega (f-u)^2 d\mu\right) du \\ &= \frac{1}{2}L\left(\int_\Omega f^2 d\mu - (m+M)\int_\Omega f d\mu + \frac{m^2 + mM + M^2}{3}\right). \end{aligned} \tag{2.16}$$

If the condition (2.11) is satisfied, then

$$0 \leq \int_\Omega \Phi \circ f d\mu - \frac{\Phi(m) + \Phi(M)}{2} - \Phi'(t)\left(\int_\Omega f d\mu - t\right) \leq L\left(\int_\Omega f^2 d\mu - 2t\int_\Omega f d\mu + t^2\right). \tag{2.17}$$

3. Further Results for Twice Differentiable Functions

The case of twice differentiable functions is as follows:

Theorem 4. *Let* $\Phi : I \to \mathbb{R}$ *be a twice differentiable convex function on the interior of* I *denoted by* $\mathring{I}$*. Assume that* $f : \Omega \to I$ *is* μ*-measurable and such that* f, $\Phi \circ f$,

$f\Phi'\circ f\in L(\Omega,\mu)$. *Then for all* $v\in I$,

$$0\leq\int_\Omega\Phi\circ f d\mu-\Phi(v)-\Phi'(v)\left(\int_\Omega f d\mu-v\right)\tag{3.1}$$
$$\leq\int_\Omega(f-v)^2\left(\int_0^1\Phi''((1-t)f+tv)\,dt\right)d\mu$$
$$\leq\begin{cases}\operatorname{essup}_\Omega(f-v)^2\int_\Omega\left(\int_0^1\Phi''((1-t)f+tv)\,dt\right)d\mu,\\[2ex] \left(\int_\Omega(f-v)^{2p}d\mu\right)^{1/p}\left(\int_\Omega\left(\int_0^1\Phi''((1-t)f+tv)\,dt\right)^q d\mu\right)^{1/q}\\ p,\ q>1\ \textit{with}\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \operatorname{essup}_\Omega\left(\int_0^1\Phi''((1-t)f+tv)\,dt\right)\int_\Omega(f-v)^2d\mu,\end{cases}$$

provided the integrals in the last term are finite.

In particular, we have the reverse of Jensen's inequality

$$0\leq\int_\Omega\Phi\circ f d\mu-\Phi\left(\int_\Omega f d\mu\right)\tag{3.2}$$
$$\leq\int_\Omega\left(f-\int_\Omega f d\mu\right)^2\left(\int_0^1\Phi''\left((1-t)f+t\int_\Omega f d\mu\right)dt\right)d\mu$$
$$\leq\begin{cases}\operatorname{essup}_\Omega\left(f-\int_\Omega f d\mu\right)^2\int_\Omega\left(\int_0^1\Phi''\left((1-t)f+t\int_\Omega f d\mu\right)dt\right)d\mu\\[2ex] \left(\int_\Omega\left(f-\int_\Omega f d\mu\right)^{2p}d\mu\right)^{1/p}\\ \times\left(\int_\Omega\left(\int_0^1\Phi''\left((1-t)f+t\int_\Omega f d\mu\right)dt\right)^q d\mu\right)^{1/q}\\ p,\ q>1\ \textit{with}\ \frac{1}{p}+\frac{1}{q}=1\\[2ex] \operatorname{essup}_\Omega\left(\int_0^1\Phi''((1-t)f+tv)\,dt\right)\left[\int_\Omega f^2d\mu-\left(\int_\Omega f d\mu\right)^2\right].\end{cases}$$

Proof. Since Φ is twice differentiable, then for all $x\in\Omega$ we have

$$\Phi'\circ f(x)-\Phi'(v)=\int_v^{f(x)}\Phi''(\tau)\,d\tau=(f(x)-v)\int_0^1\Phi''((1-t)f(x)+tv)\,dt.$$

Therefore

$$\int_\Omega[\Phi'\circ f-\Phi'(v)](f-v)\,d\mu=\int_\Omega(f-v)^2\left(\int_0^1\Phi''((1-t)f+tv)\,dt\right)d\mu$$

and by (2.1) we get the first inequality in (3.1).

By using the Hölder's integral inequality we have

$$\int_{\Omega}(f-v)^{2}\left(\int_{0}^{1}\Phi''((1-t)f+tv)\,dt\right)d\mu$$
$$\leq\begin{cases}\operatorname{essup}_{\Omega}(f-v)^{2}\int_{\Omega}\left(\int_{0}^{1}\Phi''((1-t)f+tv)\,dt\right)d\mu\\ \\ \left(\int_{\Omega}(f-v)^{2p}d\mu\right)^{1/p}\left(\int_{\Omega}\left(\int_{0}^{1}\Phi''((1-t)f+tv)\,dt\right)^{q}d\mu\right)^{1/q}\\ p,\ q>1\text{ with }\frac{1}{p}+\frac{1}{q}=1\\ \\ \operatorname{essup}_{\Omega}\left(\int_{0}^{1}\Phi''((1-t)f+tv)\,dt\right)\int_{\Omega}(f-v)^{2}d\mu,\end{cases}$$

which proves the second part of (3.2). □

Remark 5. *Let $\Phi: I\to\mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$ and $x_k\in[m,M]$, $p_k\geq 0$ for $k\in\{1,...,n\}$ with $\sum_{k=1}^{n}p_k=1$. Then for all $v\in I$,*

$$0\leq\sum_{k=1}^{n}p_k\Phi(x_k)-\Phi(v)-\Phi'(v)\left(\sum_{k=1}^{n}p_kx_k-v\right)\tag{3.3}$$
$$\leq\sum_{k=1}^{n}p_k(x_k-v)^{2}\left(\int_{0}^{1}\Phi''((1-t)x_k+tv)\,dt\right)$$
$$\leq\begin{cases}\max_{k\in\{1,\dots,n\}}(x_k-v)^{2}\sum_{k=1}^{n}p_k\left(\int_{0}^{1}\Phi''((1-t)x_k+tv)\,dt\right),\\ \\ \left(\int_{\Omega}(x_k-v)^{2p}d\mu\right)^{1/p}\left(\sum_{k=1}^{n}p_k\left(\int_{0}^{1}\Phi''((1-t)x_k+tv)\,dt\right)^{q}\right)^{1/q}\\ p,\ q>1\ \textit{with}\ \frac{1}{p}+\frac{1}{q}=1,\\ \\ \max_{k\in\{1,\dots,n\}}\left(\int_{0}^{1}\Phi''((1-t)x_k+tv)\,dt\right)\sum_{k=1}^{n}p_k(x_k-v)^{2}.\end{cases}$$

In particular, we have

$$0\leq\sum_{k=1}^{n}p_k\Phi(x_k)-\Phi\left(\sum_{i=1}^{n}p_ix_i\right)\tag{3.4}$$
$$\leq\sum_{k=1}^{n}p_k\left(x_k-\sum_{i=1}^{n}p_ix_i\right)^{2}\left(\int_{0}^{1}\Phi''\left((1-t)x_k+t\sum_{i=1}^{n}p_ix_i\right)dt\right)$$

$$\leq \begin{cases} \max_{k\in\{1,\dots,n\}}\left(x_k-\sum_{i=1}^n p_i x_i\right)^2 \\ \times\sum_{k=1}^n p_k\left(\int_0^1 \Phi''\left((1-t)\,x_k+t\sum_{i=1}^n p_i x_i\right)dt\right), \\ \\ \left(\int_\Omega \left(x_k-\sum_{i=1}^n p_i x_i\right)^{2p} d\mu\right)^{1/p} \\ \times\left(\sum_{k=1}^n p_k\left(\int_0^1 \Phi''\left((1-t)\,x_k+t\sum_{i=1}^n p_i x_i\right)dt\right)^q\right)^{1/q} \\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \max_{k\in\{1,\dots,n\}}\left(\int_0^1 \Phi''\left((1-t)\,x_k+t\sum_{i=1}^n p_i x_i\right)dt\right) \\ \times\left[\sum_{k=1}^n p_k x_k^2-\left(\sum_{i=1}^n p_i x_i\right)^2\right]. \end{cases}$$

The above result can be used to provide further upper bounds if several assumptions on the behavior of the second derivative are made:

Corollary 6. *With the assumptions of Theorem 4 and if* $\Phi'' \leq \Gamma$ *almost everywhere on* $\mathring{I}$, *where* $\Gamma > 0$, *then for all* $v \in I$,

$$0 \leq \int_\Omega \Phi\circ f d\mu - \Phi(v) - \Phi'(v)\left(\int_\Omega f d\mu - v\right) \leq \Gamma\int_\Omega (f-v)^2. \tag{3.5}$$

In particular,

$$0 \leq \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \leq \Gamma\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]. \tag{3.6}$$

Corollary 7. *With the assumptions of Theorem 4 and if* Φ'' *is convex on* $\mathring{I}$, *then for all* $v \in I$,

$$\begin{aligned} 0 &\leq \int_\Omega \Phi\circ f d\mu - \Phi(v) - \Phi'(v)\left(\int_\Omega f d\mu - v\right) \\ &\leq \int_\Omega (f-v)^2\left(\frac{\Phi''\circ f+\Phi''(v)}{2}\right)d\mu \\ &\leq \begin{cases} \operatorname{essup}_\Omega (f-v)^2\int_\Omega\left(\frac{\Phi''\circ f+\Phi''(v)}{2}\right)d\mu, \\ \\ \left(\int_\Omega (f-v)^{2p}d\mu\right)^{1/p}\left(\int_\Omega\left(\frac{\Phi''\circ f+\Phi''(v)}{2}\right)^q d\mu\right)^{1/q} \\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \operatorname{essup}_\Omega\left(\frac{\Phi''\circ f+\Phi''(v)}{2}\right)\int_\Omega (f-v)^2 d\mu. \end{cases} \end{aligned} \tag{3.7}$$

In particular,

$$0 \leq \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \tag{3.8}$$
$$\leq \int_\Omega (f-v)^2 \left(\frac{\Phi'' \circ f + \Phi''\left(\int_\Omega f d\mu\right)}{2}\right) d\mu$$
$$\leq \begin{cases} \operatorname{essup}_\Omega \left(f - \int_\Omega f d\mu\right)^2 \int_\Omega \left(\frac{\Phi''\circ f+\Phi''\left(\int_\Omega f d\mu\right)}{2}\right) d\mu, \\ \\ \left(\int_\Omega \left(f-\int_\Omega f d\mu\right)^{2p} d\mu\right)^{1/p} \left(\int_\Omega \left(\frac{\Phi''\circ f+\Phi''\left(\int_\Omega f d\mu\right)}{2}\right)^q d\mu\right)^{1/q} \\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \operatorname{essup}_\Omega \left(\frac{\Phi''\circ f+\Phi''\left(\int_\Omega f d\mu\right)}{2}\right) \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]. \end{cases}$$

If Φ'' is concave on $\mathring{I}$, then for all $v \in I$

$$0 \leq \int_\Omega \Phi \circ f d\mu - \Phi(v) - \Phi'(v)\left(\int_\Omega f d\mu - v\right) \tag{3.9}$$
$$\leq \int_\Omega (f-v)^2 \Phi''\left(\frac{f+v}{2}\right) d\mu$$
$$\leq \begin{cases} \operatorname{essup}_\Omega (f-v)^2 \int_\Omega \Phi''\left(\frac{f+v}{2}\right) d\mu, \\ \\ \left(\int_\Omega (f-v)^{2p} d\mu\right)^{1/p} \left(\int_\Omega \left(\Phi''\left(\frac{f+v}{2}\right)\right)^q d\mu\right)^{1/q} \\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \operatorname{essup}_\Omega \left(\Phi''\left(\frac{f+v}{2}\right)\right) \int_\Omega (f-v)^2 d\mu. \end{cases}$$

In particular,

$$0 \leq \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \tag{3.10}$$
$$\leq \int_\Omega \left(f - \int_\Omega f d\mu\right)^2 \Phi''\left(\frac{f+\int_\Omega f d\mu}{2}\right) d\mu$$
$$\leq \begin{cases} \operatorname{essup}_\Omega \left(f-\int_\Omega f d\mu\right)^2 \int_\Omega \Phi''\left(\frac{f+\int_\Omega f d\mu}{2}\right) d\mu, \\ \\ \left(\int_\Omega \left(f-\int_\Omega f d\mu\right)^{2p} d\mu\right)^{1/p} \left(\int_\Omega \left(\Phi''\left(\frac{f+\int_\Omega f d\mu}{2}\right)\right)^q d\mu\right)^{1/q} \\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \operatorname{essup}_\Omega \left(\Phi''\left(\frac{f+\int_\Omega f d\mu}{2}\right)\right) \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]. \end{cases}$$

4. Some Examples

We consider the convex function $\Phi_{\alpha}:\mathbb{R}\rightarrow(0,\infty)$ defined by $\Phi_{\alpha}(t)=\exp(\alpha t)$. Then by (2.1) we get

$$\begin{aligned}0&\leq\int_{\Omega}\exp(\alpha f)\,d\mu-\left(\alpha\int_{\Omega}fd\mu-\alpha v+1\right)\exp\alpha v\\&\leq\alpha\int_{\Omega}[\exp(\alpha f)-\exp(\alpha v)](f-v)\,d\mu\end{aligned}\tag{4.1}$$

for all $v\in\mathbb{R}$, provided the involved integrals exist.

If we consider the convex function $\Phi:(0,\infty)\rightarrow\mathbb{R}$, $\Phi(t)=-\ln t$, then by (2.1) we obtain

$$0\leq\ln(v)+\frac{1}{v}\left(\int_{\Omega}fd\mu-v\right)-\int_{\Omega}\ln fd\mu\leq\int_{\Omega}\frac{(f-v)^{2}}{fv}d\mu\tag{4.2}$$

for all $v\in(0,\infty)$, provided the involved integrals exist.

Further, if we consider the convex function $\Phi:(0,\infty)\rightarrow\mathbb{R}$, $\Phi(t)=t\ln t$, then by (2.1) we derive

$$\begin{aligned}0&\leq\int_{\Omega}f\ln fd\mu-v\ln v-(\ln v+1)\left(\int_{\Omega}fd\mu-v\right)\\&\leq\int_{\Omega}(\ln f-\ln v)(f-v)\,d\mu\end{aligned}\tag{4.3}$$

for all $v\in(0,\infty)$, provided the involved integrals exist.

The power function $\Phi_{p}:(0,\infty)\rightarrow(0,\infty)$, $\Phi_{p}(x)=x^{p}$ is convex for $p\in(-\infty,0)\cup[1,\infty)$. By (2.1) we have

$$0\leq\int_{\Omega}f^{p}d\mu-v^{p}-pv^{p-1}\left(\int_{\Omega}fd\mu-v\right)\leq p\int_{\Omega}\left(f^{p-1}-v^{p-1}\right)(f-v)\,d\mu\tag{4.4}$$

for all $v\in(0,\infty)$, provided the involved integrals exist.

For the convex function $\Phi_{\alpha}:\mathbb{R}\rightarrow(0,\infty)$ defined by $\Phi_{\alpha}(t)=\exp(\alpha t)$ we have for $[m,M]\subset\mathbb{R}$ that

$$\begin{aligned}L&:=\sup_{t\in[m,M]}\Phi_{\alpha}''(t)=\alpha^{2}\sup_{t\in[m,M]}\exp(\alpha t)\\&=\alpha^{2}\begin{cases}\exp(\alpha m),\ \alpha<0\\ \\ \exp(\alpha M),\ \alpha>0\end{cases}=:E(\alpha,[m,M]).\end{aligned}$$

Assume that $m\leq f\leq M$ μ-a.e. on Ω, then by (2.15) for Φ_{α} we have that

$$\begin{aligned}0&\leq\int_{\Omega}\exp(\alpha f)\,d\mu-\exp\left[\alpha\left(\frac{m+M}{2}\right)\right]\left(1+\int_{\Omega}fd\mu-\frac{m+M}{2}\right)\\&\leq E_{2}(\alpha,[m,M])\int_{\Omega}\left(f-\frac{m+M}{2}\right)^{2}d\mu\leq\frac{1}{4}E(\alpha,[m,M])(M-m)^{2},\end{aligned}\tag{4.5}$$

while from (2.16) we have

$$(4.6)\quad 0\leq \frac{1}{2}\left\{\int_{\Omega}\exp\left(\alpha f\right)d\mu + \frac{1}{M-m}\left[\exp\left(\alpha M\right)\left(M-\int_{\Omega}fd\mu\right)+\exp\left(\alpha m\right)\left(\int_{\Omega}fd\mu-m\right)\right]\right\} - \frac{\left(\exp\left(\alpha M\right)-\exp\left(\alpha M\right)\right)}{\alpha\left(M-m\right)} \leq \frac{1}{2}E\left(\alpha,[m,M]\right)\left(\int_{\Omega}f^{2}d\mu-(m+M)\int_{\Omega}fd\mu+\frac{m^{2}+mM+M^{2}}{3}\right).$$

We consider the convex function $\Phi:(0,\infty)\rightarrow\mathbb{R}$, $\Phi\left(x\right)=-\ln x$. Then $\Phi''\left(x\right)=\frac{1}{x^{2}}$, which is convex and decreasing on $(0,\infty)$. If $x\in[m,M]\subset(0,\infty)$, then we also have

$$\Phi''\left(x\right)\leq\frac{1}{m^{2}}.$$

Assume that $0<m\leq f\leq M$ μ-a.e. on Ω, then by (2.15) for Φ we have that

$$(4.7)\quad 0\leq\ln\left(\frac{m+M}{2}\right)+\left(\frac{m+M}{2}\right)^{-1}\left(\int_{\Omega}fd\mu-\frac{m+M}{2}\right)-\int_{\Omega}\ln fd\mu \leq\frac{1}{m^{2}}\int_{\Omega}\left(f-\frac{m+M}{2}\right)^{2}d\mu\leq\frac{1}{4}\left(\frac{M}{m}-1\right)^{2},$$

while from (2.16) we have

$$(4.8)\quad 0\leq\frac{M\ln M-m\ln m}{M-m}-1 -\frac{1}{2}\left\{\int_{\Omega}\ln fd\mu+\frac{1}{M-m}\left[\ln\left(M\right)\left(M-\int_{\Omega}fd\mu\right)+\ln\left(m\right)\left(\int_{\Omega}fd\mu-m\right)\right]\right\} \leq\frac{1}{2m^{2}\left(M-m\right)}\int_{m}^{M}\left(\int_{\Omega}\left(f-u\right)^{2}d\mu\right)du =\frac{1}{2m^{2}}\left(\int_{\Omega}f^{2}d\mu-(m+M)\int_{\Omega}fd\mu+\frac{m^{2}+mM+M^{2}}{3}\right).$$

The power function $\Phi_{p}:(0,\infty)\rightarrow(0,\infty)$, $\Phi_{p}\left(x\right)=x^{p}$ is convex for $p\in(-\infty,0)\cup[1,\infty)$. We have $\Phi_{p}''\left(x\right)=p\left(p-1\right)x^{p-2}$, $x\in(0,\infty)$. If $p\in(-\infty,0)\cup[1,2)$ then Φ_{p}'' is decreasing and if $p\in[2,\infty)$ then Φ_{p}'' is increasing.

For $p\in(-\infty,0)\cup[1,\infty)$ and $[m,M]\subset(0,\infty)$ we define the bound

$$(4.9)\quad B\left(p,[m,M]\right):=p\left(p-1\right)\begin{cases}m^{p-2}\text{ if }(-\infty,0)\cup[1,2),\\ \\ M^{p-2}\text{ if }p\in[2,\infty).\end{cases}$$

Assume that $0<m\leq f\leq M$ μ-a.e. on Ω, then by (2.15) for Φ_{p} we have that

$$(4.10)\quad 0\leq\int_{\Omega}f^{p}d\mu-\left(\frac{m+M}{2}\right)^{p}-p\left(\frac{m+M}{2}\right)^{p-1}\left(\int_{\Omega}fd\mu-\frac{m+M}{2}\right) \leq B\left(p,[m,M]\right)\int_{\Omega}\left(f-\frac{m+M}{2}\right)^{2}d\mu\leq\frac{1}{4}B\left(p,[m,M]\right)\left(M-m\right)^{2},$$

while from (2.16) we have

$$0 \le \frac{1}{2}\left\{\int_\Omega f^p d\mu + \frac{1}{M-m}\left[M^p\left(M - \int_\Omega f d\mu\right) + m^p\left(\int_\Omega f d\mu - m\right)\right]\right\} - \frac{M^{p+1}-m^{p+1}}{(p+1)(M-m)} \le \frac{1}{2}B(p,[m,M])\left(\int_\Omega f^2 d\mu - (m+M)\int_\Omega f d\mu + \frac{m^2+mM+M^2}{3}\right). \tag{4.11}$$

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